Clique graphs of Helly circular-arc graphs

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Abstract: Clique graphs of several classes of graphs have been already characterized. Trees, interval graphs, chordal graphs, block graphs, clique-Helly graphs are some of them. However, no characterization of clique graphs of circular-arc graphs and some of their subclasses is known. In this paper, we present a characterization theorem of clique graphs of Helly circular-arc graphs and prove that this subclass of circular-arc graphs is properly contained in the intersection between proper circular-arc graphs, clique-Helly circular-arc graphs and Helly circular-arc graphs. Furthermore, we prove properties about the 2-nd iterated clique graph of this family of graphs.

Keywords: Circular-arc graphs, clique graphs, Helly circular-arc graphs, intersection graphs.

1- Introduction

Consider a finite family of non-empty sets. The intersection graph of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect. Intersection graphs have received much attention in the study of algorithmic graph theory and their applications [6]. Well-known special classes of intersection graphs include interval graphs, chordal graphs, circular-arc graphs, permutation graphs, circle graphs, and so on.

In this paper a characterization of clique graphs of Helly

ARS COMBINATORIA 60(2001), pp. 255-271

circular-arc graphs is presented. We prove that clique graphs of this subclass of circular-arc graphs are properly contained in the intersection of three subclasses of circular-arc graphs and we analize properties about the 2-nd iterated clique graph of Helly circular-arc graphs.

We shall denote the graph G by a pair (V(G),E(G)), where V(G) denotes a finite set of vertices of G and E(G) the set of edges connecting vertices of G. Let $n = |V(G)|$ and $m = |E(G)|$. The neighborhood of a vertex v is the set N(v) consisting of all the vertices which are adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. A vertex v is an universal vertex if $N[v] = V(G)$.

A clique in a graph G is a maximal complete subgraph of G. The clique graph K(G) of G is the intersection graph of the cliques of G. We can define $K^j(G)$ as the j-th iterated clique graph of G, where $K^1(G) = K(G)$ and $K^j(G) = K(K^{j-1}(G))$, $j \geq 2$. A characterization for the class of clique graphs has been formulated by Roberts and Spencer [14], based on Hamelink's paper [7]. However, no efficient algorithm for the recognition problem is known. In fact, it is an open question whether or not this problem is NP-complete. A problem of interest, in the context of intersection graph theory and specially in the study of clique graphs, is to characterize clique graphs of special classes of graphs. This task has already been performed for trees [9], interval graphs [10], clique-Helly graphs [4], disk-Helly graphs [1], chordal graphs [17], and so on. However, no characterization of clique graphs of circular-arc graphs and their subclasses is known.

A graph G is called a circular-arc graph if there exists a set of arcs on a circle and a one-to-one correspondence between vertices of G and arcs such that two distinct vertices are adjacent if and only if their corresponding arcs intersect. That is, a circular-arc graph is the intersection graph of a set of arcs on a circle. This set of arcs is called the circular-arc representation of G. Without loss of generality, we may suppose that the arcs are open.

Circular-arc graphs admit some interesting subclasses:

1) Proper circular-arc graphs: a graph G is a proper circular-arc (PCA) graph if there is a circular-arc representation of G such that no arc is properly contained in any other. Tucker [18] proposed a characterization and an efficient algorithm, using matrix characterizations, for recognizing PCA graphs. He also gave a characterization for this subclass by forbidden subgraphs [19].

2) Unit circular-arc graphs: a graph G is a unit circular-arc (UCA) graph if there is a circular-arc representation of G such that all arcs are of the same length. Clearly, it can be easily proved that UCA $\subseteq$ PCA. In [19], it is shown that this inclusion is proper and a characterization for the UCA subclass by forbidden subgraphs is presented. Golumbic also showed in [6] a graph which belongs to PCA and does not belong to UCA.

3) Helly circular-arc graphs: first, we define the Helly property. A family of subsets S satisfies the Helly property when every subfamily of it consisting of pairwise intersecting subsets has a common element. Then, a graph G is a Helly circular-arc (HCA) graph if there is a circular-arc representation of G such that the arcs satisfy the Helly property. Gavril [5] gave a characterization of these graphs using the clique matrix of a graph. This characterization leads to an efficient algorithm for recognizing HCA graphs.

