Directed Bipartite Graphs Containing Every Possible Pair of Directed Cycles

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ABSTRACT. Let $D = (V_1, V_2; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \geq 2$. Suppose that $d_D(x) + d_D(y) \geq 3n$ for all $x \in V_1$ and $y \in V_2$. Then, with one exception, D contains two vertex-disjoint directed cycles of lengths $2s$ and $2t$, respectively, for any two positive integers s and t with $s+t \leq n$.

1 Introduction

We discuss only finite simple graphs and strict directed graphs. The terminology and notation concerning graphs is that of [4], except as indicated. A directed graph D is called a directed bipartite graph if there exists a partition $\{V_1, V_2\}$ of $V(D)$ such that the two induced directed subgraphs $D[V_1]$ and $D[V_2]$ of D contain no arcs of D. We denote by $(V_1, V_2; A)$ a directed bipartite graph with $\{V_1, V_2\}$ as its bipartition and A as its arc set. Similarly, $(V_1, V_2; E)$ represents a bipartite graph with $\{V_1, V_2\}$ as its bipartition and E as its edge set. In [11], C. Little, K. Teo and H. Wang investigated two vertexdisjoint directed cycles in D. To state the result, a directed bipartite graph B_n of order $2n$ for every integer $n \geq 2$ is constructed as follows. We use $K^*_{a,b}$ to denote the complete directed bipartite graph $(V_1, V_2; A)$ with $|V_1| = a$ and $|V_2| = b$ such that both (x, y) and (y, x) belong to A for all $x \in V_1$ and $y \in V_2$. Let $D_1 = (X_1, Y_1; A_1)$ and $D_2 = (X_2, Y_2; A_2)$ be two vertex-disjoint directed bipartite graphs such that D_1 is isomorphic to $K^*_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$ and D_2 is isomorphic to $K^*_{\lceil n/2 \rceil, \lceil n/2 \rceil}$. Then B_n consists of D_1 and D_2 and all arcs (u, v) and (x, y) for $u \in X_1$, $v \in Y_2$, $x \in Y_1$ and $y \in X_2$. C. Little, Kee Teo and H. Wang proved the following.

*This research was supported by UIRC SEED GRANTS-KDY932.

Theorem A. [11] *Let $D = (V_1, V_2; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \geq 2$. Suppose that $d_D(x) + d_D(y) \geq 3n$ for all $x \in V_1$ and $y \in V_2$. Then D contains two vertex-disjoint directed cycles of lengths $2n_1$ and $2n_2$, respectively, for any positive integer partition $n = n_1 + n_2$, unless n is even and D is isomorphic to B_n.*

In this paper, we strengthen the above result, proving the following:

Theorem B. *Let $D = (V_1, V_2; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \geq 2$. Suppose that $d_D(x) + d_D(y) \geq 3n$ for all $x \in V_1$ and $y \in V_2$. Then D contains two vertex-disjoint directed cycles of lengths $2s$ and $2t$, respectively, for any two positive integers s and t with $s + t \leq n$, unless n is even and D is isomorphic to B_n.*

For convenience, we mention some terminology and notation. Let G be a graph and D a directed graph. We use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. We denote $|E(G)|$ by $e(G)$. We use $V(D)$ and $A(D)$ to denote the vertex set and the arc set of D, respectively. For a vertex $x \in V(D)$, $N_D^+(x)$ is the set of all vertices y of D with $(x, y) \in A(D)$. We similarly define $N_D^-(x)$ and let $N_D(x) = N_D^+(x) \cup N_D^-(x)$. We also define $d_D^+(x) = |N_D^+(x)|$, $d_D^-(x) = |N_D^-(x)|$ and $d_D(x) = d_D^+(x) + d_D^-(x)$. For two vertices x and y of D, we say that x is joined to y in D if either (x, y) or (y, x) is an arc of D. For a vertex $u \in V(G)$ and a subgraph H of G, we define $d_G(u, H) = |N_G(u) \cap V(H)|$ where $N_G(u)$ denotes the set of vertices that are adjacent to u in G. Hence $d_G(u, G) = d_G(u)$, the degree of u in G. For a subset $U \subseteq V(G)$, $G[U]$ is the subgraph of G induced by U. For a subset $X \subseteq V(D)$, $D[X]$ is the directed subgraph of D induced by X. A graph or directed graph is said to be traceable if it contains a hamiltonian path or directed hamiltonian path, respectively. A graph or direcded graph is called hamiltonian if it contains a hamiltonian cycle or directed hamiltonian cycle, respectively. For any two vertices x and y of G, we define $\epsilon(xy) = 1$ if xy is an edge of G and $\epsilon(xy) = 0$ otherwise. For any two vertices u and v of D, we define $\tau(u, v) = 1$ if u is not joined to v in D and $\tau(u, v) = 0$ otherwise. We set $\Delta^+(D) = \max\{d_D^+(x) \mid x \in V(D)\}$ and $\Delta^-(D) = \max\{d_D^-(x) \mid x \in V(D)\}$.

To conclude our introduction, we propose the following conjecture.

Conjecture C. *Let $D = (V_1, V_2; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \geq 2$. Suppose that $d_D(x) + d_D(y) \geq 3n$ for all $x \in V_1$ and $y \in V_2$. If $H = (U_1, U_2; A')$ is a directed bipartite graph with $|U_1| = |U_2| = n$, $\Delta^+(H) \leq 1$ and $\Delta^-(H) \leq 1$, then D contains a directed subgraph isomorphic to H, unless n is even and D is isomorphic to B_n.*

2 Lemmas

In the following, $G = (V_1, V_2; E)$ is a bipartite graph with $|V_1| = |V_2| = n \geq 2$.

