FAMILIES OF 4-SETS WITHOUT PROPERTY B

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ABSTRACT. A family $\mathcal{F}$ of finite sets is said to have property B if there exists a set S such that $0 < |S \cap F| < |F|$ for all $F \in \mathcal{F}$. Denote by $m_N(n)$ the least integer m for which there exists a family $\mathcal{F}$ of m n-element subsets of a set V of size N such that $\bigcup \mathcal{F} = V$ and which does not have property B. We give constructions which yield upper bounds for $m_N(4)$ for certain values of N.

A family $\mathcal{F}$ of finite sets is said to have property B if there exists a set S such that $0 < |S \cap F| < |F|$ for all $F \in \mathcal{F}$. The set S is called a *B-set* for $\mathcal{F}$. Let n and N be positive integers, $n \geq 3$ and $N \geq 2n - 1$. Denote by $m_N(n)$ the least integer m for which there exists a family $\mathcal{F}$ of m n-element subsets of a set V of size N such that $\bigcup \mathcal{F} = V$ and which does not have property B but every proper sub-collection of $\mathcal{F}$ has property B. In the language of hypergraph theory, $m_N(n)$ is the least possible number of edges a critically 3-chromatic n-uniform hypergraph of order N may have. Observe that $m_N(2)$ makes sense when N is odd and, in fact, $m_{2t+1}(2) = 2t + 1$. This is just the statement that the only 3-critical 2-graphs are the cycles of odd length.

A large number of papers have been published giving bounds for $m_N(n)$. See [1] and some of the references given there. We are concerned in this paper with a rather special problem having its roots in papers of Burstein[4], Seymour[10] and Woodall[12]. Seymour and Woodall proved that for all $n \geq 3$ and all $N \geq 2n-1$, $m_N(n) \geq N$. Burstein showed that there exists a least integer $B(n)$ such that for all $N \geq B(n)$ one has $m_N(n) = N$. Liu [8] showed that $B(3) = 10$. No other value of $B(n)$ has been determined. Burstein described a general construction from which it follows that

$$B(n) \leq \frac{7}{24}(n+1)! + n(n+1)^2 \tag{1}$$

and showed that $m_N(n) = N$ for several other values of N less than the right member in (1). (1) implies that $B(4) \leq 135$. In [2] we showed by a refinement of techniques of Burstein and Liu that the term $n(n+1)^2$ can be replaced by $n - 1$.

Date: August 12, 1999.

1991 *Mathematics Subject Classification.* 05A05.

Both authors gratefully acknowledge the support of the Natural Science and Engineering Research Council of Canada.

For large n, the improvement over (1) is slight, but for small n it gives something significant. It shows, for example, that $B(4) \leq 38$. This, together with a result from Burstein's paper shows that $m_N(4) = N$ when $N = 35$ or $N \geq 38$.

The question we consider in this paper is that of obtaining upper bounds for $m_N(4)$ for the remaining values of N. Some results in this direction have been obtained by Abbott and Liu[3], Exoo[6], Goldberg and Russell[7], Seymour[9] and Toft[11]. It was pointed out by Erdős[5] that $m_{2n-1}(n) = m_{2n}(n) = \binom{2n-1}{n}$ so that $m_7(4) = m_8(4) = 35$. For $9 \leq N \leq 15$ the known upper bounds are given in Table 1.

N	Upper bound on $m_N(4)$	Reference
9	26	Abbott and Liu[3]
10	25	Exoo[6]
11	23	Seymour[9], Toft[11]
12	26	Goldberg and Russell [7]
13	26	Goldberg and Russell [7]
14	26	Goldberg and Russell [7]
15	27	Burstein[4] and Liu[8]

TABLE 1

The results in the table for $N = 12, 13, 14$ are attributed by Goldberg and Russell in [7] to a referee of that paper. The last entry is obtained from the following general inequality of Burstein and Liu

$$m_{N+n}(n) \leq m_N(n) + n \tag{2}$$

and the result $m_{11}(4) \leq 23$ of Seymour and Toft.

It is often the case in combinatorics that the hard part of about showing the existence of an object with certain specified properties is the actual finding of the object. Once it is found, the independent verification that it has all of the desired properties is fairly straightforward, although it may be time consuming. This is the case here. Because of this, we leave many such verifications to the reader.

