On the Face Lattice of a Poset Polyhedron

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Abstract

The convex polyhedron of all real-valued monotone functions defined on a finite poset is an unbounded variant of the order polytope described by Stanley. If the undirected covering graph of the poset is acyclic, then the lattice of non-empty faces of this polyhedron is a Boolean lattice. In every other case both semimodularity and dual semimodularity fail.

For a finite partially ordered set (poset) P, we consider the set of all real-valued *monotone* functions on P, i. e. functions $f : P \to \mathbf{R}$ such that

$$x \leq y \text{ in } P \Longrightarrow f(x) \leq f(y) \text{ in } \mathbf{R}. \tag{1}$$

As a subset of the real vector space $\mathbf{R}^P$ (the space of all real-valued functions on P), the monotone functions constitute a convex cone, because the sum of any two monotone functions is monotone, and any non-negative scalar multiple of a monotone function is monotone. Indeed this cone is a polyhedral cone. One description by inequalities is as follows. Write $a \prec b$ if b covers a in P, i. e. $a < b$ and there is no $x \in P$ such that $a < x < b$. Let H be the *covering relation* (also called the *covering diagram*) of P, i. e.

$$H = \{(a, b) : a \prec b\}.$$

Obviously a function $f : P \to \mathbf{R}$ is monotone if and only if it satisfies all these inequalities (for all $(a, b) \in H$), which are finite in number. We therefore call the set of all real-valued monotone functions on P the *monotone polyhedron* of P, and we denote it by $\mathcal{M}(P)$.

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We are interested here in the inclusion-ordered set of non-empty faces of $\mathcal{M}(P)$, which is a lattice. For the basic facts about unbounded polyhedra we refer to part III of Schrijver [3], and for lattice-theoretical notions to Crawley and Dilworth [1].

We note that the monotone polyhedron of P is related to, but clearly distinct from, the *order polytope* of P studied by Stanley in [4]: this latter consists of all monotone functions whose range is contained in $[0,1]$. (For a recent reference on related questions, see Pitman and Stanley [2].) The monotone polyhedron is not a polytope, since it is obviously unbounded. The monotone polyhedron has no vertices, because it contains the infinite line C formed by all real-valued constant functions on P. It is in fact the unique Minkowski sum of this line C and the polyhedral cone $\mathcal{M}^+$ of all non-negative monotone functions with zero minimum, i. e. of all $f : P \to \mathbf{R}$ such that (1) holds and

$$\forall x \in P \quad f(x) \geq 0,$$

$$\min_{x \in P} f(x) = 0.$$

In other words, $f \in \mathcal{M}(P)$ if and only if there is a constant function $f_1 \in C$ and an $f_2 \in \mathcal{M}^+$ such that $f = f_1 + f_2$, and then f_1 and f_2 are obviously unique. The cone $\mathcal{M}^+$ is the convex hull of a finite number of rays, where each ray consists of the non-negative scalar multiples of the characteristic function of some upper section of P (recall that $U \subseteq P$ is an *upper section* if $x \leq y, x \in U$ implies $y \in U$, and its characteristic function $\mathbf{1}_U$ is given by $\mathbf{1}_U(x) = 1$ for $x \in P$, $\mathbf{1}_U(x) = 0$ otherwise).

This is somewhat analoguous to the description of the vertices of the order polytope in Stanley [4]. However, for the order polytope the correspondence between vertices and upper sections of P is bijective, while the correspondence we have described above establishes a bijection between upper sections and extreme rays only if P has a maximum.

Consider now the lattice F of non-empty faces of the monotone polyhedron $\mathcal{M}(P)$. We have seen that $\mathcal{M}(P)$ is the set of solutions of the system of inequalities

$$f(a) - f(b) \leq 0 \quad (a,b) \in H$$

where H is the covering relation of P. It follows that the non-empty faces are the solution sets of the various systems

$$f(a) - f(b) \; = \; 0 \quad (a,b) \in S, \tag{2}$$

$$f(a) - f(b) \; \leq \; 0 \quad (a,b) \in H \setminus S, \tag{3}$$

where $S \subseteq H$. Every subset S of H defines a system (2) – (3) and defines thus a non-empty face, but different subsets S may define the same face.