Lemma 2.1. *Let $P = x_1y_1 \dots x_ky_k$ be a path of G. Let $y \in V(G)$ be a vertex not on P such that $\{x_1, y\} \not\subseteq V_i$ for each $i \in \{1, 2\}$. Suppose that $d_G(x_1, P) + d_G(y, P) \geq k + 1$. Then G has a path P' from y to y_k such that $V(P') = V(P) \cup \{y\}$.*

Proof: The condition implies that $\{x_iy, x_1y_i\} \subseteq E$ for some $i \in \{1, 2, \dots, k\}$. Then the path $P' = yx_iy_{i-1} \dots x_2y_1x_1y_ix_{i+1}y_{i+1} \dots x_ky_k$ satisfies the requirement. □

Lemma 2.2. [10] *Let $P = x_1y_1 \dots x_ky_k$ be a path of G. Let $x \in V_1$ and $y \in V_2$ be vertices not on P. Then the following two statements hold:*

(a) *If $d_G(x, P) + d_G(y, P) \geq k + 2 - \epsilon(xy)$, then G contains a path P' from x_1 to y_k such that $V(P') = V(P) \cup \{x, y\}$.*

(b) *If $d_G(x, P) + d_G(y, P) \geq k + 1 - \epsilon(xy)$, then G contains a path P' such that $V(P') = V(P) \cup \{x, y\}$.*

Lemma 2.3. [3] *The following two statements hold:*

(a) *Let $P = x_1y_1 \dots x_ky_k$ be a path of G with $k \geq 2$. If $d_G(x_1, P) + d_G(y_k, P) \geq k + 1$, then G has a cycle C such that $V(C) = V(P)$.*

(b) *If $d_G(x) + d_G(y) \geq n + 1$ for any two non-adjacent vertices x and y with $x \in V_1$ and $y \in V_2$, then G is hamiltonian.*

Lemma 2.4. [9] *If $d_G(x) + d_G(y) \geq n$ for all $x \in V_1$ and $y \in V_2$, then either G is traceable, or n is even and G is the vertex-disdoint union of two subgraphs isomorphic to $K_{n/2,n/2}$.*

Lemma 2.5. [1,10] *Let $C = x_1y_1 \dots x_ky_kx_1$ be a cycle of G. Let $i, j \in \{1, 2, \dots, k\}$. Suppose that $d_G(x_i, C) + d_G(y_j, C) \geq k + 2$. Then G has a path P from y_i to x_{j+1} such that $V(P) = V(C)$, where subscripts are reduced modulo k.*

Lemma 2.6. [10] *Suppose that G has a hamiltonian path and for any two endvertices u and v of a hamiltonian path of G, $d_G(u) + d_G(v) \geq k$ holds, where k is an integer greater than n. Then for every $x \in V_1$ and every $y \in V_2$, $d_G(x) + d_G(y) \geq k$.*

Let $V_1 = \{x_1, \dots, x_n\}$ and $V_2 = \{y_1, \dots, y_n\}$ be such that $d(x_1) \leq \dots \leq d(x_n)$ and $d(y_1) \leq \dots \leq d(y_n)$. Schmeichel and Mitchem [8] proved that if

$n > 3$ and $d(x_k) \le k$ implies $d(y_{n-k}) \ge n-k+1$ for each $1 \le k \le n$, then G contains a cycle of length $2l$ for each $l \in \{2, 3, \ldots, n\}$. As a corollary, we have the following.

Lemma 2.7. *If $d(x) + d(y) \ge n+1$ for each $x \in V_1$ and $y \in V_2$, then G contains a cycle of length $2k$ for each $k \in \{2, 3, \ldots, n\}$, unless $n = 3$ and G is a cycle of length 6.*

3 Proof of Theorem B

Besides using the idea of [11], our proof also depends heavily on Lemma 2.7 and the following Lemma 3.1.

Let $D = (V_1, V_2; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \ge 2$ such that $d_D(x) + d_D(y) \ge 3n$ for all $x \in V_1$ and $y \in V_2$. Suppose, for a contradiction, that D does not contain two vertex-disjoint directed cycles of lengths $2s$ and $2t$, respectively, for some positive integers s and t with $s + t \le n$. Furthermore, D is not isomorphic to B_n when n is even. By Theorem A, $s + t \le n$.

We construct a bipartite graph $G = (V_1, V_2; E)$ from D such that $xy \in E$ if and only if both (x, y) and (y, x) belong to A. Then G does not contain two vertex-disjoint cycles of lengths $2s$ and $2t$, respectively. For each subgraph G' of G and each vertex $x \in V(G) - V(G')$, we let $d_D^+(x, G') = |N_D^+(X) \cap V(G')|$. Similarly, we define $d_D^-(x, G')$ and $d_D(x, G')$.

Lemma 3.1. *Let p and q be two positive integers with $p + q < n$. Let F_1 be an induced subgraph of order $2p$ in G and $F_2 = G - V(F_1)$. Let $m = n - p$. Let $x_1yy_1, \ldots, x_{m-1}y_{m-1}$ be $m-1$ independent edges of F_2 and put $\{x_m, y_m\} = V(F_2) - \{x_i, y_i \mid 1 \le i \le m-1\}$. Suppose that F_1 is traceable and that $D[V(F_2 - x_i - y_i)]$ contains a directed cycle of length $2q$ for each $i \in \{1, 2, \ldots, m\}$. Then D contains two vertex-disjoint directed cycles of lengths $2(p+1)$ and $2q$, respectively.*

Proof: On the contrary, suppose that the assertion in the lemma is false. Let u and v be any two endvertices of a hamiltonian path of F_1. Let $r = m$ if $x_my_m \in E(G)$ and otherwise $r = m - 1$. Then

$$d_D(u, x_iy_i) + d_D(v, x_iy_i) \le 2 \text{ for all } i \in \{1, 2, \ldots, r\}.$$

Thus

$$d_D(u, F_2) + d_D(v, F_2) \le 2r + \lambda$$

where $\lambda = 4$ if $r = m - 1$ and otherwise $\lambda = 0$. Consequently,

$$d_D(u, F_1) + d_D(v, F_1) \ge 3n - 2r - \lambda = 3p + m + 2(m - r) - \lambda.$$

This yields that $d_G(u, F_1) + d_G(v, F_1) \ge p + m + 2(m - r) - \lambda$. By the assumption of the lemma, we see that we can choose $r = 2$ if $m = 2$.

Thus we have that $d_G(u, F_1) + d_G(v, F_1) \geq p + 1$. By Lemma 2.3, F_1 is hamiltonian if $p \geq 2$. Let $C = u_1 v_1 \ldots u_p v_p$ be a hamiltonian path of F_1 such that $u_1 v_p \in E$. Then $d_D(x_i, u_j v_j) + d_D(y_i, u_j v_j) \leq 2$ for all $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, p\}$ for otherwise we readily see that D contains two required directed cycles. Thus $d_D(x_i, F_1) + d_D(y_i, F_1) \leq 2p$ for each $i \in \{1, 2, \ldots, r\}$. Let I be the number of arcs of D between F_1 and F_2. Then

$$I = \sum_{w \in V(F_1)} d_D(w) - 2|A(D[V(F_1)])| \geq 3np - 4p^2. \tag{1}$$

We now break into the following two cases. Say $\{x_1, \ldots, x_m\} \subseteq V_1$.

