Transformation of Spanning Trees in a 2-Connected Graph

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Let T be a spanning tree of a graph G. This paper is concerned with the following operation : we remove an edge $e \in E(T)$ from T, and then add an edge $f \in E(G) - E(T)$ so that $T - e + f$ is a spanning tree of G. We refer to this operation of obtaining $T - e + f$ from T as the *transfer* of e to f. We prove that if G is a 2-connected graph with $|V(G)| \geq 5$, and if T_1 and T_2 are spanning trees of G which are not stars, then T_1 can be transformed into T_2 by repeated applications of a transfer of a nonpendant edge (an edge xy of a tree T is called a nonpendant edge of T if both of x and y have degree at least 2 in T).

1. Introduction

We consider finite undirected graphs without loops or multiple edges. Transformation of edges in a spanning tree is studied and used by various authors. For example, there are researches on spanning trees the number of whose endvertices is specified; in particular, Heinrich and Liu [3] gave a bound on the number of spanning trees with a specified number of endvertices. For other works, see [4], [2] or [1].

We are concerned with the following operation : we remove an edge $e \in E(T)$ from T, and then add an edge $f \in E(G) - E(T)$ joining the two components of $T - e$ (note that the resulting graph, $T - e + f$, is again a spanning tree of G). We refer to this operation of obtaining $T - e + f$ from T by saying that we *transfer* e to f (*in* T). A vertex x of a tree T is called an *endvertex* of T if x has degree 1 in T. An edge xy of a tree T is called a *pendant edge* of T if x or y is an endvertex of T; otherwise it is called a *nonpendant edge* of T.

The following two facts are already known (see Exercise 6.7 of [1]).

(i) Let T_1 and T_2 be spanning trees of a connected graph G. Then T_1 can be transformed into T_2 by repeated applications of a transfer of an edge.

(ii) Let G, T_1, T_2 be as in (i), and suppose G is 2-connected. Then T_1 can be transformed into T_2 by repeated applications of a transfer of a pendant edge.

In this paper, we consider a transfer of nonpendant edges, and prove the following theorem.

Theorem 1. *Let G be a 2-connected graph with $|V(G)| \geq 5$. Let T_1 and T_2 be spanning trees of G which are not stars. Then T_1 can be transformed into T_2 by repeated applications of a transfer of a nonpendant edge.*

In the theorem, we cannot drop the assumption that $|V(G)| \geq 5$. Also Theorem 1 does not generally hold for a connected graph which is not 2-connected. In fact the following corollary holds (for a spanning tree T of a connected graph G and for a block B of G, we let $T|_B$ denote the spanning tree of B defined by $E(T|_B) = E(T) \cap E(B)$).

Corollary 2. *Let G be a connected graph. Let T_1 and T_2 be spanning trees of G. Then T_1 can be transformed into T_2 by repeated applications of a transfer of a nonpendant edge if and only if one of the following five conditions is satisfied:*

(i) $|V(G)| \leq 2$;

(ii) $|V(G)| = 3$ *and* $T_1 = T_2$;

(iii) *G is 2-connected and $|V(G)| = 4$, and $\{e \mid e$ is a pendant edge of $T_1\} = \{f \mid f$ is a pendant edge of $T_2\}$;*

(iv) *G is 2-connected and $|V(G)| \geq 5$, and either T_1 and T_2 are not stars or $T_1 = T_2$; or*

(v) *G is not 2-connected and $|V(G)| \geq 4$, and for any endblock B of G, if one of $T_1|_B$ and $T_2|_B$ is the star having as its center the cutvertex of G contained in B, then so is the other one (i.e., $T_1|_B = T_2|_B$).*

We prove Theorem 1 in Section 2 and prove Corollary 2 in Section 3.