We shall give three special constructions. The first is a special case of a constsruction of Burstein which establishes $m_{35}(4) = 35$. The second is a slight modification of the first. It will show that $m_{37}(4) \leq 38$. The third will show that $m_{25}(4) \leq 29$. We shall make frequent use of the following easily established theorem.

Theorem 1. Let $\mathcal{F}$ be a family of sets without property B. Let a and b be elements of $\bigcup \mathcal{F}$ such that $\{a, b\} \not\subseteq F$ for each $F \in \mathcal{F}$. Then the family $\mathcal{F}'$ obtained from $\mathcal{F}$ by identifying b with a does not have property B. □

If $\mathcal{F}$ is critical in the sense that every proper subfamily of $\mathcal{F}$ has property B, it may not be the case that $\mathcal{F}'$ is critical. However, when we apply the theorem to a specific family $\mathcal{F}$ we may, by making an appropriate choice for a and b, take advantage of properties of $\mathcal{F}$ which may not be available in general and, in so doing, obtain an $\mathcal{F}'$ which is critical.

We shall also make frequent use of (2) and the fact that the family $\{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,6\}, \{2,5,7\}, \{3,4,7\}, \{3,5,6\}\}$ does not have property B and is critical. The sets are the lines in the Fano plane $PG(2,2)$. It is the example showing $m_7(3) = 7$.

We now describe the three special families. To save space, in much of what follows we drop brackets and commas in describing sets so that, for example, $\{1,2,3\}$ will be written 123. It will also be convenient if we denote by $\mathcal{F}_N$ a family which establishes an upper bound for $m_N(4)$.

Construction 2. $\mathcal{F}_{35}$ is the family of 4-element sets obtained from the Fano plane by replacing each set ijk by the five sets $ijka$, $ijkb$, $ijkc$, $ijkd$, $abcd$. This is the example which shows that $m_{35}(4) = 35$ and is a special case of the construction given in [4]. Its members are

$123a_1$	$145a_2$	$167a_3$	$246a_4$	$257a_5$	$347a_6$	$356a_7$
$123b_1$	$145b_2$	$167b_3$	$246b_4$	$257b_5$	$347b_6$	$356b_7$
$123c_1$	$145c_2$	$167c_3$	$246c_4$	$257c_5$	$347c_6$	$356c_7$
$123d_1$	$145d_2$	$167d_3$	$246d_4$	$257d_5$	$347d_6$	$356d_7$
$a_1b_1c_1d_1$	$a_2b_2c_2d_2$	$a_3b_3c_3d_3$	$a_4b_4c_4d_4$	$a_5b_5c_5d_5$	$a_6b_6c_6d_6$	$a_7b_7c_7d_7$

□

Construction 3. $\mathcal{F}_{37}$ is obtained from the Fano plane by replacing each set ijk other than 356 as in Construction 2 and replacing 356 by the eight sets $356a_7$, $356b_7$, $356c_7$, $1a_7b_7c_7$, $356a_8$, $356b_8$, $356c_8$, $2a_8b_8c_8$. The members of $\mathcal{F}_{37}$ are

$123a_1$	$145a_2$	$167a_3$	$246a_4$	$257a_5$	$347a_6$
$123b_1$	$145b_2$	$167b_3$	$246b_4$	$257b_5$	$347b_6$
$123c_1$	$145c_2$	$167c_3$	$246c_4$	$257c_5$	$347c_6$
$123d_1$	$145d_2$	$167d_3$	$246d_4$	$257d_5$	$347d_6$
$a_1b_1c_1d_1$	$a_2b_2c_2d_2$	$a_3b_3c_3d_3$	$a_4b_4c_4d_4$	$a_5b_5c_5d_5$	$a_6b_6c_6d_6$

$356a_7$	$356a_8$
$356b_7$	$356b_8$
$356c_7$	$356c_8$
$1a_7b_7c_7$	$2a_8b_8c_8$

This is the example which shows that $m_{37}(4) \leq 38$. It will be instructive if we sketch an argument showing that it does not have property B and is critical. Suppose that S meets each member of $\mathcal{F}_{37}$. We need to show that S contains a member.

Case 1: S hits 356.

