Some Theorems on Generalized Stirling Numbers

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ABSTRACT. Recently, Hsu and Shiue [10] obtained a kind of generalized Stirling number pairs with three free parameters and proved some of its properties. Here, some properties analogous to those of ordinary Stirling numbers are investigated, viz. horizontal recurrence relations, vertical recurrence relations, rational generating function, and explicit formulas. Futhermore, a kind of infinite sum which is useful in some combinatorial applications of the generalized Stirling numbers, is evaluated.

1 Introduction

In a recent paper of Hsu and Shiue [10] it was shown that a kind of generalized Stirling number pair with three free parameters can be introduced via a pair of linear transformations between generalized factorials, viz.

$$(t \mid \alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t-\gamma \mid \beta)_k \tag{1}$$

$$(t \mid \beta)_n = \sum_{k=0}^{n} S(n,k;\beta,\alpha,-\gamma)(t+\gamma \mid \alpha)_k \tag{2}$$

where $n \in N$ (set of nonnegative integers), α, β, γ may be real or complex numbers with $(\alpha,\beta,\gamma) \neq (0,0,0)$, and $(t \mid \alpha)_n$ denotes the generalized factorial of the form

$$(t \mid \alpha)_n = \prod_{j=0}^{n-1}(t-j\alpha), \quad n \geq 1, \quad (t \mid \alpha)_0 = 1$$

and, in particular, $(t \mid 1)_n = (t)_n$ with $(t)_0 = 1$. It was also mentioned that various well-known generalizations could be obtained by special choices of the parameters α, β, and γ (c.f. [10]). Furthermore, the generalization of some theorems of the classical Stirling numbers such as triangular recurrence relation, the vertical generating function, the congruence property, the Dobinski-type formula and a kind of asymptotic expansion were investigated. However, there are still many properties of the generalized Stirling numbers that are worthy of consideration.

Just like the classical Stirling numbers the generalized Stirling numbers have also combinatorial and statistical applications for integral values of α, β, and γ. In fact, we have found one such application which is discussed in a separate paper entitled "Combinatorial and Statistical Applications of Generalized Stirling Numbers" and will be submitted for publication elsewhere.

The object of this paper is to give some new properties or theorems of the generalized Stirling numbers. Throughout this paper we will use $S^1(n,k)$, $S^2(n,k)$ to denote $S(n,k;\alpha,\beta,\gamma)$ and $S(n,k;\beta,\alpha,-\gamma)$, respectively. Take note that relation (1) implies the following

$$S^1(0,0)=1, \quad S^1(n,n)=1, \quad S^1(1,0)=\gamma$$

and as convention we assume $S^1(n,k)=0$ for $k \geq n+1$. We use $s(n,k)$ and $S(n,k)$ to denote the ordinary Stirling numbers of the first and second kind, respectively.

2 Equivalence Theorem

A number of papers related to the generalization of Stirling numbers have appeared in the literature. The ways of generalization are based on the use of differences of generalized factorials, linear transformations between generalized factorials, triangular recurrence relation, and vertical generating function. For instance, the joint paper of Charalambides and Koutras [5] is based on the differences of generalized factorials, the joint work of Hsu and Shiue [10] is based on the linear transformations between generalized factorials with Stirling number pairs as connection coefficients, and the works of Howard [8] and Carlitz [3], [4] about degenerate weighted Stirling numbers are based on the vertical generating function, etc. Among the four, the second approach is observed to be more simple and transparent in generalizing Stirling numbers in the sense that it allows us to recognize easily some implicative relations among Stirling-type numbers. The following theorem asserts that these four approaches are equivalent to each other.

