Weak Exponent of Indecomposability of an Irreducible Boolean Matrix*

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ABSTRACT. We provide upper estimates on the weak exponent of indecomposability of an irreducible Boolean matrix.

1 Introduction

A Boolean matrix is a matrix whose entries are 0 or 1; the arithmetic underlying the matrix multiplication and addition is Boolean, that is, it is the usual integer arithmetic except that $1+1=1$.

Let B_n be the set of all $n \times n$ Boolean matrices, and let r be an integer with $-n < r < n$. A matrix $A \in B_n$ is r-indecomposable if it contains no $k \times l$ zero submatrix with $1 \leq k,\ l \leq n$ and $k+l = n-r+1$. In particular, A is $(1-n)$-indecomposable if and only if $A \neq 0$, while A is $(n-1)$-indecomposable if and only if $A = J_n$, the all-1's matrix. A 1-indecomposable matrix is also said to be fully indecomposable, and a 0-indecomposable matrix is also called a Hall matrix.

A matrix $A \in B_n$ is reducible if there is a permutation matrix P such that

$$PAP^{-1} = (A)_1 0 A_{21} A_2,$$

where A_1 and A_2 are nonvacuous square matrices; otherwise A is irreducible. In particular, the 1×1 matrix (0) is reducible and (1) is irreducible.

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Let IB_n be the set of all irreducible matrices in B_n. It is well known that

$$A + A^2 + \cdots + A^n = J_n$$

for any $A \in IB_n$. Note that J_n is r-indecomposable for any r with $-n < r < n$. Hence, for any $A \in IB_n$ and any integer r with $-n < r < n$, there exists a minimum positive integer p such that $A + A^2 + \cdots + A^p$ is r-indecomposable; such an integer p is called the weak exponent of r-indecomposablity of A, and is denoted by $w_r(A)$.

In [1], Brualdi and Liu use $f_w(A)$, $h_w(A)$ for $w_1(A)$ and $w_0(A)$ and call them the weak fully indecomposable exponent and weak Hall exponent of A respectively.

In [2], Liu proved that $f_w(A) \leq \lfloor \frac{n}{2} \rfloor + 1$ and $h_w(A) \leq \lceil \frac{n}{2} \rceil$ for any $A \in IB_n$, where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer $\leq x$ and the smallest integer $\geq x$ respectively.

In the present paper, the above results are extended to the general cases. We prove that

$$w_r(A) \leq \left\lfloor \frac{n+r+1}{2} \right\rfloor,$$

for any matrix $A \in IB_n$ and integer r with $-n < r < n$, and we also prove this upper estimate is best possible.

2 Preliminaries

To obtain our results, we need some notation and terminology (see [3] for basic results on matrices and directed graphs). For a matrix $A = (a_{ij}) \in B_n$, the directed graph of A, $D(A)$, is the graph with vertex set $V(D(A)) = \{1, 2, \ldots, n\}$ and arc set $E(D(A)) = \{(i, j) \colon a_{ij} \neq 0\}$. It is well known that the (i, j) entry of A^k is nonzero if and only if there is a walk of length k from vertex i to vertex j in $D(A)$.

Let $A \in B_n$ and let $X \subseteq V(D(A))$. By $R_t(A, X)$, we denote the set of all vertices reachable from a vertex in X via a walk of length t. Clearly, $R_1(A^i, X) = R_i(A, X)$.

By the definition of r-indecomposability, a matrix $A \in B_n$ is r-indecomposable if and only if, for each k such that $\max\{1, 1-r\} \leq k \leq \min\{n, n-r\}$, every $k \times n$ submatrix of A has at least $k+r$ columns with nonzero entries. Equivalently, $A \in B_n$ is r-indecomposable if and only if, for each $X \subseteq V(D(A))$ with $\max\{1, 1-r\} \leq |X| \leq \min\{n, n-r\}$, $|R_1(A, X)| \geq |X| + r$.

We need the following lemma, which has appeared in [2], for completeness, however, a proof is included here.

Lemma 1. *Suppose that $A \in IB_n$, $X \subseteq V(D(A))$, and $1 \leq t \leq n$. If*

$R_1(\sum_{i=1}^{t} A^i, X) \neq V(D(A))$, *then*

$$\left|R_1(\sum_{i=1}^{t} A^i, X)\right| \geq |R_1(A, X)| + t - 1.$$

Proof: The case $t = 1$ is trivial. Suppose $t > 1$. Let $V_1 = R_1(\sum_{i=1}^{t-1} A^i, X)$, $V_2 = V(D(A)) \backslash V_1$. Since $V_1 \neq V(D(A))$, we have $V_2 \neq \emptyset$. Note that $V_1 = \bigcup_{i=1}^{t-1} R_i(A, X)$.