4) Clique-Helly circular-arc graphs: a graph G is a clique-Helly circular-arc (CH-CA) graph if G is a circular-arc graph and a clique-Helly graph. A graph is clique-Helly when its cliques satisfy the Helly property. Recently, Szwarcfiter [16] described a characterization of clique-Helly graphs leading to a polynomial time algorithm for recognizing them. This method together with a polynomial algorithm for circular-arc graphs [13,20] results in an efficient algorithm for recognizing CH-CA graphs.

We have shown in [3] minimal examples belonging to all possible intersections of these subclasses, except in one region, which is empty.

This paper is organized in the following way. In Section 2, some theorems about circular-arc graphs are reviewed. In Section 3, we present a characterization theorem of clique graphs of HCA graphs and prove that clique graphs of Helly circular-arc graphs are strictly contained in the intersection between proper circular-arc graphs, clique-Helly circular-arc graphs and Helly circular-arc graphs. Furthermore, we analize properties about $K^2(G)$ when G is a HCA graph.

Definitions not given here can be found in [6] or [8].

2- Preliminaries

First, a characterization of connected proper circular-arc graphs by local tournaments [2,11,12] and round orientations [2] is reviewed.

A tournament is an orientation of a complete graph. A local tournament is a directed graph in which the out-set as well as the in-set of every vertex are tournaments.

A round enumeration of a directed graph D is a circular ordering $S = \{v_0,\dots,v_{n-1}\}$ of its vertices such that for each i there exist non-negative integers r_i,s_i such that the vertex v_i has an inset $N^S_{in} = \{v_{i-1},v_{i-2},\dots,v_{i-ri}\}$ and an outset $N^S_{out} = \{v_{i+1},v_{i+2},\dots,v_{i+si}\}$ (additions and subtractions are modulo n). A directed graph which admits a round enumeration is called round. An undirected graph is said to have a round orientation if it admits an orientation which is a round directed graph.

Theorem 1 [2,15]: The following statements are equivalent for a connected graph G

(1) G is orientable as a local tournament.
(2) G has a round orientation.
(3) G is a proper circular arc graph.

A characterization of Helly circular-arc graphs is reviewed [5]. A matrix has a circular 1's form if the 1's in each column appear in a circular consecutive order. A matrix has the circular 1's property if by a permutation of the rows it can be transformed into a matrix with a circular 1's form. Let G be a graph and $M_1, M_2, \ldots, M_k$ the cliques of G. We will denote by A_G the k x n clique matrix, that is the entry (i,j) is 1 if the vertex $v_j \in M_i$ and 0, otherwise.

Theorem 2 [5]: A graph G is a Helly circular-arc graph if and only if A_G has the circular 1's property.

3- Clique graphs of Helly circular-arc graphs

A characterization theorem of clique graph of Helly circular-arc graphs is formulated. Let G be a graph and S_G a circular ordering of its vertices. We define a circular complete subgraph in a circular ordering S_G as a set of consecutive vertices of S_G, which form a complete subgraph of G. A family of subgraphs of G, F = $\{F_1, \ldots, F_r\}$, covers G if every vertex and edge of G lies in any F_i.

Theorem 3: A graph H $\in$ K(HCA) if and only if H admits a circular ordering S_H such that there exists a family of circular complete subgraphs of S_H, F = $\{F_1, \ldots, F_r\}$, which satisfies:

1) F covers H.
2) F verifies the Helly property.

Proof:

$\Rightarrow$) Let H be a connected graph belonging to K(HCA), then there is a graph G in HCA so that K(G) = H. Let |V(G)| = k and |V(H)| = n. We call R the Helly circular-arc representation of G and $A_1,\ldots,A_k$ the arcs of R corresponding to the vertices $v_1,\ldots,v_k$ of G. In R, the n cliques of G are represented by intersection points p_i on the circle. A clockwise ordering of these points defines a circular ordering S_H of the vertices of H, $S_H = \{p_1,\ldots,p_n\}$. The set of intersection points which are covered by an arc A_i of R represents a set of vertices of H which induces a complete subgraph of H. We denote this set of intersection points by $C(A_i)=\{p_{i1},p_{i1+1},\ldots,p_{i1+i2}\}$ and these points are consecutive in the circular ordering S_H. So, the complete subgraph induced by $C(A_i)$ in H is a circular complete subgraph of S and A_i covers all the intersection points of $C(A_i)$ (this fact is denoted by $C(A_i) \subseteq A_i$). Let $F=\{C(A_1),C(A_2),\ldots, C(A_k)\}$ be a family of circular complete subgraphs of S_H. We are going to prove that F verifies properties 1 and 2.