We can, however, describe a bijective correspondence between non-empty faces and certain subsets of H. This may be viewed as a variation on the correspondence described by Stanley [4] between faces of the poset polytope and certain partitions of P.

Any $S \subseteq H$ is called a *subdiagram*. We denote by $K(S)$ the *set of connected blocks* of the relation S on P, i. e. the set of equivalence classes of the smallest equivalence relation containing S. Let H/S denote the relation on $K(S)$ defined by

$$H/S = \{(A,B) \in K(S)^2 : \exists a \in A, b \in B, (a,b) \in H \setminus S\}.$$

Informally speaking, H/S is obtained from H by contracting the couples (a,b) in S. We say that S is a *closed subdiagram* if H/S is irreflexive, antisymmetric and acyclic. Closed subdiagrams are related to the *"connected compatible partitions"* of P considered by Stanley [4] as follows. If partitions are viewed as equivalence relations on P (i. e. subsets of P^2), then each connected compatible partition arises from a unique closed subdiagram S (recall that $S \subseteq H \subseteq P^2$) as the smallest equivalence relation on P containing S, and the correspondence is bijective since each closed subdiagram is the intersection of H with a unique connected compatible partition. Connected compatible partitions and closed subdiagrams constitute two isomorphic lattices. We prefer to work with subdiagrams because the intersection of two closed subdiagrams is a closed subdiagram, while this property does not hold for connected compatible partitions.

The following variant of Theorem 1.2 of Stanley [4] (reported by L. Geissinger, see [4]) is then not difficult to verify:

Lemma. *Every non-empty face of the monotone polyhedron of a poset P is defined by a unique closed subdiagram S of the covering relation H of P, via the system (2) - (3). The lattice of non-empty faces of the monotone polyhedron is dually isomorphic to the lattice of closed subdiagrams. In the lattice of closed subdiagrams, the meet of two closed subdiagrams is their intersection.* □

Consider the *covering graph* of a poset P, i. e. the undirected graph obtained from the covering relation by replacing couples (a,b) by edges $\{a,b\}$. If this graph is acyclic (i. e. a forest), then every subdiagram of the covering relation H is closed, and the lattice of closed subdiagrams is simply the power set lattice $\mathcal{P}(H)$. This is a Boolean lattice, and thus the non-empty face lattice of $\mathcal{M}(P)$ is Boolean if the covering graph is acyclic. (Note that this is not true for the order polytope described by Stanley [4].) We show below that acyclicity is also necessary for that face lattice to be Boolean, and in the rather strong sense that if acyclicity fails, then already semimodularity as well as dual semimodularity must fail. (Note that face lattices of general polyhedra can be semimodular without being Boolean or even modular.)

Recall that a lattice is *semimodular* if for all elements x, y

$$x \wedge y \prec x \Longrightarrow y \prec x \vee y.$$

The converse implication is referred to as *dual semimodularity.* Both of these conditions are weaker than modularity, and much weaker than the distributivity required in the case of Boolean lattices.

Theorem. *Let P be a finite poset and let L be the lattice of non-empty faces of the polyhedron of real-valued monotone functions on P. The following conditions are equivalent:*

(i) the undirected covering graph of P is acyclic (i. e. it is a forest),

(ii) L is a Boolean lattice,

(iii) L is a modular lattice,

(iv) L is semimodular,

(v) L is dually semimodular.

Proof. Based on the foregoing discussion, it is enough to prove that if the covering graph G of P is not a forest, then *(iv)* and *(v)* fail.

The fact that G is not a forest means that G contains some cycles. For any simple cycle C (one without self-intersection), a node x of C is called a *peak* if both neighbors of x in C are below x in P, and it is called a *valley* if both neighbors are above x in P. Let us call the total number of peaks and valleys the *change number* of C. This is an even number, since the number of peaks must equal the number of valleys, and obviously it is at least 2.