Theorem 1. *The following are equivalent*

(i) *The generalized Stirling number $S^1(n,k)$ equals*

$$S^1(n,k) = \frac{1}{\beta^k k!}\left[\Delta^k(\beta t+\gamma \mid \alpha)_n\right]_{t=0} = \frac{1}{\beta^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(\beta j+\gamma \mid \alpha)_n;$$

(ii) *The following relation holds*

$$(t \mid \alpha)_n = \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)(t-\gamma \mid \beta)_k;$$

(iii) *For the numbers $S^1(n,k)$, we have the triangular recurrence relation*

$$S(n+1,k) = S(n,k-1) + (k\beta - n\alpha + \gamma)S(n,k)$$

where $n \geq k \geq 1$. In particular, $S(n,0) = (\gamma \mid \alpha)_n$;

(iv) *The vertical generating function for $S^1(n,k)$ is given by*

$$(1+\alpha t)^{\gamma/\alpha}\left[\frac{(1+\alpha t)^{\beta/}-1}{\beta}\right]^k = k!\sum_{n\geq 0} S^1(n,k)\frac{t^n}{n!}.$$

Proof: To prove (i) ⇔ (ii), we make use of (i) with t replaced by $\beta t+\gamma$ and a precise application of Newton's interpolation formula [12], we obtain (ii), and vice versa. For (ii) ⇔ (iii), (iii) ⇒ (iv), and (iv) ⇒ (iii) the reader may see the respective references, [10, p. 370], [10, p. 372], and [11, p. 227]. □

3 Recurrence Relations

Recurrence relations are useful in constructing tables of values. In Comtet's book [8], the ordinary Stirling numbers have three types of recurrence relations, namely, the triangular, horizontal, and vertical recurrence relations. Among the three, only the first type was given a generalization. In this section, the last two types will be extended to the general case.

First, let us consider the horizontal recurrence relation which is given in two different forms.

Theorem 2. *The first form of horizontal recurrence relation for $S^1(n,k)$ is given by*

$$S^1(n,k) = \sum_{j=0}^{n-k}(-1)^j((k+1)\beta - n\alpha+\gamma \mid -\beta)_j S^1(n+1,k+j+1). \quad (3)$$

Proof: Using theorem 1(iv),

$$\begin{aligned}
\text{RHS of (3)} &= \sum_{j=0}^{n-k}(-1)^j((k+1)\beta - n\alpha + \gamma \mid -\beta)_j S^1(n, k+j) \\
&+ \sum_{j=0}^{n-k}(-1)^j((k+1)\beta - b\alpha + \gamma \mid -\beta)_{j+1} S^1(n, k+j+1) \\
&= \sum_{j=-1}^{n-k}(-1)^{j+1}((k+1)\beta - n\alpha + \gamma \mid -\beta)_{j+1} S^1(n, k+j+1) \\
&+ \sum_{j=0}^{n-k}(-1)^j((k+1)\beta - n\alpha + \gamma \mid -\beta)_{j+1} S^1(n, k+j+1) \\
&= S^1(n, k).
\end{aligned}$$

□

To see the usefulness of equation(3) we rewrite it in the form

$$S^1(n+1, k+1) = S^1(n,k) - \sum_{j=1}^{n-k}(-1)^j((k+1), \beta - n\alpha + \gamma \mid -\beta)_j \\ S^1(n+1, k+j+1)$$

where k is evaluated from n down to 0.

Using the assignments $(\alpha, \beta, \gamma) = (1, 0, 0)$ and $(\alpha, \beta, \gamma) = (0, 1, 0)$, Theorem 2 will reduce to the horizontal recurrence relations for ordinary Stirling numbers (c.f. [8], [6]). Moreover, the fact that Broder's r-Stirling numbers of the first and second kind can be obtained by choosing $\alpha = -1$, $\beta = 0$, $\gamma = r$ and $\alpha = 0$, $\beta = 1$, $\gamma = r$, respectively with n replaced by $n - r$ and k by $k - r$, in particular,

$$\begin{bmatrix} n \\ k \end{bmatrix}_r = S(n-r, k-r; -1, 0, r), \quad \begin{Bmatrix} n \\ k \end{Bmatrix}_r = S(n-r, k-r; 0, 1, r), \quad (4)$$

using the above assignments Theorem 2 will give a kind of horizontal recurrence relations for both kinds of r-Stirling numbers. Interested reader may try to do the numerical test using the tables given in Broder's paper [1]. Furthermore, for $(\alpha, \beta\gamma) = (0, 0, 1)$ we have

$$\binom{n}{k} = \binom{n+1}{k+1} - \binom{n+1}{k+2} + \cdots + (-1)^{n-k}\binom{n+1}{n+1}.$$