Case 1. $r = m$.

We clearly have that $I \leq 2pm$. Together with (1), this yields that $p \geq m$. We also see that, for each $i \in \{1, \ldots, m\}$, $4m \geq d_D(x_i, F_2) + d_D(y_i, F_2) \geq 3n - 2p = 3m + p$. It follows that $p \leq m$. Consequently, we obtain that $D[V(F_1)] \cong D[V(F_2)] \cong K^*_{n/2, n/2}$ and the lemma follows as $D \not\cong B_n$.

Case 2. $r = m - 1$.

By Theorem A, $m \geq q + 2$ and so $m \geq 3$. Let $F_2 = F_2 - x_m - y_m$. We clearly have that $I \leq 2p(m-1) + d_D(x_m, F_1) + d_D(y_m, F_1)$. As $d_D(x_m, F_1) + d_D(y_m, F_1) \leq 4p$, we see, together with (1), that $p \geq m - 2$. Suppose that $p = m - 2$. Then we must have that $D[V(F_1)] \cong K^*_{p,p}$ and $d_D(x_m, F_1) + d_D(y_m, F_1) = 4p$. Then we see that $p = 1$ and $\tau(x_m, y_m) = 1$ for otherwise $D[V(F_1 + x_m + y_m)]$ is hamiltonian. Therefore $m = 3$ and $q = 1$. Then by the degree condition on D, it is easy to see that D contains two directed cycles of lengths 2 and 4, respectively. Hence $p \geq m - 1$.

As $m \geq 3$, $p \geq 2$. Thus F_1 is hamiltonian. As $D[V(F_1 + x_m + y_m \cdot)]$ is not hamiltonian, we see that $d_G(x_m, F_1) + d_G(y_m, F_1) \leq p + \tau(x_m, y_m)$ by Lemma 2.2(a). Then $d_D(x_m, F_1) + d_D(y_m, F_1) \leq 3p + \tau(x_m, y_m)$. It follows that $d_D(x_m, F_2) + d_D(y_m, F_2) \geq 3m - \tau(x_m, y_m)$ and $d_G(x_m, F_2) + d_G(y_m, F_2) \geq (m-1) + 1$. This implies that there exists $i \in \{1, \ldots, m-1\}$, say $i = 1$, such that $\{x_1 y_m, y_1 x_m\} \subseteq E$. Then $\{x_2 y_2, \ldots, x_{m-1} y_{m-1}, x_m y_1, y_m x_1\}$ is a perfect matching of F_2. Hence $q \geq 2$ by Case 1. Thus $m \geq q+2 \geq 4$. Since $p \geq m-1$ and $d_D(x_i, F_1) + d_D(y_i, F_1) \leq 2p$ for each $i \in \{1, \ldots, m-1\}$, we can readily show that $4m \geq d_D(x_i, F_2) + d_D(y_i, F_2) \geq 3m + p \geq 4m-1$ and $d_G(x_i, F_2) + d_G(y_i, F_2) \geq 2m-1$ for each $i \in \{1, \ldots, m-1\}$. Then $d_G(x_i, F_2') + d_G(y_j, F_2') \geq 2(m-1) - 2 \geq (m-1) + 1$ for all $i, j \in \{1, \ldots, m-1\}$. By Lemma 2.3(b), F_2' is hamiltonian. Let $a_1 b_1 \ldots a_{m-1} b_{m-1} a_1$ be a hamiltonian cycle of F_2'. As $d_G(x_m, F_2') + d_G(y_m, F_2') \geq (m-1) + 1$, we may assume w.l.o.g. that $\{a_1 y_m, b_1 x_m\} \subseteq E$. As $d_D(x_i, F_2) + d_D(y_i, F_2) \geq 4m - 1$ for each $i \in \{1, \ldots, m-1\}$, we see that x is joined to y in D for each $x \in \{x_1, \ldots, x_m\}$ and $y \in \{y_1, \ldots, y_m\}$ with $\{x, y\} \neq \{x_m, y_m\}$. As $q \leq m-2$, we then see that each of $D[V(F_2 - x_m - b_1)]$ and $D[V(F_2 - y_m - a_1)]$

contains a directed cycle of length $2q$, and $D[V(F_2' - a_i - b_i)]$ contains a directed cycle of length $2q$ for each $i \in \{2, \dots, m-1\}$. Thus we go back to Case 1 again. This proves the lemma. □

We will divide our proof into two parts after the following four claims.

Claim 1. If C is a cycle of length 6, and x and y are two vertices not on C in G with $x \in V_1$ and $y \in V_2$ such that $d_G(x, C) + d_G(y, C) \geq 5$, then $C + x + y$ has a perfect matching M such that $C + x + y - V(e)$ contains a cycle of length 4 for all $e \in M$.

Proof of Claim 1: W.l.o.g., say $C = u_1v_1u_2v_2u_3v_3u_1$ such that $\{v_1, v_2, v_3\} \subseteq N_G(x)$ and $\{u_1, u_2\} \subseteq N_G(y)$. We see that the claim is true by choosing $M = \{u_1v_1, u_3v_3, u_2y, v_2x\}$. □

Claim 2. For all $x \in V_1$ and $y \in V_2$, $d_G(x) + d_G(y) \geq n + 2\tau(x, y)$.

Proof of Claim 2.: We have

$$d_G(x) + d_G(y) = d_D(x) + d_D(y) - (|N_D(x)| + |N_D(y)|) \tag{2}$$
$$\geq 3n - (2n - 2\tau(x, y)) = n + 2\tau(x, y) \tag{3}$$

□

Let $r = n - s$. By Claim 2 and Lemma 2.4, either G is traceable, or $G = G' \cup G''$ with $G' \cong G'' \cong K_{n/2,n/2}$. In the latter case, it is easy to see, by the condition on the degrees of D, that the theorem holds for D since D is not isomorphic to B_n. Therefore we may assume that G is traceable. Then we can choose two vertex-disjoint induced subgraphs of G, say $G_1 = (A_1, B_1; E_1)$ and $G_2 = (A_2, B_2; E_2)$, of orders $2s$ and $2r$, respectively such that

$$\text{both } G_1 \text{ and } G_2 \text{ are traceable.} \tag{4}$$

Subject to (4), we may further choose G_1 and G_2 such that

$$e(G_1) + e(G_2) \text{ is maximum.} \tag{5}$$

Claim 3. [10]. Let u and v be two endvertices of a hamiltonian path of G_1 and let x and y be two endvertices of a hamiltonian path of G_2. Suppose that $uy \in E$ and $vx \in E$. Then

$$\begin{aligned} &d_G(u, G_1) + d_G(v, G_1) + d_G(x, G_2) + d_G(y, G_2) \\ \geq &d_G(u, G_2) + d_G(v, G_2) + d_G(x, G_1) + d_G(y, G_1). \end{aligned} \tag{6}$$