We may suppose that S does not contain any set of the type $a_ib_ic_id_i$ since otherwise we have finished. Also S hits each set of this type. Without loss of generality, we may suppose that S misses $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ and contains $\{b_1, b_2, b_3, b_4, b_5, b_6\}$. S must then hit each of 123, 145, 167, 246, 257, 347, 356. Since the Fano plane does not have property B, S contains one of these sets. If it contains one of the first six sets then we are done because $S \supset \{b_1, b_2, ..., b_6\}$. We may therefore suppose that S contains 356. Since S doesn't contain 123, S doesn't contain 1 or 2, say $1 \notin S$. Then, since S hits $1a_7b_7c_7$, we must have, without loss of generality, $a_7 \in S$. But then S contains $356a_7$.

Case 2: S misses 356.

Then S contains a_7, b_7, c_7, a_8, b_8, c_8. If S hits $\{1, 2\}$ we are done. Thus we suppose $1, 2 \notin S$. Thus S misses 123. It then follows that S contains $a_1b_1c_1d_1$.

In order to show that $\mathcal{F}_{37}$ is critical we must exhibit, for each member F, a B-set for the family $\mathcal{F}_{37} - \{F\}$. One may, of course, take advantage of various symmetries so that not all members F need to be checked.The essentially different cases and a B-set in each case are shown in Table 2. □

Set F deleted	B-set for $\mathcal{F}_{37} - \{F\}$
$123a_1$	$F \cup \{a_2, a_3, a_4, a_5, a_6\}$
$a_1b_1c_1d_1$	$F \cup \{4, 5, 6, 7, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$
$145a_2$	$F \cup \{a_1, a_3, a_4, a_5, a_6, a_8\}$
$a_2b_2c_2d_2$	$F \cup \{2, 3, 6, 7, a_1, a_3, a_4, a_5, a_6, a_7\}$
$347a_6$	$F \cup \{a_1, a_2, a_3, a_4, a_5, a_7, a_8\}$
$a_6b_6c_6d_6$	$F \cup \{1, 2, 5, 6, a_1, a_2, a_3, a_4, a_5\}$
$356a_7$	$F \cup \{2, a_1, a_2, a_3, a_4, a_5, a_6\}$
$1a_7b_7c_7$	$F \cup \{4, 6, 7, a_1, a_2, a_3, a_4, a_5, a_6, a_8\}$

TABLE 2

Construction 4. $\mathcal{F}_{25}$ consists of the following sets:

$ab12$	$ac12$	$bc12$	$135a_1$	$146a_2$	$245a_3$	$236a_4$
$ab34$	$ac34$	$bc34$	$135b_1$	$146b_2$	$245b_3$	$236b_4$
$ab56$	$ac56$	$bc56$	$135c_1$	$146c_2$	$245c_3$	$236c_4$
			$135d_1$	$146d_2$	$245d_3$	$236d_4$
			$a_1b_1c_1d_1$	$a_2b_2c_2d_2$	$a_3b_3c_3d_3$	$a_4b_4c_4d_4$

This is an example which shows that $m_{25}(4) \leq 29$.

In order to show that $\mathcal{F}_{25}$ doesn't have property B we must show that any S which hits each member must contain a member.

Case 1: S misses one of ab, ac, bc.

Then S must hit each of 12, 34, 56, and must therefore contain one of 135, 146, 245, 236, 246, 235, 136, 145. If S contains 135 then, since S must hit $a_1b_1c_1d_1$, it must contain $135a_1$, say. We may therefore suppose that S doesn't contain 135 and, by the same reasoning, doesn't contain 146, 245 or 236. If S contains 246, then we may suppose that S misses 135 since, otherwise, we arrive at a situation already discussed. However, it then follows that S must contain $a_1b_1c_1d_1$. The same argument applies to 235, 136 and 145.

Case 2: S hits each of ab, ac, bc.

Then S must contain one of ab, ac, bc. Observe now that S is a B-set if and only if $\overline{S}$ is a B-set and the argument used in Case 1 applies.

It follows that $\mathcal{F}_{25}$ doesn't have property B.

In order to see that every proper subfamily of $\mathcal{F}_{25}$ has property B, note that if we delete $ab12$ then $ab12a_1a_2a_3a_4$ is a B-set, if we delete $135a_1$ then $135a_1a_2a_3a_4$ is a B-set, and if we delete $a_1b_1c_1d_1$ then $a_1b_1c_1d_1a_2a_3a_4246$ is a B-set. These are the only essentially different possibilities. □

We now use (2) and the Theorem to obtain upper bounds for $m_N(4)$ for the remaining values of N, $16 \leq N \leq 36$. Whenever it is convenient to do so we make the identification described in the Theorem so that two identical sets result, one copy of which may then be discarded. How this is done is described in Table 3.