Theorem 3. *The second form of horizontal recurrence relation for* $S^1(n, k)$

is given by

$$S^1(n,k) = \frac{1}{\beta^k k!}\left[(\beta k+\gamma \mid \alpha)_n - \sum^{k-1} j = 0 (k)_j \beta^j S^1(n,j)\right], \ \beta \neq 0.$$

Proof: Recall the well-known reciprocal relation

$$g_n = \sum_{k=0}^{n} \binom{n}{k} f_k \Leftrightarrow f_n = \sum_{k=0}^{k-1} (-1)^{n-k} \binom{n}{k} g_k \tag{5}$$

which is known as the inversion of binomial transforms. Applying this inversion formalae (5) to Theorem 1(i) we easily obtain

$$(\beta k+\gamma \mid \alpha)_n = \sum_{j=0}^{k} \binom{k}{j} j! \beta^j S^1(n,j). \tag{6}$$

Thus, we get Theorem 3. □

One of the horizontal recurrence relations given in Comtet's book [8] can be obtained from Theorem 3 with $\alpha = 0$, $\beta = 1$, and $\gamma = 0$. Using the congruence relation (c.f. [10])

$$S^1(p,j) \equiv 0 \pmod p, \quad 2 \leq j \leq p-1.$$

and Theorem 3 with n being replaced by an odd prime number p, we find

$$(\beta k+\gamma \mid \alpha)_p \equiv k(\beta+\gamma \mid \alpha)_p + (1-k)(\gamma \mid \alpha)_p \pmod p.$$

Taking $\alpha = 0$, $\beta = 1$, and $\gamma = 0$ we get $k^p \equiv k \pmod p$. This is Fermat's Little Theorem.

Similar to that of the horizontal recurrence relation the vertical recurrence relation for $S^1(n,k)$ may be given in two different forms. However, the second form is not as general as the first form since it applies only when $\alpha = 0$. The following theorem will give us the first form of this relation.

Theorem 4. *The first form of vertical recurrence relation for $S^1(n,k)$ is given by*

$$k\, S^1(n,k) = \sum_{j=k-1}^{n-1} \binom{n}{j} (\beta-\alpha \mid \alpha)_{n-j-1} S^1(j,k-1) \tag{7}$$

Proof: To prove equation (7) we need to recall the double generating function for $S^1(n,k)$ (c.f. [10]) which is given by

$$\begin{aligned}\psi(t,u) &= \sum_{n,k\geq 0} S^1(n,k)\frac{t^n}{n!}u^k \\ &= (1+\alpha t)^{\gamma/\alpha} \exp\left[u\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right].\end{aligned} \tag{8}$$

Taking the partial derivative on both sides of (8) with respect to u, we have

$$\sum_{k\geq 1}\sum_{n\geq 0} kS^1(n,k)\frac{t^n}{n!}u^{k-1} = \psi(t,u)\left[u\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right]$$
$$= \sum_{n,k\geq 0}\sum_{i\geq 1}\frac{S^1(n,k)\alpha^i}{\beta}\binom{\beta/\alpha}{i}\frac{t^{n+i}}{n!}u^{k-1}.$$

Replacing $n+i$ with j, we get

$$\sum_{k\geq 1}\sum_{n\geq 0} kS^1(n,k)\frac{t^n}{n!}u^{k-1}$$
$$= \sum_{k\geq 1}\sum_{n\geq k-1}\sum_{j\geq n+1}\binom{j}{n}(\beta-\alpha\mid\alpha)_{j-n-1}S^1(n,k-1)\frac{t^j}{j!}u^{k-1}$$
$$= \sum_{k\geq 1}\sum_{j\geq k}\sum_{n=k-1}^{j-1}\binom{j}{n}(\beta-\alpha\mid\alpha)_{j-n-1}S^1(n,k-1)\frac{t^j}{j!}u^{k-1}$$
$$= \sum_{k\geq 1}\sum_{n\geq k}\sum_{j=l-1}^{n-1}\binom{n}{j}(\beta-\alpha\mid\alpha)_{n-j-1}S^1(j,k-1)\frac{t^n}{n!}u^{k-1}.$$