Clearly, F satisfies the property 1. Let w_i be the vertex of H corresponding to p_i and A_j be an arc which represents a vertex of G lying in the clique of G correspoding to p_i, then $p_i \in C(A_j)$. Now, let w_i and w_j be adjacent vertices of H corresponding to p_i and p_j, so there is an arc A_t, which represents a vertex of G lying in the cliques of G corresponding to p_i and p_j. Then p_i and $p_j \in C(A_t)$.

It remains to verify that F satisfies the Helly property. Let F' be a subfamily of F / $\forall$ $C(A_i),C(A_j) \in$ F', $C(A_i) \cap C(A_j) \neq \emptyset$ and A $=\{A_m /\ C(A_m) \in F'\}$. Then, $\forall$ $A_i,A_j \in$ A, $C(A_i) \cap C(A_j) \subseteq A_i \cap A_j \neq \emptyset$. So, the arcs of A form a complete subgraph in G. This complete subgraph is contained in a clique of G, the clique corresponding, for example, to the intersection point p_t. Then, $p_t \in A_m$, $\forall A_m \in$ A. In consequence, $p_t \in \cap\, C(A_m)$, $C(A_m) \in F'$, and so F verifies the Helly property.

Clearly, the subfamily of F, $F^{*}=\{C(A_i) \in F \ / \ C(A_i) \not\subset C(A_j), \forall C(A_j)$ such that $C(A_i) \neq C(A_j)\}$, also verifies properties 1 and 2.

$\Leftarrow$) Let $S_H=\{v_1,v_2,\ldots,v_n\}$ be a circular ordering of the vertices of H and $F=\{F_1,F_2,\ldots,F_r\}$ a family of circular complete subgraphs of S_H that verifies 1 and 2. We are going to construct a HCA representation R of some graph G such that K(G)=H:

First, we draw $\{v_1,v_2,\ldots,v_n\}$ as a set of points situated in the positions $\{2\pi/n, 4\pi/n,\ldots,2\pi\}$ of the circle (vertex v_j is situated in the position $2\pi j/n$).

By each $F_i=\{v_{i1},v_{i1+1},\ldots,v_{i1+i2}\}$, we draw in R an arc $A_i=[2\pi i_1/n, 2\pi(i_1+i_2)/n]$ on the circle (if F_i is composed by a single vertex v_{i1}, we may suppose that $A_i=[2\pi i_1/n, (2\pi i_1/n)+\varepsilon]$, with ε a small positive real number). By each v_j, we draw an arc $B_j=[2\pi j/n-\pi/2n, 2\pi j/n+\pi/2n]$ on the circle.

First, we must verify that this circular arc representation R of G is Helly.

Let F' be a subfamily of arcs A_i such that for every pair $A_i,A_j \in F'$, $A_i \cap A_j \neq \emptyset$ and $F''=\{F_i \in F \ / \ A_i \in F'\}$. Clearly, for each pair $F_i, F_j \in F''$, $F_i \cap F_j \neq \emptyset$ because $A_i \cap A_j \neq \emptyset$. As F verifies Helly (property 2), $\cap F_j \neq \emptyset \quad \forall F_j \in F''$. It means that there is a vertex v_t in every $F_j \in F''$ and v_t lies in every $A_j \in F'$.

Then $\cap A_j \neq \emptyset \ \forall A_j \in F'$, so F' is Helly.

Now, let F' be a subfamily of arcs that contains an arc B_j and for every pair of arcs of F', they have a nonempty intersection. Particularly, every arc has a common intersection with B_j. Then v_j lies in every arc and v_j belongs to the intersection of all the arcs of F', which implies that F' is also Helly. Both results imply that the circular arc representation R of G satisfies the Helly property.