Claim 4. [10]. Let u and v be two endvertices of a hamiltonian path of G_1 and let x and y be two endvertices of a hamiltonian path of G_2 such that

$u \in V_1$ and $x \in V_1$. Let $G_1 = G_1 - u - v + x + y$ and $G_2 = G_2 - x - y + u + v$. If both G_1 and G_2 are traceable, then

$$\begin{aligned} &d_G(u, G_1) + d_G(v, G_1) + d_G(x, G_2) + d_G(y, G_2) \\ \geq &d_G(u, G_2) + d_G(v, G_2) + d_G(x, G_1) + d_G(y, G_1) \\ &- 2(\epsilon(uy) + \epsilon(vx)) + 2(\epsilon(uv) + \epsilon(XY)) \end{aligned} \tag{7}$$

In particular, if $d_G(u, G_2) + d_G(v, G_2) \geq r+2$ and $d_G(x, G_1) + d_G(y, G_1) \geq s+2$, then (7) holds.

Let $\{H_1, H_2\} = \{G_1, G_2\}$ and set $2n_1 = |V(H_1)|$ and $2n_2 = |V(H_2)|$. Then $\{n_1, n_2\} = \{s, r\}$. Let $P_1 = u_1 v_1 \ldots u_{n_1} v_{n_1}$ and $P_2 = x_1 y_1 \ldots x_{n_2} y_{n_2}$ be two hamiltonian paths of H_1 and H_2, respectively. Furthermore, we choose P_2 with $d_G(x_1, H_2) + d_G(y_{n_2}, H_2)$ as small as possible. We may assume that $u_1 \in A_1$, $x_1 \in A_2$, $A_1 \cup A_2 = V_1$ and of course $B_1 \cup B_2 = V_2$. Since D does not contain two vertex-disjoint directed cycles of lengths $2s$ and $2t$, respectively, we set H_1 and H_2 such that if $H_2 = G_1$ then H_2 does not contain a directed cycle of length $2s$, and if $H_2 = G_2$ then H_2 does not contain a directed cycle of length $2t$. Then we have the following.

$$N_D(x_1) \subseteq V_2 - \{y_s\} \text{ and } N_D(y_s) \subseteq V_1 - \{x_1\} \text{ if } H_2 = G_1; \tag{8}$$

$$N_D(x_1) \subseteq V_2 - \{y_t\} \text{ and } N_D(y_r) \subseteq V_1 - \{x_{r-t+1}\} \text{ if } H_2 = G_2. \tag{9}$$

By (2), (8) and (9), we obtain

$$d_G(x_1) + d_G(y_{n_2}) \geq n + 2. \tag{10}$$

We now divide our proof into the following two parts.

Part I. $d_G(x_1, H_2) + d_G(y_{n_2}, H_2) \leq n_2$.

By (10), we have

$$d_G(x_1, H_1) + d_G(y_{n_2}, H_1) \geq n_1 + 2. \tag{11}$$

When $H_2 = G_1$, it is easy to see that $\{(x_1, y_i), (y_{n_2}, x_i)\} \not\subseteq A$ and $\{(y_i, x_1), (x_i, y_{n_2})\} \not\subseteq A$ for all $i \in \{1, 2, \ldots, n_2\}$ for otherwise $D[V(G_1)]$ is hamiltonian. This implies that $n_2 \geq 2$, $d_D^+(x_1, H_2) + d_D^+(y_{n_2}, H_2) \leq n_2$ and $d_D^-(x_1, H_2) + d_D^-(y_{n_2}, H_2) \leq n_2$, and therefore $d_D(x_1, H_1) + d_D(y_{n_2}, H_1) \geq 3n - 2n_2 = 3n_1 + n_2$. It follows that

$$\text{if } H_2 = G_1, \text{ then } d_G(x_1, H_1) + d_G(y_{n_2}, H_1) \geq n_1 + n_2 \geq n_1 + 2. \tag{12}$$

Claim 5. For any two vertices u and v of H_1 with $u \in V_1$ and $v \in V_2$, $d_G(u, H_1) + d_G(v, H_1) \geq n_1 + 1$. Furthermore, if u and v are two endvertices of a hamiltonian path of H_1 and u is not joined to v in D, then $d_G(u, H_1) + d_G(v, H_1) \geq n_1 + 2$.

Proof of Claim 5: Suppose, for a contradiction, that the claim is not true. Then at least one of the two assertions is false. If the first assertion is false, then, by Lemma 2.6, there exist two endvertices u and v of a hamiltonian path of H_1 such that $d_G(u, H_1) + d_G(v, H_1) \leq n_1$. If the first assertion is true and the second false, then there exist two endvertices u and v of a hamiltonian path of H_1 such that $d_G(u, H_1) + d_G(v, H_1) = n_1 + 1$ and u is not joined to v in D. In both cases, we may assume that $\{u, v\} = \{u_1, v_{n_1}\}$. Let $\tau = \tau(u_1, v_{n_1})$. By (3), we have

$$\begin{aligned} d_G(u_1, H_2) + d_G(v_{n_1}, H_2) &= d_G(u_1) + d_G(v_{n_1} - d_G(u_1, H_1) - d_G(v_{n_1}, H_1) \\ &\geq n + 2\tau - (n_1 + \tau) = n_2 + \tau. \end{aligned} \tag{13}$$