Comparing the coefficients of the term $\frac{t^n}{n!}u^{k-1}$, we obtain (7). □

The vertical recurrence relations for both kinds of ordinary Stirling numbers can be obtained by letting $(\alpha,\beta,\gamma)=(1,0,0)$ and $(\alpha,\beta,\gamma)=(0,1,0)$. Using the assignments given in (4), Theorem 4 will give the following recurrence relations for both kinds of r-Stirling numbers

$$\begin{bmatrix} n \\ r \end{bmatrix}_r = \frac{1}{k-r}\sum_{j=0}^{n-r-1}\binom{n-r}{j+1}\begin{bmatrix} n-j-1 \\ k-1 \end{bmatrix}_r j!,\ k\neq r,$$
$$\begin{bmatrix} n \\ r \end{bmatrix}_r = (r\mid -1)_{n-r}$$

and

$$\begin{Bmatrix} n \\ r \end{Bmatrix}_r = \frac{1}{k-r}\sum_{j=0}^{n-r-1}\binom{n-r}{j+1}\begin{Bmatrix} n-j-1 \\ k-1 \end{Bmatrix}_r,\ k\neq r,$$
$$\begin{Bmatrix} n \\ r \end{Bmatrix}_r = r^{n-r}.$$

Note that the structure of these relations is almost similar to that obtained by Broder [1, p. 248]. It is verified that these formulas generate the same values as those given in tables 1-3 in Broder's paper. Moreover, using Theorem 4, the two kinds of Howard's degenerate weighted Stirling numbers

[9] can be expressed as

$$S_1(n,k,\lambda \mid \theta) = \frac{1}{k} \sum_{j=k-1}^{n-1} \binom{n}{j} (1-\theta \mid -1)_{n-j-1} S_1(j,k-1,\lambda \mid \theta), \ k \neq 0$$

$$S(n,k,\lambda \mid \theta) = \frac{1}{k} \sum_{k=k-1}^{n-1} \binom{n}{j} (1-\theta \mid \theta)_{n-j-1} S(j,k-1,\lambda \mid \theta), \ k \neq 0$$

where $S_1(n,k,\lambda \mid \theta) = S(n,k;-1,-\theta,\lambda-\theta)$ and $S(n,k,\lambda \mid \theta) = S(n,k;\theta,1,\lambda)$. It is also verified that tables given in Howard's paper can be generated using these recurrence relations.

The next theorem is the rational generating function for $S(n,k;0,\beta,\gamma)$ which is useful in proving the second form of vertical recurrence relation. It is a generalization of the rational generating function for ordinary Stirling numbers of the second kind (c.f. [8]).

Theorem 5. *The $S(n,k;0,\beta,\gamma)$ have the following rational generating function*

$$\phi_k(\beta,\gamma) = \sum_{n\geq k} S(n,k;0,\beta,\gamma)t^n = \frac{t^k}{\prod_{j=0}^{k}[1-(\beta j+\gamma)t]}. \tag{9}$$

Proof: Consider the equation

$$\frac{1}{\prod_{j=0}^{k}[1-(\beta j+\gamma)t]} = \sum_{j=0}^{k} \frac{A_j}{1-(\beta j+\gamma)t}. \tag{*}$$

Using the method of partial fractions, equation (10) may be rewritten in the form

$$\sum_{j=0}^{k} \left[A_j \prod_{j_1=0}^{j-1} [1-(\beta j_1+\gamma)t] \prod_{j_2=j+1}^{k} [1-(\beta j_2+\gamma)t] \right] = 1.$$

Taking $t = (\beta j+\gamma)^{-1}$, we obtain

$$A_j = \frac{1}{\beta^k k!}(-1)^{k-j}\binom{k}{j}(\beta j+\gamma)^k, \quad j = 0,1,2,\ldots,k.$$

Substituting this to equation $(*)$, we have

$$\frac{t^k}{\prod_{j=0}^{k}[1-(\beta j+\gamma)t]} = \sum_{j=0}^{k} \frac{(-1)^{k-j}\binom{k}{j}(\beta j+\gamma)^k t^k}{\beta^k k!} \frac{1}{1-(\beta j+\gamma)t}$$

$$= \sum_{\nu\geq 0} \left[\frac{1}{\beta^k k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (\beta j+\gamma)^{k+\nu} \right] t^{k+\nu}.$$