It remains to verify that K(G) is isomorphic to H, where G is the graph with R as its HCA representation. We have to prove that $\{v_1,v_2,\ldots,v_n\}$ are the intersection points corresponding to the cliques of G. All of them are needed because every arc B_j just contains one vertex v_k. And no other intersection point is needed because the endpoints of every arc A_i are vertices v_k. If v_i and v_j are adjacents in H then they belong to the same F_k (property 1). By construction, v_i and v_j are both in A_k, and so, their respective cliques in G have a vertex in common. Clearly, the converse is true by the same argument.

Notes:

1) In a sense, the above characterization is similar to Roberts and Spencer's general characterization of clique graphs [14]. In that case, they only ask the condition of a family of complete subgraphs which satisfies 1 and 2.

2) It is important to mention that this characterization apparently does not lead to determining the complexity of the corresponding recognition problem.

Definition: A family of circular complete subgraphs of S_H, $F^*=\{C(A_1),C(A_2),\ldots, C(A_k)\}$ is a dominant family if $C(A_i) \not\subset C(A_j)$, $\forall\ i \neq j$.

We are going to prove the following lemma which identifies the cliques of a graph $H \in K(HCA)$.

Let G be a Helly circular-arc graph; $A_1,\ldots,A_k$ the arcs of a circular-arc representation of G and $H = K(G)$. Let $F=\{C(A_1),C(A_2),\ldots, C(A_k)\}$ be a family of circular complete subgraphs of S_H and $F^* = \{F_1,\ldots,F_r\} = \{C(A_i) \in F \,/\, C(A_i) \not\subset C(A_j),\ \forall C(A_j) \text{ such that } C(A_i) \neq C(A_j)\}$ a dominant family of circular complete subgraphs of S_H as we have defined in Theorem 3. Let $U = \{u_1,\ldots,u_p\}$ be a set of universal vertices of H.

Lemma 1: The sets $C'(A_i) = C(A_i) \cup U$, for each $C(A_i) \in F^*$, induce the only cliques of H.

Proof: Let $C'(A_i) = C(A_i) \cup U$, for $C(A_i) \in F^*$. We know that $C'(A_i)$ is a complete subgraph of H. It remains to verify the maximality. Let p_t be an intersection point not belonging to $C'(A_i)$ such that p_t (as a vertex of H) is adjacent to every vertex of $C'(A_i)$. So, there are two arcs A_j and A_k as in the Figure 1 because F^* is a dominant family and the arcs A_i, A_j and A_k must verify the Helly property. Hence, p_t is a universal vertex of H, which is a contradiction.

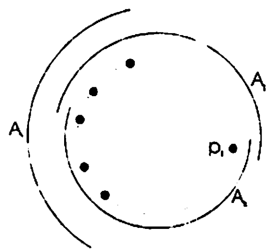

Figure 1

Then $C'(A_i) = C(A_i) \cup U$, for each $C(A_i) \in F^*$, induces a clique of H. Let us verify that there is not another clique in H. Suppose that there is a clique C in H, such that $C' = C \setminus U$ and $C' \not\subset C(A_i)$, for any $C(A_i) \in F^*$. As $C' \not\subset C(A_i)$, the intersection points of C' can be drawn as in Figure 2.

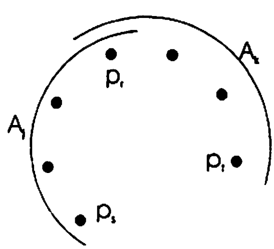

Figure 2

If A_j and A_k cover all the circle, then p_r is an universal vertex of H, which is a contradiction. Otherwise, there is an arc A_m joining p_s with p_t. Then, A_j, A_k and A_m do not verify the Helly property, or we have one of the situations of Figure 3, which imply that p_s or p_t are universal vertices of H. Both cases lead again to a contradiction.

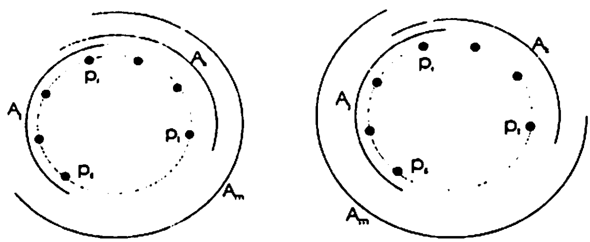

Figure 3

Corollary 1: Let G be a Helly circular-arc graph. Then K(G) is a Helly circular-arc graph, a clique-Helly circular-arc graph and a proper circular-arc graph

Proof:

a) K(G) is a Helly circular-arc graph.