Remark 1. *Let T be a spanning tree of a connected graph G, and let T' be the spanning tree which we obtain by transferring a nonpendant edge e of T to f (so $T' = T - e + f$). Then f is a nonpendant edge of T' (because the two components of $T - e$, both of which have order at least 2, are precisely* *the components of $T' - f$). Hence we can obtain T by transferring (a nonpendant edge) f to e in T'. Thus if T_1 can be transformed into T_2 by repeated applications of a transfer of a nonpendant edge, then T_2 can be transformed into T_1 by repeated applications of a transfer of a nonpendant edge.*

2. Proof of Theorem 1

Let G, T_1, T_2 be as in Theorem 1. Let S be a largest common subtree of T_1 and T_2 (it is possible that S consists of a single vertex). For spanning trees T, T' of G, we write $T \to T'$ when T can be transformed into T' by repeated applications of a transfer of a nonpendant edge. We prove $T_1 \to T_2$ by backward induction on $| V(S) |$. Let $n = | V(G) |$.

1. We consider the case where $| V(S) | = n - 1$. Let z be the remaining vertex, i.e., the vertex which does not belong to S (z is an endvertex of both T_1 and T_2), and let x be the vertex adjacent to z in T_1 and let y be the vertex adjacent to z in T_2. We divide the proof into three cases according to the length of the (unique) x-y path in S. In general, if H is a graph, then for $a, b \in V(H)$, we let $d_H(a,b)$ denote the length of a shortest a-b path in H, and for $a \in V(H)$ and $B \subseteq V(H)$, we let $d_H(a, B)$ denote $\min\{d_H(a,b) \mid b \in B\}$.

Case 1. $d_S(x,y) \geq 3$.

On the x-y path in S, there exists an edge ab such that $a \neq x, y$ and $b \neq x, y$. We can transfer ab to yz in T_1, and then we can transfer xz to ab in $T_1 - ab + yz$. Then we obtain T_2. Therefore $T_1 \to T_2$.

Case 2. $d_S(x,y) = 2$.

Let a be the only vertex adjacent to x and y in T_1 and in T_2. Since $| V(G) | \geq 5$, $N_S(x) - \{a\} \neq \emptyset$ or $N_S(y) - \{a\} \neq \emptyset$ or $N_S(a) - \{x, y\} \neq \emptyset$.

Subcase 2.1. $N_S(x) - \{a\} \neq \emptyset$ or $N_S(y) - \{a\} \neq \emptyset$.

By symmetry (see the remark at the end of Section 1), we need to consider only the case where $N_S(x) - \{a\} \neq \emptyset$. Then we can transfer xa to yz in T_1, and then we can transfer xz to xa in $T_1 - xa + yz$. Then we obtain T_2. Therefore $T_1 \to T_2$.

Subcase 2.2. $N_S(a) - \{x, y\} \neq \emptyset$.

In this case, we can transfer xa to yz in T_1, and then we can transfer ya to xa in $T_1 - xa + yz$, and finally we can transfer xz to ya in $T_1 + yz - ya$. Then we obtain T_2. Therefore $T_1 \to T_2$.

Case 3. $d_S(x,y) = 1$.

Since $| V(G) | \geq 5$, at least one of x and y is not an endvertex of S.

Subcase 3.1. Neither x nor y is an endvertex of S.

We can transfer xy to yz in T_1, and then we can transfer xz to xy in $T_1 - xy + yz$. Then we obtain T_2. Therefore $T_1 \rightarrow T_2$.

Subcase 3.2. x or y is an endvertex of S.

By symmetry, we may assume that y is an endvertex of S. Then x is not an endvertex of S. Let

$$A = \{v \in V(T_1) \mid v \in N_{T_1}(x),\ v \text{ is an endvertex of } T_1\},$$

$$B = V(T_1) - \{x\} - A.$$

Since T_1 is not a star, $B \neq \emptyset$. Note that $z, y \in A$. We prove $T_1 \rightarrow T_2$ by induction on $d_{G-x}(z, B)$.

First we consider the case where $d_{G-x}(z, B) = 1$. Choose $v \in N_G(z) \cap B$. By the definition of B, $d_S(x, v) \geq 2$ or v is not an endvertex of S. It follows that $T_1 = S + zx \rightarrow S + zv$ by Case 1 or Case 2 when $d_S(x, v) \geq 2$ and by Subcase 3.1 when $d_S(x, v) = 1$ and v is not an endvertex of S. Since $d_S(y, v) \geq 2$, $S + zv \rightarrow S + zy = T_2$ by Case 1 or Case 2. Therefore $T_1 \rightarrow T_2$.