Making use of Theorem 1(i) and replacing $k+\nu$ with n, we get (9). □

It can easily be verified that for $\beta = 1$, and $\gamma = 0$ Theorem 5 will yield

$$\sum_{n=k}^{\infty} S(n,k)t^n = \frac{t^k}{\prod_{j=0}^{k}(1-jt)},$$

which is the rational generating function for Stirling numbers of the second kind (c.f. [8], [6]). Furthermore, the vertical generating function for r-Stirling numbers of the second kind $\left\{ {j \atop m} \right\}_r$ (c.f. [1, p. 248]) can also be obtained from this theorem by letting $n = j - r$, $k = m - r$, $\beta = 1$, and $\gamma = r$. More precisely, we have

$$\sum_{j=m}^{\infty} \left\{ {j \atop m} \right\}_r t^j = \frac{t^m}{(1-rt)(1-(r+1)t)\dots(1-mt)}.$$

Moreover, if we let $\beta = 0$, $\gamma = 1$, and $t = \frac{1}{2}$, we can easily obtain the following interesting power series identity

$$\sum_{n=k}^{\infty} \binom{n}{k} 2^{-(n+1)} = 1. \tag{10}$$

Theorem 6. *The second form of vertical recurrence relation for $S(n,k;0,\beta,\gamma)$ is given by*

$$S(n+1,k+1;0,\beta,\gamma) = \sum_{j=k}^{n} S(j,k;0,\beta,\gamma)(\beta(k+1)+\gamma)^{n-j}.$$

Proof: Usina equation (9) we obtain

$$\begin{aligned}\sum_{n\geq k} S(n,k;0,\beta,\gamma)t^n &= t(1-(\beta k+\gamma)t)^{-1}\phi_{k-1}(\beta,\gamma)\\ &= \sum_{j\geq k}\sum_{m\geq 0} S(j-1,k-1;0,\beta,\gamma)(\beta k+\gamma)^m t^{j+m}.\end{aligned}$$

Replacing $j+m$ with n, we have

$$\begin{aligned}\sum_{n\geq k} S(n,k;0,\beta,\gamma)t^n &= \sum_{j\geq k}\sum_{n\geq j} S(j-1,k-1;0,\beta,\gamma)(\beta k+\gamma)^{n-j}t^n\\ &= \sum_{n\geq k}\sum_{j=k}^{n} S(j-1,k-1;0,\beta,\gamma)(\beta k+\gamma)^{n-j}t^n.\end{aligned}$$

Comparing the coefficients of the term t^n, we obtain Theorem 6. □

For $\alpha = 0$, $\beta = 1$, and $\gamma = 0$. Theorem 4 will reduce to a vertical recurrence relation for ordinary Stirling numbers of the second kind. One may try to derive another form of vertical recurrence relation for the second kind of r-Stirling numbers and verify that this generates the table given in Broder's paper. Furthermore, for $\alpha = 0$, $\beta = 0$, and $\gamma = 1$, the theorem will give

$$\binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}$$

which is known to be the Chu-Shih-Chieh's identity [7].

4 Explicit Formula for $S^1(n,k)$

The explicit formula for $S^1(n,k)$ given in Theorem 1(i) will only work when $\beta \neq 0$. The next theorem will give us a formula that is applicable whatever the value of β. We call this as Schlömilch-type formula.

Theorem 7. (Schlömilch-type Formula)

$$S^1(n,k) = \sum_{m=k}^{n}\sum_{h=0}^{m-k}(-1)^h\binom{n}{m}\binom{m+h-1}{m+h-k}\binom{2m-k}{m-h-k}\psi(m,h) \quad (11)$$

where $\psi(m,h) = \frac{(\gamma|\alpha)_{n-m}}{\alpha^h h!}\sum_{j=0}^{h}(-1)^j\binom{h}{j}(\alpha(h-j) \mid \beta)_{m-k+h}$, $\alpha \neq 0$, *and define* $0^0 = 1$.

