Bounds on domination number of complete grid graphs

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Abstract

In a paper of Cockayne et al., the authors establish an upper and a lower bound for the dominating number of the complete grid graph $G_{n,n}$ of order n^2. Namely, they proved a "formula", and cited two questions of Paul Erdős. One of these questions was *"Can we improve the order of the difference between lower and upper bounds from $\frac{n}{5}$ to $\frac{n}{2}$?"*. Our aim here is to give a positive answer to this question.

1 Introduction

Let G be a simple graph (that is, without loops nor multiple edges), $V(G)$ will denote its vertex set and $E(G)$ its edge set. We denote by $N(x)$ the neighborhood of x, and $N[x] = N(x) \cup \{x\}$.
We say that a set D of vertices in a graph G is *dominating*, if every vertex of G is either in D, or adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ of a graph G is the smallest cardinality of such a set.
The *Cartesian product* of two graphs G and H is the graph denoted by $G\Box H$, with $V(G\Box H) = V(G) \times V(H)$ (where $\times$ denotes the Cartesian product of sets) and $((u,u'); (v,v')) \in E(G\Box H)$ if and only if $u = v$ and $(u',v') \in E(H)$ or $u' = v'$ and $(u,v) \in E(G)$.

The domination number of the Cartesian product of two paths $P_k\Box P_n$ (called complete grid graph $G_{k,n}$) has intensively investigated. D.S Johnson [7] has attributed the (unpublished) proof of NP-completeness of the

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decision version problem of the domination problem (that is, given a graph G and an integer m, is there a dominating set of G of size m or less ?) for arbitrary grid graphs to F.T. Leighton. Nevertheless the complexity of determining the value of $\gamma(P_k \Box P_n)$ remains unknown. Until now, only a few cases were settled when $k \leq 16$ ([6], [1] and [4]).

In [3], the authors gave the following bounds, for all $n \geq 8$:

$$\frac{n^2+n-3}{5} \leq \gamma(P_n \Box P_n) \leq \left\lfloor \frac{n^2+4n-20}{5} \right\rfloor.$$

In the same paper, the authors mention two questions of Paul Erdős :

- *Can we improve the order of the difference between the lower and upper bounds from $\frac{n}{5}$ to $\frac{n}{2}$?*
- *What can you say about 3-dimensional complete grid graphs ?*

In the next section we prove the following :

Theorem 1.1 *For $n \geq k \geq 14$, we have :*

$$\left\lceil \frac{kn + \frac{5}{4}(k+n) - 16}{5} \right\rceil \leq \gamma(P_k \Box P_n) \leq \left\lfloor \frac{kn + 2(k+n) - 20}{5} \right\rfloor.$$

When $k = n \geq 14$, Theorem 1.1 give a positive answer to the first Erdős question.
For the second one, the reader can find in [5], the asymptotical values of domination numbers of d-dimensional complete grid graphs, for every $d \geq 2$.

2 Proof of the main result

We decompose the proof of Theorem 1.1 in two lemmas.

Lemma 2.1 (Lower bound) *For any $n \geq k \geq 14$, we have :*

$$\gamma(P_k \Box P_n) \geq \left\lceil \frac{kn + \frac{5}{4}(k+n) - 16}{5} \right\rceil.$$

A similar proof to the one given in [3], raises :

Lemma 2.2 (Upper bound, [2]) *For any $k \geq n \geq 8$, we have :*

$$\gamma(P_k \Box P_n) \leq \left\lfloor \frac{(k+2)(n+2)}{5} \right\rfloor - 4. \ \Box$$

The proof Lemma 2.1 proceeds by a refinement of the technique given in [3]. We need some additional definitions.

Let $G = (V, E)$ be a graph of maximum degree Δ and let D be a subset of V. The *redundancy* of a vertex x is $|N[x] \cap D| - 1$ (that is, the number of extra times that the vertex x is dominated by D). The *degree deficiency* of a vertex x belonging to D is $\Delta - d(x)$. Finally, the *deficiency* of a vertex x is given by :

$$F_D(x) = \begin{cases} \Delta - d(x) + |N(x) \cap D| \text{ if } x \in D \\ |N(x) \cap D| - 1, \text{ otherwise} \end{cases}.$$

For any $X \subseteq V$, $F_D(X) = \sum_{x \in X} F_D(x)$.

Lemma 2.3 *For any graph $G = (V, E)$ and for any dominating set D of V, we have that*

$$|D| = \frac{|V| + F_D(V)}{\Delta(G) + 1}.$$

Proof. Let Δ be the maximum degree of G. Give charge $\Delta + 1$ to each vertex in D. So the total charge of G is $(\Delta + 1)|D|$.
Now, we redistribute the charge according to the following rule :

Every vertex in D sends charge 1 to each of its neighbors.