If $\epsilon(u_1 y_{n_2}) + \epsilon(v_{n_1} x_1) = 0$, then we see, using Lemma 2.2(b) and (11), that both H_1' and H_2' are traceable, where $H_1' = H_1 - u_1 - v_{n_1} + x_1 + y_{n_2}$ and $H_2' = H_2 - x_1 - y_{n_2} + u_1 + v_{n_1}$. Thus by Claim 4, we have $\epsilon(u_1 y_{n_2}) + \epsilon(v_{n_1} x_1) \geq 1$, a contradiction. Therefore $\epsilon(u_1 y_{n_2}) + \epsilon(v_{n_1} x_1) \geq 1$. By Claim 3, we then have $\epsilon(u_1 y_{n_2}) + \epsilon(v_{n_1} x_1) \geq 1$ for otherwise (6) is violated. Therefore $\epsilon(u_1 y_{n_2}) + \epsilon(v_{n_1} x_1) = 1$. W.1.o.g., we may assume that $v_{n_1} x_1 \in E$ and $u_1 y_{n_2} \notin E$. If $d_G(u_1, H_2) + d_G(v_{n_1}, H_2) \geq n_2 + 1 + \tau$, then both H_1' and H_2' are traceable by Lemma 2.2, and then by Claim 4, we obtain a contradiction, that is, $n_2 + n_1 + \tau \geq (n_2 + 1 + \tau) + (n_1 + 2) - 2$. Hence we must have

$$\begin{aligned} d_G(u_1, H_1) + d_G(v_{n_1}, H_1) &= n_1 + \tau \text{and} \\ d_G(u_1, H_2) + d_G(v_{n_1}, H_2) &= n_2 + \tau. \end{aligned} \tag{14}$$

By (2), (3) and (14), we obtain

$$N_D(u_1) = V_2 \text{and } N_D(v_{n_1}) = V_1 \text{ if } \tau = 0; \tag{15}$$

$$N_D(u_1) = V_2 - \{v_{n_1}\} \text{ and } N_D(v_{n_1}) = V_1 - \{u_1\} \text{ if } \tau = 1. \tag{16}$$

By (11) and (14) and the hypothesis of Part I, we have either $d_G(u_1, H_2) + d_G(x_1, H_1) > d_G(u_1, H_1) + d_G(x_1, H_2)$, or $d_G(v_{n_1}, H_2) + d_G(y_{n_2}, H_1) > d_G(v_{n_1}, H_1) + d_G(y_{n_2}, H_2)$. We distinguish these two cases in the following. First, we note that $u_{n_1} y_{n_2} \notin E$ for otherwise we see, from (15) and (16), that $D[V(H_1 - v_{n_1} + y_{n_2})]$ is hamiltonian and $D[V(H_2 - y_{n_2} + v_{n_1})]$ contains a directed cycle of length $2t$ if $H_1 = G_1$ and $H_2 = G_2$, and otherwise $D[V(H_2 - y_{n_2} + v_{n_1})]$ is hamiltonian and $D[V(H_1 - v_{n_1} + y_{n_2})]$ contains a directed cycle of length $2t$.

Case 1. $d_G(u_1, H_2) + d_G(x_1, H_1) > d_G(u_1, H_1) + d_G(x_1, H_2)$.

In this case, let $H_1'' = H_1 - u_1 + x_1$ and $H_2'' = H_2 - x_1 + u_1$. Then H_1'' is traceable and $e(H_1'') + e(H_2'') > e(H_1) + e(H_2)$. By (4) and (5), H_2'' should not be traceable. Therefore $u_1 y_1 \notin E$ and $d_G(u_1, H_2) + d_G(y_{n_2}, H_2) = d_G(u_1, H_2 - x_1 - y_1) + d_G(y_{n_2}, H_2 - x_1 - y_1) + \epsilon(x_1 y_{n_2}) \leq n_2 - 1 + 1 = n_2$

by Lemma 2.1. Clearly, we have that either $N_D(u_1) \neq V_2$ or $N_D(y_{n_2}) \neq V_1$ for otherwise D contains two required directed cycles. Then by (2), $d_G(u_1)+d_G(y_{n_2}) \geq n+1$. Therefore $d_G(u_1, H_1)+d_G(y_{n_2}, H_1) \geq n_1+1$. By (14), $d_G(y_{n_2}, H_1) \geq d_G(v_{n_1}, H_1)+1-\tau$ and $d_G(v_{n_1}, H_2) \geq d_G(y_{n_2}, H_2)+\tau$. Furthermore, we have $d_G(u_1, H_1-u_{n_1}-v_{n_1})+d_G(y_{n_2}, H_1-u_{n_1}-v_{n_1}) \geq n_1$ as $u_{n_1}y_{n_2} \notin E$, and therefore we see, by Lemma 2.1, that $H_1 - v_{n_1} + y_{n_2}$ is traceable. Clearly, $H_2 - y_{n_2} + v_{n_1}$ is traceable. This is in contradiction with (5) $e(H_1 - v_{n_1} + y_{n_2}) + e(H_2 - y_{n_2} + v_{n_1}) \geq e(H_1) + e(H_2) + 1$.

Case 2. $d_G(v_{n_1}, H_2) + d_G(y_{n_2}, H_1) > d_G(v_{n_1}, H_1) + d_G(y_{n_2}, H_2)$.

In this case, let $H_1'' = H_1 - v_{n_1} + y_{n_2}$ and $H_2'' = H_2 - y_{n_2} + v_{n_1}$. Then H_2'' is traceable and $e(H_1'')+e(H_2'') > e(H_1)+e(H_2)$. Therefore by (4) and (5), H_1'' should not be traceable. Then by Lemma 2.1, we see that $d_G(u_1, H_1 - u_{n_1} - v_{n_1}) + d_G(y_{n_2}, H_1 - u_{n_1} - v_{n_1}) \leq n_1 - 1$. As $u_{n_1}y_{n_2} \notin E$, we obtain $d_G(u_1, H_1) + d_G(y_{n_2}, H_1) \leq n_1$. By (2), (8) and (9), $d_G(u_1) + d_G(y_{n_2}) \geq n+1$. Therefore $d_G(u_1, H_2)+d_G(y_{n_2}, H_2) \geq n_2+1$. Together with (14), we see that $d_G(v_{n_1}, H_1) \geq d_G(y_{n_2}, H_1)$ and $d_G(y_{n_2}, H_2) \geq d_G(v_{n_1}, H_2)$. This is in contradiction with the assumption of this case. Thus Claim 5 holds. □