Let $F^* = \{F_1,...,F_r\} = \{C(A_i) \in F \,/\, C(A_i) \not\subset C(A_j), \forall C(A_j)$ such that $C(A_i) \neq C(A_j)\}$ be a dominant family of circular complete subgraphs of $S_{K(G)}$ as we defined in Theorem 3, and U, the set of universal vertices of H. We are going to analize the matrix B with the members of F^* in the rows (ordering them in the consecutive way given by the circular-arc representation), and $\{p_1,...,p_n\}$ the vertices of K(G) in the columns. In each row, we write 1 if p_j belongs to the corresponding $C(A_i)$, and 0, otherwise. By construction, the matrix B has a circular 1's form. As we need the cliques in the rows to construct A_G, by Lemma 1 we have to add 1's in the columns corresponding to the vertices of U. Clearly, A_G preserves the circular 1's form (it is possible that two rows represent the same clique of K(G); in that case we have to eliminate one of them but the property of circular 1's is still valid for A_G). Then, by Theorem 2, K(G) is a Helly circular-arc graph.

b) K(G) is a clique-Helly circular-arc graph.

As K(G) is a circular-arc graph, we must show that it is clique-Helly. Let us divide the proof in two cases:

1) If G is clique-Helly, then K(G) is clique-Helly too [4].
2) If G is not clique-Helly, let $M_1,...,M_k$ be a family of cliques of G minimally non Helly and $p_1,...,p_k$ the corresponding intersection points. As this family is minimally non Helly, for every subset of j intersection points (j < k) we have an arc in the circular-arc representation of G which covers all the intersection points in the subset. But the graph G lies in HCA, then we cannot have the situation of Figure 4.

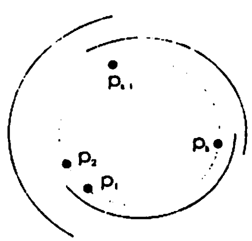

Figure 4

So, we have two arcs which cover all the circle and there is a clique M_t which intersects with any clique in G (Figure 5). Hence, the vertex w_t in K(G) (corresponding to clique M_t in G and the intersection point p_t in the representation) is a universal vertex and we know that every graph with a universal vertex is a clique-Helly graph.

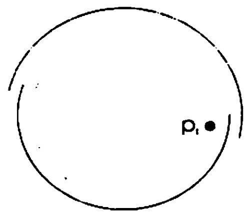

Figure 5

c) K(G) is a proper circular-arc graph

We may assume that K(G) is connected. If it is not connected, this means that G is an interval graph. So, K(G) is a proper interval graph [10], which is a subclass of proper circular-arc graph.

Let $F^* = \{F_1,...,F_r\} = \{C(A_i) \in F \,/\, C(A_i) \not\subset C(A_j), \forall C(A_j)$ such that $C(A_i) \neq C(A_j)\}$ be a dominant family of circular complete subgraphs of $S_{K(G)}$. Let $\{p_1,...,p_n\}$ be the vertices of $S_{K(G)}$. We define $F^+(p_i)$ as the largest subsequence $\{p_i,p_{i+1},...\}$ of the circular ordering such that there exists only one F_j of F^* which contains it (F^+ is well defined because it covers K(G) and is a dominant family). Similarly, we define $F^-(p_i)$ as the largest subsequence $\{...,p_{i-1},p_i\}$ of the circular ordering such that there exists only one F_j of F^* which contains it. As F^* covers K(G), $N[p_i] = F^+(p_i) \cup F^-(p_i)$. For each p_i, we orient the edge $p_i \rightarrow p_k$, if p_k belongs to $F^+(p_i) \setminus F^-(p_i)$ and $p_k \rightarrow p_i$, if p_k belongs to $F^-(p_i) \setminus F^+(p_i)$ (each edge is oriented in only one direction). It remains to orient the edges (p_i,p_k), when p_k belongs to $F^+(p_i) \cap F^-(p_i)$. These edges may be oriented in an arbitrary way. Clearly, the inset of each p_i is contained in $F^-(p_i)$ and the outset, is contained in $F^+(p_i)$. So, this orientation transforms the graph in a local tournament because $F^+(p_i)$ and $F^-(p_i)$ are complete subgraphs of K(G). Then, K(G) is a proper circular-arc graph (Theorem 2).