Any vertex x of G receives $|N(x) \cap D|$ charge. Any vertex x in D keeps $\Delta + 1 - d(x)$ charge.
Thus, $(\Delta+1)|D| = \sum_{x \notin D}(|N(x) \cap D|) + \sum_{x \in D}(|N(x) \cap D| + \Delta + 1 - d(x)) = |V - D| + \sum_{x \notin D}(|N(x) \cap D| - 1) + |D| + \sum_{x \in D}(|N(x) \cap D| + \Delta - d(x)) = |V| + F_D(V)$. □

Lemma 2.4 *Let $G = (V, E)$ be a graph and let D be a dominating set of G. For any partition $D_1, \ldots, D_r$ of D, we have :*

$$F_D(V) \geq \sum_{i=1}^{r} F_{D_i}(N[D_i]).$$

Proof. For every $i = 1, \ldots, r$, let Δ_i be the maximum degree of the graph induced by $N[D_i]$. Since $D_1, \ldots, D_r$ is a partition of D, we have $|D| = \sum_{i=1}^{r} |D_i|$ and since D is a dominating set of G, we have $|V| \leq \sum_{i=1}^{r} |N[D_i]|$. Now, by Lemma 2.3 we obtain that

$$\frac{|V| + F_D(V)}{\Delta + 1} = |D| = \sum_{i=1}^{r} \frac{|N[D_i]| + F_{D_i}(N[D_i])}{\Delta_i + 1} \geq \frac{|V| + \sum_{i=1}^{r} F_{D_i}(N[D_i])}{\Delta + 1}. \quad \square$$

Proof of Theorem 1.1 Let D be a dominating set of $P_k \square P_n$, for $n \geq k \geq 14$. Let $D1$ (resp. $D4$) be the set of vertices in D which are in the first 7 rows (resp. columns) and the $n-7$ first columns (resp. $k-7$ last rows) of $P_k \square P_n$. Let $D2$ (resp. $D3$) be the set of vertices in D which are in the last 7 rows (resp. columns) and the $n-7$ last columns (resp. $k-7$ first rows) of $P_k \square P_n$ (see Figure 1).

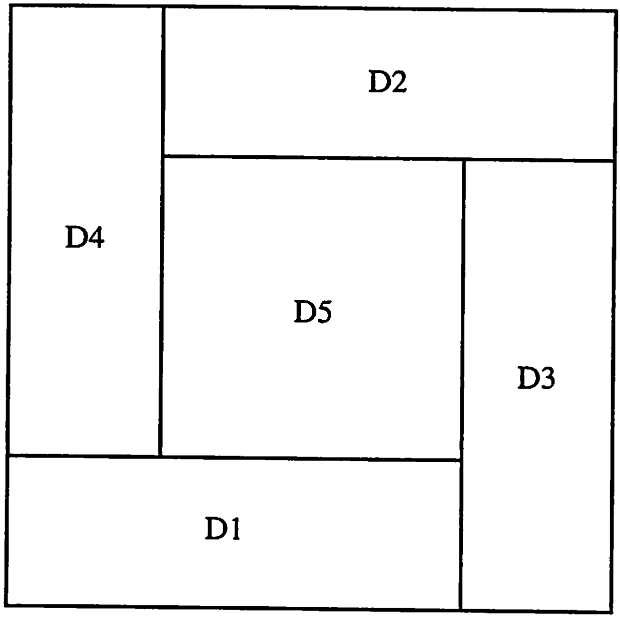

Figure 1. Partition of $P_k \square P_n$

We obtain a lower bound of $F_{Di}(N[Di])$. Let $D1_l$ be the set of vertices in $D1$ which are in the first l columns.
Using a computer, we can check that for all $m = 2, \ldots, 23$, we have $F_{D1_m}(N[D1_m]) \geq \lceil \frac{15m+9}{24} \rceil$ and that $F_{D1_{l+24}}(N[D1_{l+24}]) \geq 15 + F_{D1_l}(N[D1_l])$.
Hence, for all $m \geq 2$ we have :

$$F_{D1_m}(N[D1_m]) \geq \left\lceil \frac{15m+9}{24} \right\rceil.$$

Hence $F_{D1}(N[D1]) = F_{D1_{n-7}}(N[D1_{n-7}]) \geq \lceil \frac{15(n-7)+9}{24} \rceil$. Similarly, we can deduce a lower bound for $F_{Di}(N[Di])\ \forall i = 1, \ldots, 4$.
Let $D5 = D - (D1 \cup D2 \cup D3 \cup D4)$. Since $D5$ is a dominating set of $N[D5]$

then $F_{D5}(N[D5]) \geq 0$. By Lemma 2.4, we obtain

$$\begin{aligned} F_D(V(P_k \square P_n)) &\geq \sum_{i=1}^{5} F_{Di}(N[D_i]) \\ &\geq 2\left\lceil \frac{15(n-7)+9}{24} \right\rceil + 2\left\lceil \frac{15(k-7)+9}{24} \right\rceil \geq \frac{5(k+n)}{4} - 16. \end{aligned}$$

Finally, by Lemma 2.3, we obtain :

$$|D| = \frac{kn + F_D(V(P_k \square P_n))}{5} \geq \frac{kn + \frac{5}{4}(k+n) - 16}{5}. \square$$

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