By (11), $n_1 \geq 2$. By Claim 5 and Lemma 2.3, H_1 is hamiltonian. We may assume $C = P_1 + u_1v_{n_1}$ is a hamiltonian cycle of H_1. Let $F_1 = H_2 - x_1 - y_{n_2}$ if $H_2 = G_1$ and otherwise $F_1 = G[\{y_1, x_2, \ldots, y_{t-1}, x_t\}]$. Set $F_2 = G - V(F_1)$. Clearly, $F_2 - \{x_1, y_{n_2}\}$ has a perfect matching. By Lemma 3.1, There exists an edge on C, say $u_1v_{n_1}$, such that if $H_2 = G_2$ then $F_2-\{u_1, v_{n_1}\}$ does not contain a directed cycle of length $2s$, and if $H_2 = G_1$ then either $F_2 - x_1 - y_{n_2}$ or $F_2 - \{u_1, v_{n_1}\}$ does not contain a directed cycle of length $2t$. Let us first assume that $H_2 = G_1$ and $F_2 - x_1 - y_{n_2}$ does not contain a directed cycle of length $2t$. As $H_1 = G_2$ and by Claim 5 and Lemma 2.7, we see that $t = 2$ and G_2 is a cycle of length 6. By (11), $d(x_1, G_2)+d(y_{n_2}, G_2) \geq 5$. By Claim 1 and Lemma 3.1, we are done. Thus $H_2 = G_2$. The following claim completes the rest of the proof in this part.

Claim 6. For each edge $uv \in E(C)$, $D[V(H_1 - u - v + x_1 + y_{n_2})]$ is hamiltonian if $H_1 = G_1$ and otherwise $D[V(H_1 - u - v + x_1 + y_{n_2})]$ contains a directed cycle of length $2t$.

Proof of Claim 6: Suppose that the claim is not true. We may assume w.l.o.g. that $D[V(H_1 - u_1 - v_{n_1} + x_1 + y_{n_2})]$ is not hamiltonian. Let $H = H_1 - u_1 - v_{n_1}$. Then $|V(H)| = 2(n_1 - 1)$. We distinguish two cases as follows.

Case 1. $dG(x_1, H) + d_G(y_{n_2}, H) \geq (n_1 - 1) + 2$.

In this case, we see that $n_1 \geq 3$. Suppose that $D[V(H)]$ is hamiltonian. If $H_1 = G_1$, then by Lemma 2.2(a), $D[V(H + x_1 + y_{n_2})]$ is hamiltonian. If $H_1 = G_2$, we may assume that $d_G(x_1, H) \geq (n_1 + 1)/2$ and then it is easy to see that $D[V(H+x_1)]$ contains a directed cycle of length $2t$ as $t \leq n_1 - 1$, a contradiction. Therefore $d_G(v_1, H) + d_G(n_{n_1}, H) \leq n_1 - 1$ and v_1 is not joined to u_{n_1} in D. Then we see that $d_G(v_1, H_1) + d_G(u_{n_1}, H_1) \leq n_1 + 1$. By Claim 5, we further obtain $d_G(v_1, H_1) + d_G(u_{n_1}, H_1) = n_1 + 1$ and H_1 has no hamiltonian path from v_1 to u_{n_1}. By Lemma 2.2(a), this implies that $d_G(u_1, H) + d_G(v_{n_1}, H) \leq n_1 - 1$ and consequently,

$$d_G(u_1, H_1) + d_G(v_{n_1}, H_1) = n_1 + 1 \tag{17}$$

Furthermore, we have

$$d_G(u_1, e) + d_G(v_{n_1}, e) \leq 1, \text{ for any edge } e \in E(P_1 - u_1 - v_{n_1}). \tag{18}$$

Let i_0 be the largest integer in $\{1, 2, \ldots, n_1 - 1\}$ and j_0 the least integer in $\{2, 3, \ldots, n_1\}$ such that $\{u_1 v_{i_0}, v_{n_1} u_{j_0}\} \subseteq E$. By (17) and (18), we see that $N_G(u_1, P_1) = \{v_1, v_2, \ldots, v_{i_0}, v_{n_1}\}$ and $N_G(v_{n_1}, P_1) = \{u_1, u_{j_0}, u_{j_0+1} \ldots, u_{n_1}\}$ with $j_0 = i_0 + 2$. We have $u_{i_0} v_j \notin E$ for all $j \in \{j_0 - 1, j_0, j_0 + 1, \ldots, n_1 - 1\}$, for otherwise for some $j \geq j_0 - 1$ we see that

$$v_1 u_2 \ldots v_{i_0-1} u_{i_0} v_j u_j v_{j-1} \ldots u_{i_0+1} v_{i_0} u_1 v_{n_1} u_{j+1} v_{j+1} u_{j+2} \ldots v_{n_1-1} u_{n_1}$$

is a hamiltonian path from v_1 to u_{n_1} in H_1, a contradiction. Similarly, we must have $u_i v_{j_0} \notin E$ for all $i \in \{2, 3, \ldots, i_0 + 1\}$. Moreover, if $i_0 \neq 1$ then $u_{i_0} v_{n_1} \notin E$, and if $j_0 \neq n_1$ then $u_1 v_{j_0} \notin E$. In both cases, $d_G(u_{i_0}, H_1) + d_G(v_{j_0}, H_1) \leq n_1$, contradicting Claim 5. If $i_0 = 1$ and $j_0 = n_1$, then $n_1 = 3$ and we verify immediately that $D[V(H_1 - u_1 - v_{n_1} + x_1 + y_{n_2})]$ contains a desired directed cycle.

Case 2. $d_G(x_1, H) + d_G(y_{n_2}, H) \leq n_1$.

In this case, by (11), we see that $\{x_1 v_{n_1}, u_1 y_{n_2}\} \subseteq E$, $d_G(x_1, H_1) + d_G(y_{n_2}, H_1) = n_1 + 2$ and $d_G(x_1, H) + d_G(y_{n_2}, H) = n_1$. Thus $d_G(x_1, H_2) + d_G(y_{n_2}, H_2) = n_2$ by (10). By (8), (9) and (10), we see that equality holds in (8), (9) and (10). Then $D[V(H_1 - u_1 + x_1)]$ contains a directed cycle of length $2s$ or $2t$ if $H_1 = G_1$ or $H_1 = G_2$, respectively. Therefore $\{y_1, \ldots, y_{n_2-1}\} \not\subseteq N_D(u_1)$ for otherwise we obtain two required cycles. Similarly, $\{x_2, \ldots, x_{n_2}\} \not\subseteq N_D(v_{n_1})$. By (2), $d_G(u_1) + d_G(v_{n_1}) \geq n + 2$. If $d_G(u_1, H_1) + d_G(v_{n_1}, H_1) = n_1 + 1$, then $d_G(u_1, H_2) + d_G(v_{n_1}, H_2) \geq n_2 + 1$, and by Claim 3, $n_1 + 1 + n_2 \geq n_1 + 2 + n_2 + 1$, a contradiction. Hence $d_G(u_1, H_1) + d_G(v_{n_1}, H_1) \geq n_1 + 2$ and thus $d_G(u_1, H) + d_G(v_{n_1}, H) \geq (n_1 - 1) + 1$. By Lemma 2.2(a), H_1 has a hamiltonian path from v_1 to u_{n_1}.