Remarks:

1) The inclusions of Corollary 1 are proper. The graph H (Figure 6) is a PCA, CH-CA and HCA graph but it does not belong to K(HCA).

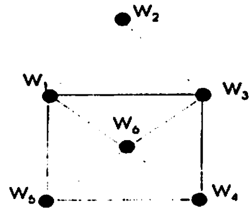

Figure 6

It is easy to find a proper circular-arc representation of H and clearly, it is a clique-Helly graph. Let us prove that H is a HCA graph. Let $M_1=\{w_1,w_2,w_3\}$, $M_2=\{w_1,w_5,w_6\}$, $M_3=\{w_4,w_5,w_6\}$, $M_4=\{w_3,w_4,w_6\}$ and $M_5=\{w_1,w_3,w_6\}$ be the cliques of H. Then, A_H is the following matrix, which has a circular 1's form:

	w_1	W_2	w_3	w_4	w_5	w_6
M_1	1	1	1	0	0	0
M_2	1	0	0	0	1	1
M_3	0	0	0	1	1	1
M_4	0	0	1	1	0	1
M_5	1	0	1	0	0	1

So, G $\in$ HCA (Theorem 2).

It remains to verify that H $\notin$ K(HCA). Suppose the contrary, let G be a HCA graph and K(G) = H. We select a Helly circular-arc representation of G. Let $A_1,\ldots,A_n$ be the arcs of this representation and $v_1,\ldots,v_n$ the respective vertices of G. Each clique C_i of G is represented (because G $\in$ HCA) by a set of arcs which have a common intersection. There is a point of the circle p_i which lies in this intersection, we call it intersection point. We identify each intersection point p_i with a vertex w_i of H and a clique C_i of G. The edge $(w_i,w_j) \in E(H)$ if and only if the cliques C_i and C_j have some arc A_t in common. In the Helly circular-arc representation of G, we first draw the intersection points p_1, p_3, p_4 and p_5, corresponding to the cliques C_1, C_3, C_4 and C_5 of G, respectively (Figure 7).

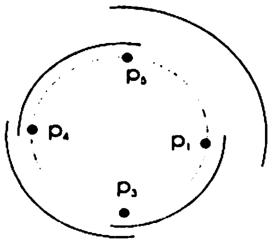

Figure 7

Now the clique C_6 must intersect C_1, C_3, C_4 and C_5. We need that the arcs A and A′ cover all the circle because the graph G belongs to HCA (Figure 8).

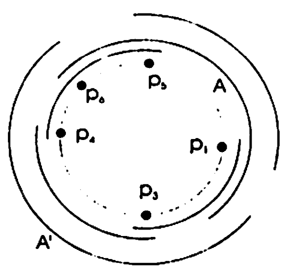

Figure 8

We cannot draw the intersection point p_2 in the representation so that the clique C_2 does not intersect C_6. Clearly, this fact shows us a contradiction because w_6 is not adjacent to w_2 in H.

2) It is interesting to analyze the relation between K(HCA) and UCA because both are subclasses of PCA. In figure 9, we show a graph in K(HCA) \ UCA, another graph in UCA \ K(HCA) and the trivial graph (only one vertex), which lies in K(HCA) $\cap$ UCA.

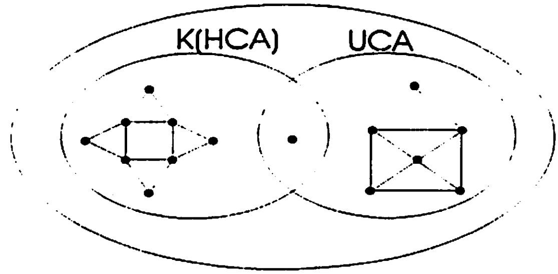

Figure 9

Corollary 2: If G is a Helly circular-arc graph, then either $K^2(G)$ is an induced subgraph of G or $K^2(G)$ is a complete graph.