Let us consider a simple cycle C with smallest possible change number, and among all such cycles let us suppose that the order-convex hull $[C]$ of the set of vertices of C is minimal with respect to set inclusion. We shall distinguish two cases, but first let us introduce some notation and make some observations. For $a < b$ in P, the set of all $x \in P$ with $a \leq x \leq b$ is denoted by $[a, b]$. The restriction of the covering relation H of P to $[a, b]$ is denoted by $H[a, b]$. Thus

$$H[a, b] = \{(x, y) : a \leq x \prec y \leq b\}.$$

It is easy to verify that $H[a, b]$ is a closed subdiagram.

Case 1. The change number of C is 2. Let a be the unique valley and b the unique peak of C. Then $H[a,b]$ is the union of $n \geq 2$ pairwise disjoint directed paths $P_1, P_2, \ldots, P_n$ from a to b, each P_i of length at least 2, say

$$P_i = \{(a, c_i) \ldots (d_i, b)\}.$$

Moreover, for $i \neq j$ P_i and P_j have only a and b as common vertices, due to the minimality assumption on the order-convex hull $[C]$.

Clearly, the subdiagrams $\emptyset$, $\{(a, c_1)\}$ – denoted ac_1 for short, $H[c_1, b]$ and $H[a, b]$ are closed. Further, it can be seen that

$$K = H[c_1, b] \cup \{(d_2, b)\}$$

is closed as well. In the lattice of closed subdiagrams we have

$$\begin{aligned} ac_1 \wedge H[c_1, b] &= \emptyset, \\ ac_1 \vee H[c_1, b] &= H[a, b]. \end{aligned}$$

Observe that ac_1 covers $\emptyset$, but $H[a,b]$ does not cover $H[c_1, b]$ because

$$H[c_1, b] \subset K \subset H[a, b].$$

This shows that the lattice of closed subdiagrams is not semimodular, i. e. that *(v)* fails.

The subdiagrams

$$I = \{(a, c_1), (a, c_2), \ldots, (a, c_n)\},$$

$$J = H[a, b] \setminus I = H[c_1, b] \cup H[c_2, b] \cup \ldots \cup H[c_n, b]$$

are also closed. We have

$$\begin{aligned} I \wedge J &= \emptyset, \\ I \vee J &= H[a, b]. \end{aligned}$$

Now J is covered by $H[a, b]$, but I does not cover $\emptyset$ because

$$\emptyset \subset ac_1 \subset I.$$

This shows that the lattice of closed subdiagrams is not dually semimodular, i. e. that *(iv)* fails.

Case 2. The change number of C is greater than 2. In this case the subdiagram

$$K = \{(x, y) \in H : \{x, y\} \in C\}$$

is closed. Choose one of the two circular enumerations of the vertices of C. Let K_1 consist of those members (x, y) of K for which y immediately follows x in this circular enumeration, and let $K_2 = K \setminus K_1$. Clearly each K_i has at least two members. Choose $\alpha \in K_1, \beta \in K_2$ arbitrarily.

The subdiagrams $\emptyset$, α, $K_1 - \alpha$, $K_1 - \alpha + \beta$ are closed, and we have

$$\begin{aligned} \alpha \wedge (K_1 - \alpha) &= \emptyset, \\ \alpha \vee (K_1 - \alpha) &= C. \end{aligned}$$

Now $\emptyset$ is covered by α, but $(K_1 - \alpha)$ is not covered by C because

$$(K_1 - \alpha) \subset (K_1 - \alpha + \beta) \subset C.$$

Thus the closed subdiagram lattice is not semimodular and *(v)* fails.

The subdiagrams $\{\alpha, \beta\}$ and $C \setminus \{\alpha, \beta\}$ are also closed. We have

$$\begin{aligned} \{\alpha, \beta\} \wedge (C \setminus \{\alpha, \beta\}) &= \emptyset, \\ \{\alpha, \beta\} \vee (C \setminus \{\alpha, \beta\}) &= C. \end{aligned}$$

Clearly, $C \setminus \{\alpha, \beta\}$ is covered by C, but $\emptyset$ is not covered by $\{\alpha, \beta\}$. Therefore the closed subdiagram lattice is not dually semimodular and *(iv)* fails. □

References

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