Suppose $R_t(A, X) \cap V_2 = \emptyset$. Then $R_t(A, X) \subseteq V_1 = R_1(\sum_{i=1}^{t-1} A^i, X)$. Since $A \in IB_n$, $D(A)$ is strongly connected. Hence there is a vertex $x \in V_2$ and a vertex $y \in V_1$ such that $(y, x) \in E(D(A))$, which implies that $x \in R_1(\sum_{i=1}^{t} A^i, X)$. Note that $x \notin R_1(\sum_{i=1}^{t-1} A^i, X)$. We have $x \in R_t(A, X)$, which is a contradiction. Thus $R_t(A, X) \cap V_2 \neq \emptyset$, and there is at least one vertex, say $z \in R_t(A, X)$ but $z \notin V_1$. We have

$$\begin{aligned} \left|R_1(\sum_{i=1}^{t} A^i, X)\right| &= \left|\bigcup_{i=1}^{t} R_i(A, X)\right| = |V_1 \cup R_t(A, X)| \\ &\geq \left|R_1(\sum_{i=1}^{t-1} A^i, X)\right| + 1, \end{aligned}$$

which implies the desired result. □

3 The result

We have

Theorem 1. *For any matrix $A \in IB_n$, and any integer r with $-n < r < n$, we have*

$$w_r(A) \leq \left\lfloor \frac{n + r + 1}{2} \right\rfloor.$$

Proof: Let $X \subseteq V(D(A))$ with $|X| = k$, and $\max\{1, 1 - r\} \leq k \leq \min\{n, n - r\}$.

Case 1. $\frac{n-r+1}{2} < k \leq \min\{n, n - r\}$.

Note that $R_1(A^i, X) = R_i(A, X)$, and $|X| = k$. Since $D(A)$ is strongly connected, any vertex in $V(D(A))$ is reachable from a vertex in X by a walk of length at most $n - k + 1$. Hence

$$R_1(\sum_{i=1}^{n-k+1} A^i, X) = \bigcup_{i=1}^{n-k+1} R_i(A, X) = V(D(A)).$$

Since $n-k+1 \le n - \frac{n-r+2}{2} + 1 = \frac{n+r}{2} < \frac{n+r+1}{2}$, we have

$$\left|R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X)\right| = |V(D(A))| = n \ge k+r.$$

Case 2. $\max\{1, 1-r\} \le k \le \frac{n-r+1}{2}$.

Case 2.1. $R_1(\sum_{i=1}^{k+r} A^i, X) = V(D(A))$.

Since $k + r \le \frac{n-r+1}{2} + r = \frac{n+r+1}{2}$ we have

$$\left|R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X)\right| = |V(D(A))| = n \ge k+r.$$

Case 2.2. $R_1(\sum_{i=1}^{k+r} A^i, X) \neq V(D(A))$.

It follows from Lemma 1 that

$$\left|R_1(\sum_{i=1}^{k+r} A^i, X)\right| = |R_1(A, X)| + (k+r) - 1.$$

Note that $D(A)$ is strongly connected. We have $|R_1(A,X)| \ge 1$. Thus

$$\left|R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X)\right| \ge \left|R_1(\sum_{i=1}^{k+r} A^i, X)\right| \ge 1 + (k+r) - 1 = k+r.$$

Combining Cases 1 and 2, we have

$$\left|R_1(\sum_{i=1}^{\lfloor (n+r+1)/2 \rfloor} A^i, X)\right| \ge k + r = |X| + r,$$

which implies that

$$w_r(A) \le \left\lfloor \frac{n+r+1}{2} \right\rfloor.$$

The proof of the theorem is complete. □

As a consequence of Theorem 1, we have

Corollary 1. *If $A \in IB_n$, then $w_1(A) \le \lfloor \frac{n}{2} \rfloor + 1$ and $w_0(A) \le \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$.*

Theorem 2. *The upper estimate in Theorem 1 is best possible.*

Proof: Let $A \in IB_n$ with $D(A) = D$, where $V(D) = \{1, 2, \ldots, n\}$ and $E(D) = \{(i, \lfloor \frac{n-r+1}{2} \rfloor + 1) \colon 1 \le i \le \lfloor \frac{n-r+1}{2} \rfloor\} \cup \{(i, i+1) \colon \lfloor \frac{n-r+1}{2} \rfloor + 1 \le i \le n\} \cup \{(n, i) \colon 1 \le i \le \lfloor \frac{n-r+1}{2} \rfloor\}$. If $t \le \lfloor \frac{n+r-1}{2} \rfloor$, it can be easily seen that all columns except columns $\lfloor \frac{n-r+1}{2} \rfloor + 1, \ldots, \lfloor \frac{n-r+1}{2} \rfloor + t$ are zero in rows $1, 2, \ldots, \lfloor \frac{n-r+1}{2} \rfloor$ of $A + A^2 \cdots + A^t$; hence $A + A^2 + \cdots + A^t$ contains a $\lfloor \frac{n-r+1}{2} \rfloor \times (n-t)$ zero submatrix with

$$\left\lfloor \frac{n-r+1}{2} \right\rfloor + (n-t) \ge \left\lfloor \frac{n-r+1}{2} \right\rfloor + n - \left\lfloor \frac{n+r-1}{2} \right\rfloor$$

$$\ge n + \left\lfloor \frac{n-r+1}{2} - \frac{n+r-1}{2} \right\rfloor = n - r + 1,$$

which implies that $A + A^2 + \cdots + A^t$ is not r-indecomposable. By the definition of weak exponent of indecomposability, we have

$$w_r(A) \ge \left\lfloor \frac{n+r-1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+r+1}{2} \right\rfloor .$$

By Theorem 1, $w_r(A) \le \lfloor \frac{n+r+1}{2} \rfloor$. Thus we have proved

$$w_r(A) = \left\lfloor \frac{n+r+1}{2} \right\rfloor ,$$

which means that the upper estimate in Theorem 1 can be achieved for every n and r. We complete the proof. □

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