Proof: Suppose that K(G) has a universal vertex u. Then, any clique of K(G) contains u, which implies that $K^2(G)$ is a complete graph. Otherwise, let K(G) be a graph without universal vertices. Let $F^* = \{F_1,\ldots,F_r\} = \{C(A_i) \in F \,/\, C(A_i) \not\subset C(A_j), \forall C(A_j)$ such that $C(A_i) \neq C(A_j)\}$ be a dominant family of circular complete subgraphs of $S_{K(G)}$ as we have defined in Theorem 3. By Lemma 1, $\{C(A_i)\}$, the members of F^*, are the only cliques of K(G), so they represent the vertices of $K^2(G)$. It is clear that for any $C(A_i)$, $C(A_j) \in F^*$ $(i \neq j)$, $C(A_i) \cap C(A_j) \neq \emptyset$ if and only if $A_i \cap A_j \neq \emptyset$, where A_i and A_j are the arcs of the circular-arc representation, corresponding to the vertices v_i and v_j of G. Hence, $K^2(G)$ is an induced subgraph of G.

Note: Escalante [4] proved a similar result for clique-Helly graphs. He showed that if G is a clique-Helly graph, then $K^2(G)$ is an induced subgraph of G.

Acknowledgements: We would like to thank Jayme Szwarcfiter who provided us the necessary references and valuable suggestions which have improved this work. We also thank Pavol Hell for his interesting comments. This paper was partially supported by UBACyT grant EX-209.

References

[1] H. Bandelt and E. Prisner, "Clique Graphs and Helly Graphs", Journal of Combinatorial Theory, Serie B 51, (1991), 34-45.

[2] X. Deng, P. Hell and J. Huang, "Linear time representation algorithms for proper circular-arc graphs and proper interval graphs", SIAM Journal of Computing, 25 (1996), 390-403.

[3] G. Durán and M. Lin, "On some subclasses of circular arc-graphs", Technical Report TR003-99 (1999), Departamento de Computación, FCEyN, Universidad de Buenos Aires, submitted.

[4] F. Escalante, "Über iterierte Clique-Graphen", Abh. Math. Sem. Univ. Hamburg. 39 (1973), 59-68.

[5] F. Gavril, "Algorithms on circular-arc graphs", Networks 4 (1974), 357-369.

[6] M. Golumbic, "Algorithm Graph Theory and Perfect Graphs", Academic Press, New York, 1980.

[7] R. Hamelink, "A partial characterization of clique graphs", Journal of Combinatorial Theory Ser. B 5 (1968), 192-197.

[8] F. Harary, "Graph Theory", Addison-Wesley, (1969).

[9] S. Hedetniemi and P. Slater, "Line graphs of triangleless graphs and iterated clique graphs", Lecture Notes in Math., Springer-Verlag (1972), 139-147.

[10] B. Hedman, "Clique graphs of time graphs", Journal of Combinatorial Theory Ser. B 37 (1984), 270-278.

[11] P. Hell, J. Bang-Jensen and J. Huang, "Local tournaments and proper circular-arc graphs", Algorithms, Lecture Notes in

Computer Science, Vol. 450 (Springer-Verlag, 1990), 101-109.

[12] P. Hell and J. Huang, "Lexicographic orientation and representation algorithms for comparability graphs, proper circular-arc graphs and proper interval graphs", Journal of Graph Theory 20 (1995), 361-374.

[13] W. Hsu, "O(m.n) algorithms for the recognition and isomorphism problems on circular-arc graphs", SIAM J. Comput., 24 (1995), 411-439.

[14] F. Roberts and J. Spencer, "A characterization of clique graphs", J. Combin. Theory, Serie B 10 (1971), 102-108.

[15] D. Skrien, "A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular-arc graphs and nested interval graphs", Journal of Graph Theory 6 (1982), 309-316.

[16] J. Szwarcfiter, "Recognizing Clique-Helly graphs", Ars Combinatoria, 45 (1997), 29-32.

[17] J. Szwarcfiter and C. Bornstein, "Clique graphs of chordal and path graphs", Siam J. Discrete Math., Vol. 7 N° 2 (1994), 331-336.

[18] A. Tucker, "Matrix characterizations of circular-arc graphs", Pacific J. Math. 38 (1971), 535-545.

[19] A. Tucker, "Structure theorems for some circular-arc graphs", Discrete Mathematics 7 (1974), 167-195.

[20] A. Tucker, "An efficient test for circular-arc graphs", SIAM J. Comput. 9 (1980), 1-24.