First, suppose that $H_2 = G_1$. By (12), $n_2 = 2$. Then By (2) and (12), we obtain $N_D(x_1) = V_2 - \{y_2\}$ and $N_D(y_2) = V_1 - \{x_1\}$. Therefore $D[V(H_1 - u_1 - v_1 - u_2 + x_1)]$ contains a directed cycle of length $2t$ as x_1 is joined to $v_{n_1} - t + 1$ in D and $D[\{u_1, v_1, u_2, y_2\}]$ is hamiltonian as y_2 is joined to u_2 in D, a contradiction.

Next, suppose $H_2 = G_2$. Then we have that $N_D(x_1) = V_2 - \{y_t\}$ and $N_D(y_{n_2}) = V_1 - \{x_{n_2} - t + 1\}$. W.1.o g., say $(x_1, y_{n_2}) \in A$. Suppose that H has a hamiltonian path whose two endvertices, say $u \in V_1$ and $v \in V_2$, are joined in D. If $(u,v) \in A$, let $a_1b_1 \dots a_{n_1-1}b_{n_1-1}$ be a hamiltonian path of H with $(a_1, b_{n_1-1}) \in E$ and $a_1 \in V_1$, and if $(v,u) \in A$, let $b_1a_2 \dots a_{n_1-1}b_{n_1-1}a_1$ be a hamiltonian path of H with $(b_1, a_1) \in A$ and $a_1 \in V_1$. As $d_G(x_1, H) + d_G(y_{n_2}, H) = n_1$, there exists $i \in \{1, 2, \dots, n_1 - 1\}$ such that $\{a_iy_{n_2}, b_ix_1\} \subseteq E$. Thus

$$(a_i, b_{i-1}, \dots, a_2, b_1, a_1, b_{n_1-1}, a_{n_1-1}, b_{n_1-2}, \dots, b_i, x_1, y_{n_2}, a_i)$$

is a hamiltonian directed cycle in $D[V(H + x_1 + y_{n_2})]$, a contradiction. Hence H does not have such a path. This implies that v_1 is not joined to u_{n_1} in D and $d_G(v_1, H) + d_G(u_{n_1}, H) \le n_1 - 1$. By Claim 5, $d_G(v_1, H_1) + d_G(u_{n_1}, H_1) = n_1 + 1$. By (2), $d_G(v_1) + d_G(u_{n_1}) \ge n + 2$. Hence $d_G(v_1, H_2) + d_G(u_{n_1}, H_2) \ge n_2 + 1$. Since we already know that H_1 has a hamiltonian path from v_1 to u_{n_1}, we see, by Claim 3, that $\epsilon(x_1v_1) + \epsilon(y_{n_2}u_{n_1}) \le 1$. By Lemma 2.2(b), $H_1 = H_1 - v_1 - u_{n_1} + x_1 + y_{n_2}$ and $H_2 = H_2 - x_1 - y_{n_2} + v_1 + u_{n_1}$ are traceable. By Claim 4, $\epsilon(x_1v_1) + \epsilon(y_{n_2}u_{n_1}) = 1$ and $d_G(v_1, H_2) + d_G(u_{n_1}, H_2) = n_2 + 1$. W.l.o.g., say $x_1v_1 \in E$. It follows from (2) that $N_D(v_1) = V_1 - \{u_{n_1}\}$. Then $D[\{x_1, y_1, \dots, x_{t-1}, y_{t-1}, x_t, v_1\}]$ is hamiltonian as $v_1x_1 \in E$ and v_1 is jointed to x_t. We also see that $D[V(H_1 - v_1 + y_{n_1})]$ is hamiltonian since $u_1y_{n_2} \in E$ and y_{n_2} is joined to u_2, a contradiction. □

Part II. $d_G(x_1, H_2) + d_G(y_{n_2}, H_2) \ge n_2 + 1$.

By the minimality of $d_G(x_1, H_2) + d_G(y_{n_2}, H_2)$, we have that $d_G(x, H_2) + d_G(y, H_2) \ge n_2 + 1$ for any two endvertices x and y of a hamiltonian path of H_2. By the choice of H_2, Lemma 2.6 and Lemma 2.7, we see that H_2 must be a cycle of length 6. Moreover, $t = 2$ and $H_2 = G_2$. Then we see that $dD(z, G_2) = 4$ for all $z \in V(G_2)$. By (2), we obtain

$$d_G(x_i) + d_G(y_j) \ge n + 2 \text{ for all } i, j \in \{1, 2, 3\}; \tag{19}$$

$$d_G(x_i, G_1) + d_G(y_j, G_1) \ge s + 1 \text{ for all } i, j \in \{1, 2, 3\}. \tag{20}$$

Claim 7. For any two vertices u and v of G_1 with $u \in A_1$ and $v \in B_1$, $d_G(u, G_1) + d_G(v, G_1) \ge s + 1$. Furthermore, if u and v are two endvertices of a hamiltonian path of G_1 and u is not joined to v in D, then $d_G(u, G_1) + d_G(v, G_1) \ge s + 2$.

Proof of Claim 7: Suppose, for a contradiction, that the claim is not true. Then at least one of the two assertions is false. If the first assertion is false, then, by Lemma 2.6, there exist two endvertices u and v of a hamiltonian path of G_1 such that $d_G(u, G_1) + d_G(v, G_1) \le s$. If the first assertion is true and the second false, then there exist two endvertices u and v of a hamiltonian path of G_1 such that $d_G(u, G_1) + d_G(v, G_1) = s + 1$ and u is not joined to v in D. In both cases, we may assume that $\{u, v\} = \{u_1, v_s\}$.