The Theorem ensures that the families constructed via its use do not have property B. It does not, however, guarantee that the families are critical. This has to be verified directly by showing that the deletion of a set results in a family with a B-set. One may, of course, appeal to various symmetries so as to reduce the number of cases that need to be considered. We give the details only for the family $\mathcal{F}_{30}$. Details for the remaining cases are available from the first author. The members of $\mathcal{F}_{30}$ are listed below.

$123a_1$	$145a_1$	$167a_3$	$246a_3$	$257a_5$	$347a_6$	$356a_7$
$123b_1$	$145b_1$	$167b_3$	$246b_4$	$257b_5$	$347b_6$	$356b_7$
$123c_1$	$145c_1$	$167c_3$	$246c_4$	$257c_5$	$347c_6$	$356c_7$
$123d_1$	$145d_1$	$167d_3$	$246d_4$	$257d_5$	$347d_6$	$356d_7$
$a_1b_1c_1d_1$		$a_3b_3c_3d_3$	$a_3b_4c_4d_4$	$a_5b_5c_5d_5$	$a_6b_6c_6d_6$	$a_7b_7c_7d_7$

The essentially different cases and a B-set in each case are given in Table 4.

It would be of interest to settle the cases $N = 34$ and $N = 37$ since from the general inequality $m_N(n) \geq N$ and the results obtained here we know that $m_{34}(4) = 34$ or 35 and $m_{37}(4) = 37$ or 38.

N	Upper bound on $m_N(4)$	$\mathcal{F}_N$ obtained from
36	37	$\mathcal{F}_{37}$ by identifying a_7 and a_8 and removing a copy of $356a_8$
34	35	$\mathcal{F}_{35}$ by identifying a_1 and a_2
33	35	$\mathcal{F}_{34}$ by identifying b_1 and b_2
32	35	$\mathcal{F}_{33}$ by identifying c_1 and c_2
31	34	$\mathcal{F}_{32}$ by identifying d_1 and d_2 and removing a copy of $a_1b_1c_1d_1$
30	34	$\mathcal{F}_{31}$ by identifying a_3 and a_4
24	29	$\mathcal{F}_{25}$ by identifying a_1 and a_2
23	29	$\mathcal{F}_{24}$ by identifying b_1 and b_2
22	29	$\mathcal{F}_{23}$ by identifying c_1 and c_2
21	28	$\mathcal{F}_{22}$ by identifying d_1 and d_2 and removing a copy of $a_1b_1c_1d_1$
20	28	$\mathcal{F}_{21}$ by identifying a_3 and a_4
19	28	$\mathcal{F}_{20}$ by identifying b_3 and b_4
18	28	$\mathcal{F}_{19}$ by identifying c_3 and c_4
17	27	$\mathcal{F}_{18}$ by identifying d_3 and d_4 and removing a copy of $a_3b_3c_3d_3$
16	27	$\mathcal{F}_{17}$ by identifying a_1 and a_3
26	33	$\mathcal{F}_{22}$ and (2)
27	33	$\mathcal{F}_{23}$ and (2)
28	33	$\mathcal{F}_{24}$ and (2)
29	33	$\mathcal{F}_{25}$ and (2)

TABLE 3

Set F deleted	B-set for $\mathcal{F}_{30} - \{F\}$
$123a_1$	$F \cup \{a_3, a_5, a_6, a_7\}$
$a_1b_1c_1d_1$	$F \cup \{4, 5, 6, 7, a_3, a_5, a_6, a_7\}$
$167a_3$	$F \cup \{a_1, a_5, a_6, a_7\}$
$167b_3$	$F \cup \{a_1, a_3, a_5, a_6, a_7\}$
$a_3b_3c_3d_3$	$F \cup \{2, 3, 4, 5, a_5, a_6, a_7\}$
$257a_5$	$F \cup \{a_1, a_3, a_6, a_7\}$
$a_5b_5c_5d_5$	$F \cup \{1, 3, 4, 6, a_1, a_3, a_6, a_7\}$

TABLE 4

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