Proof: Using the vertical generating function for $S^1(n,k)$ given in Theorem 1(iv), we have

$$\begin{aligned}\sum_{n\geq 0} k!S^1(n,k)\frac{t^n}{n!} &= \left[\sum_{n\geq 0}(\gamma \mid \alpha)_n\frac{t^n}{n!}\right]\left[\sum_{n\geq 0} k!S(n,k;\alpha,\beta,0)\frac{t^n}{n!}\right]\\ &= \sum_{n\geq 0}\left[\sum_{m=k}^{n}\frac{(\gamma \mid \alpha)_{n-m}t^{n-m}}{(n-m)!}\frac{k!S(m,k;\alpha,\beta,0)t^m}{m!}\right]\\ &= \sum_{n\geq 0}\left[\sum_{m=k}^{n}\frac{k!(\gamma \mid \alpha)_{n-m}S(m,k;\alpha,\beta,0)}{(n-m)!m!}\right]t^n.\end{aligned}$$

Comparing the coefficients of the term t^n, we have

$$S^1(n,k) = \sum_{m=k}^{n}\binom{n}{m}(\gamma \mid \alpha)_{n-m}S(m,k;\alpha,\beta,0). \quad (12)$$

From the paper of Hsu and Yu [11], $S(m,k;\alpha,\beta,0)$ can be written as

$$S^1(m,k;\alpha,\beta,0) = \sum_{h=0}^{m-k} (-1)^h \binom{m+h-1}{m+h-k} \binom{2m-k}{m-h-k} S(m-k+h,h;\beta,\alpha,0).$$

Substitute this to equation (12) and apply theorem 1(i) to $S(m-k+h,h;\beta,\alpha,0)$, we obtain (11). □

Note that if we let $\alpha = 1$, $\beta = 0$, and $\gamma = 0$ equation (11) will reduce to Schlömilch-type formula for Stirling numbers of the first kind (c.f. [8]). We know that formula (11) will fail when $\alpha = 0$. Hence, the two explicit formulas (given in theorem 1(i) and equation (11)) are not applicable when $\alpha = \beta = 0$. But using the basic relation (1) we can easily obtain

$$S(n,k;0,0,\gamma) = \binom{n}{k}\gamma^{n-k}. \tag{13}$$

Moreover, for nonzero α, β, γ Theorem 1(i) and the Schlömilch-type formula can both give the exact value for $S^1(n,k)$.

The next result provides a formula that is useful to compute the exact value of $S(n,k;0,\beta,\gamma)$ for any β, and γ. Since the Schlömilch-type formula is not applicable to the case when $\alpha = 0$, we can say that Schlömilch-type formula and the formula that follows are exactly supplementary to each other.

Theorem 8. *The following explicit formula holds*

$$S(n,k;0,\beta,\gamma) = \sum_{c_0+c_1+\cdots+c_k=n-k} \gamma^{c_0}(\beta+\gamma)^{c_1}(2\beta+\gamma)^{c_2}\ldots(k\beta+\gamma)^{c_k}$$

Proof: Using the rational generating function given in Theorem 4, we have

$$\begin{aligned}\sum_{n\geq k} S(n,k;0,\beta,\gamma)t^{n-k} &= \prod_{j=0}^{k}\left[\sum_{c_j\geq 0}(\beta j+\gamma)^{c_j}t^{c_j}\right] \\ &= \sum_{c_0+c_1+\cdots+c_k\geq 0}\left[\prod_{j=0}^{k}(\beta j+\gamma)^{c_j}\right]t^{c_0+c_1+\cdots+c_k} \\ &= \sum_{n\geq k}\left[\sum_{c_0+c_1+\cdots+c_k=n-k}\prod_{j=0}^{k}(\beta j+\gamma)^{c_j}\right]t^{n-k}.\end{aligned}$$

Identifying the coeficients of the term t^{n-k}, we obtain Theorem 8. □

For $\beta = 1$ and $\gamma = 0$, Theorem 8 will give

$$S(n,k) = \sum_{c_0+c_1+\cdots+c_k=n-k} 1^{c_1}2^{c_2}\ldots k^{c_k},$$

which is the explicit formula for ordinary Stirling numbers of the second kind (c.f.[8, p. 207]). On the other hand, if we let $\beta = 0$, $\gamma = 1$. we get

$$\binom{n}{k} = \sum_{c_0+c_1+\cdots+c_k=n-k} 1$$

which means that $\binom{n}{k}$ counts the number of nonnegative integer solutions c_i of the equation

$$c_0 + c_1 + \cdots + c_k = n - k.$$

Futhermore, for any γ with $\beta = 0$, we have

$$S(n,k,0,0,\gamma) = \sum_{c_0+c_1+\cdots+c_k=n-k} \gamma^{c_0+c_1+\cdots+c_k} = \binom{n}{k}\gamma^{n-k}$$

which is exactly equation (13).