Let $\tau = \tau(u_1, v_s)$. By (3), we have

$$d_G(u_1, G_2) + d_G(v_s, G_2) = d_G(u_1) + d_G(v_s) - d_G(u_1, G_1) - d_G(v_s, G_1)$$
$$\geq n + 2\tau - (s + \tau) = 3 + \tau. \qquad (21)$$

Let us first assume that $d_G(u_1, G_2) + d_G(v_s, G_2) \geq 4 + \tau$. Together with (20), we see, by Claim 3, that for each edge $x_i y_j$ of G_2, either $x_i v_s \notin E$ or $y_j u_1 \notin E$. W.l.o.g., say $x_1 v_s \notin E$. If $y_3 u_1 \notin E$, then both $G_1 - u_1 - v_s + x_1 + y_3$ and $G_2 - x_1 - y_3 + u_1 + v_s$ are traceable by Lemma 2.2(b). Then we obtain a contradiction by Claim 4. Hence we must have that $y_3 u_1 \in E$. Similarly, we must have that $y_1 u_1 \in E$ and $x_3 v_s \notin E$, and so $x_2 v_s \notin E$ and $y_2 u_1 \in E$, a contradiction. Therefore we must have, by (2) and (3), the following:

$$d_G(u_1, G_1) + d_G(v_s, G_1) = s + \tau \text{ and } d_G(u_1, G_2) + d_G(v_s, G_2) = 3 + \tau; \qquad (22)$$

$$N_D(u_1, G_1) = V_2 - \{v_s\} \text{ and } N_D(v_s) = V_1 - \{u_1\} \text{ if } \tau = 1; \qquad (23)$$

$$N_D(u_1, G_1) = V_2 \text{ and } N_D(v_s) = V_1 \text{ if } \tau = 0. \qquad (24)$$

By (21), $s \geq 2$. First, suppose that $d(u_1, G_2) > 0$ and $d_G(v_s, G_2) > 0$. Then there exists an edge of G_2, say $x_1 y_3$, such that $\{u_1 y_3, v_s x_1\} \subseteq E$. By Claim 3, we must have that $d_G(x_1, G_1) + d_G(y_3, G_1) \leq s+1$. By (2), we must have that $N_D(x_1, G_1) = B_1$ and $N_D(y_3, G_1) = A_1$. Together with (22), (23) and (24), we see that both $D[V(G_1 - u_1 + x_1)]$ and $D[\{u_1, y_2, x_3, y_3\}]$ are hamiltonian, a contradiction.

Therefore we may assume w.l.o.g. that $d_G(v_s, G_2) = 3$ and $d_G(u_1, G_2) = 0$. Hence $\tau = 0$. If $s = 2$, then $x_1 v_1 u_2 y_3 x_1$ and $v_2 x_2 y_2 x_3 v_2$ are two vertex-disjoint cycles in G, a contradiction. Thus $s \geq 3$. Clearly, $D[\{u_1, v_1, u_2, v_2\}]$ is hamiltonian as u_1 is joined to v_2. Therefore u_3 is not joined to any y_i ($i = 1, 2, 3$) for otherwise $D[V(G) - \{u_1, v_1, u_2, v_2\}]$ contains a directed cycle of length $2t$. By (2), $d_G(x_1) + d_G(y_3) \geq n + 3$, and so $d_G(x_1, G_1 - u_s - v_S) + d_G(y_3, G_1 - u_s - v_S) \geq S$. By Lemma 2.2(a), $G_1 - u_s - v_s + x_1 + y_3$ has a hamiltonian path from u_1 to v_{s-1}. Since u_1 is joined to v_{s-1} in D, $D[V(G_1 - u_s - v_S + x_1 + y_3)]$ is hamiltonian. This is a contradiction since $v_s x_2 y_2 x_3 v_s$ is a cycle of G. So the claim holds. □

By Claim 7, it is easy to verify that there exist two required directed cycles in D if $s \in \{1, 2\}$. So $s \geq 3$. By Lemma 2.3, G_1 is hamiltonian. W.l.o.g., say $u_1 v_s \in E$. By Lemma 3.1, there exists an edge of $P_1 + u_1 v_s$, say $u_1 v_s$, such that $D[V(G_1 - u_1 - v_s + x_1 + y_3)]$ is not hamiltonian. Let $H = G_1 - u_1 - v_s$. We break into the following two cases.

Case 1. $d_G(x_1, H) + d_G(y_3, H) \geq (s - 1) + 1$.

By Lemma 2.1(a), if $D[V(H)]$ is hamiltonian, then $D[V(H + x_1 + y_3)]$ is hamiltonian, a contradiction. Therefore v_1 is not joined to u_s in D and

$d_G(v_1, H) + d_G(u_s, H) \le s - 1$. Then $d_G(v_1, G_1) + d_G(u_s, G_1) = s + 1$ by Claim 7. Furthermore, G_1 does not have a hamiltonian path from v_1 to u_s for otherwise v_1 must be joined to u_s in D. This implies that $d_G(u_1, e) + d_G(v_s, e) \le 1$ for all $e \in E(P_1 - u_1 - v_s)$. Then we define i_0 and j_0 as we did in the paragraph below (18) and repeat the argument with Claim 5, H_1 and n_1 replaced by Claim 7, G_1 and s, respectively. Consequently, we obtain a contradiction, unless $s = 3$. But when $s = 3$, G_1 must be a cycle of length 6 for otherwise $D[V(G_1)]$ contains a directed cycle of length 4 and we are done. In this case, we see that $d_D(x_1, H) + d_D(y_3, H) \ge 6$. Then it is easy to see that if x_1 is joined to v_1 and y_3 is joined to u_3 in D, then $D[V(H + y_1 + y_3)]$ is hamiltonian, a contradiction. Hence either $\tau(x_1, v_1) = 1$ or $\tau(y_3, u_3) = 1$. By (2), $d_G(x_1) + d_G(y_3) \ge n + 3 = 9$ and so $d_G(x_1, G_1) + d_G(y_3, G_1) \ge 5$. By Claim 1 and Lemma 3.1, D has two required directed cycles.

Case 2. $d_G(x_1, H) + d_D(y_3, H) \le s - 1$.

Then we must have that $d_G(x_1, G_1) + d_D(y_3, G_1) = s+1$ and $\{x_1v_s, y_3u_1\} \subseteq E$. By (2), $N_D(x_1, G_1) = B_1$ and $N_D(y_3, G_1) = A_1$. We can use any edge x_iy_j of G_2 to play the role of x_1y_3 is the above argument to show that $N_D(x_i, G_1) = B_1$ and $N_D(y_j, G_1) = A_1$. In particular, u_1 is joined to y_2 in D. Consequently, we see that both $D[V(G_1 - u_1 + x_1)]$ and $D[\{u_1, y_2, x_3, y_3\}]$ are hamiltonian. This completes the proof of the theorem.

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