5 Evaluation of Infinite Sum

Consider the following infinite sum

$$\sigma_k(x) = \sum_{n\geq k} S^1(n-1,k-1)\frac{1}{(x \mid \alpha)_n}$$

This kind of sum with $\alpha = 1$ was first investigated by Charalambides and Koutras [5]. Take note that in certain combinatorial applications of the numbers $S^1(n,k)$ the evaluation of the sum $\sigma_k(x)$ is necessary. The following lemma will help us to evaluate this sum.

Lemma. *The infinite sum $\sigma_k(x)$ has the following recurrence relation*

$$\sigma_k(x) = \frac{1}{x-\gamma-\beta(k-1)}\sigma_{k-1}(x) \tag{14}$$

where $k = 2, 3,$ and with initial condition $\sigma_1(x) = \frac{1}{x-\gamma}$.

Proof: Using the triangular recurrence relation for $S^1(n,k)$ and the fact that

$$\frac{1}{(x \mid \alpha)_n} = \frac{x - n\alpha}{(x \mid \alpha)_{n+1}}$$

we have

$$(x-n\alpha)S^1(n,k-1)\frac{1}{(x\mid\alpha)_{n+1}} = [\beta(k-1)+\gamma]S^1(n-1,k-1)\frac{1}{(x\mid\alpha)_n}$$
$$-(n-1)\alpha S^1(n-1,k-1)\frac{1}{(x\mid\alpha)_n}$$
$$+S^1(n-1,k-2)\frac{1}{(x\mid\alpha)_n}.$$

Summing up both sides over $n=k-1,k,k+1,\ldots$ we obtain the recurrence relation (14). For the initial value $\sigma_1(x)$, we have

$$\sigma_1(x)=\sum_{n\geq 1}\frac{(\gamma\mid\alpha)_{n-1}}{(x\mid\alpha)_n}=\frac{1}{x}\sum_{j\geq 0}\frac{(\gamma/\alpha)_j}{(x/\alpha-1)_j}$$
$$=\frac{1}{x}F(-\gamma/\alpha,1;-x/\alpha+1;1)$$

where $F(a,b;c;z)$ denotes the hypergeometric series for which

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{15}$$

Note that $\Gamma(z)=\left[ze^{\gamma z}\prod_{n\geq 1}\left[(1+z/n)e^{-z/n}\right]\right]^{-1}$. Making use of (15), we get

$$F(-\gamma/\alpha,1;-x/\alpha+1;1)=\frac{x/\alpha}{x/\alpha=\gamma/\alpha}=\frac{x}{x-\gamma}.$$

Thus, $\sigma_1(x)=\frac{1}{x-\gamma}$. □

Theorem 9. *The infinite sum $\sigma_k(x)$ has the following explicit formula*

$$\sigma_k(x)=\sum_{n\geq k}S^1(n-1,k-1)\frac{1}{(x\mid\alpha)_n}=\frac{1}{(x-\gamma\mid\beta)_k}. \tag{16}$$

Proof: Making use of the lemma, we can easily obtain (16). □

The number $G(n,k;\beta,\gamma)$ defined in the joint paper of Charalambides and Koutras [5] can be expressed in terms of Stirling numbers as

$$G(n,k;\beta,\gamma)=\beta^k S(n,k;1,\beta,\gamma).$$

In this case with $x=\beta y+\gamma$ equation (16) will yield

$$\sum_{n\geq k}G(n-1,k-1;\beta,\gamma)\frac{1}{(\beta y+\gamma)_n}=\frac{1}{(y)_k}$$

which is the infinite sum obtained by Charalambides and Koutras. For the case where $(\alpha, \beta, \gamma) = (0, 0, 1)$ and $x = 2$ in (16), we obtain the following power series identity

$$\sum_{n=k}^{\infty} \binom{n-1}{k-1} 2^{-n} = 1$$

which is precisely equivalent to (10).

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