Next we consider the case where $d_{G-x}(z, B) \geq 2$. Let $(z =)z_0 z_1 \cdots z_l (z_l \in B)$ be a shortest path between z and B in $G - x$. Since $d_{G-x}(z, B) \geq 2$, $z_1 \in A$. Since $d_{G-x}(z, B) > d_{G-x}(z_1, B)$, we see that $T_1 \rightarrow T_1 - z_1 x + z_1 z$ by applying the induction hypothesis with z and y replaced by z_1 and z, respectively. Moreover since xz is a nonpendant edge of $T_1 - z_1 x + z_1 z$, we can transfer xz to xz_1 in $T_1 - z_1 x + z_1 z$. Hence $T_1 \rightarrow T_1 - xz + zz_1 = S + zz_1$. If $z_1 = y$, then $S + zz_1 = T_2$ and, therefore $T_1 \rightarrow T_2$. If $z_1 \neq y$, then since $d_S(z_1, y) = 2$, we get $S + zz_1 \rightarrow S + zy = T_2$ by Case 2 and, therefore $T_1 \rightarrow T_2$. This concludes the discussion for the case where $\mid V(S) \mid = n - 1$.

2. We consider the case where $\mid V(S) \mid = n - 2$. Let x and y be the two remaining vertices. Assume for the moment that x and y are adjacent to each other in T_1 or T_2. By symmetry, we may assume that $xy \in E(T_1)$. Let e be the only edge of T_1 which joins $V(S)$ and $\{x, y\}$. Then e is a nonpendant edge of T_1. Now let T_3 be the spanning tree which we obtain by transferring e in T_1 to an edge f of T_2 which joins $V(S)$ and $\{x, y\}$. Then the order of a largest common subtree of T_3 and T_2 is greater than $\mid V(S) \mid (= n - 2)$, i.e., either the order of a largest common subtree of T_3 and T_2 is $n - 1$, or $T_3 = T_2$. Hence by the induction hypothesis, $T_3 \rightarrow T_2$. Therefore $T_1 \rightarrow T_2$.

Thus we may assume that x and y are nonadjacent both in T_1 and in T_2. Write

$$T_1 = S + e_1 + e_1' \ (e_1 = ax, e_1' = cy),$$

$$T_2 = S + e_2 + e_2' \ (e_2 = bx, e_2' = dy)$$

(it is possible that $a = c$ or $b = d$). Since T_1 is not a star, at least one of $S + e_1$ and $S + e_1'$ is not a star even if S is a star (here we make use of the fact that $| V(G) | \geq 5$). We may assume that $S + e_1$ is not a star. Now let $T_3 = S + e_1 + e_2'$. Then since T_3 is not a star and the order of a largest common subtree of T_1 and T_3 and that of T_2 and T_3 are greater than $| V(S) | (= n - 2)$, it follows from the induction hypothesis that $T_1 \rightarrow T_3$ and $T_3 \rightarrow T_2$. Therefore $T_1 \rightarrow T_2$.

3. We consider the case where $| V(S) | \leq n - 3$. Let $e_1 = ax$ be an edge of T_1 with $a \in V(S)$ and $x \in V(G - S)$, and let $e_2 = by$ be an edge of T_2 with $b \in V(S)$ and $y \in V(G - S)$. We divide the proof into two cases.

Case 1. $x \neq y$.

If $S + e_1 + e_2$ is not a star, then letting T_3 be a spanning tree which contains $S + e_1 + e_2$, we see that $T_1 \rightarrow T_3 \rightarrow T_2$ by arguing as in the last part of the proof for the case where $| V(S) | = n - 2$. Thus we may assume that $S+e_1+e_2$ is a star. Then $a = b$ and a is the center of $S+e_1+e_2$. Since $| V(S) | \leq n-3$, $| V(S+e_1+e_2) | \leq n-1$, and hence by the assumption that G is 2-connected, we see that in $G - a$, there exists an edge e which joins $V((S+e_1+e_2)-a)$ and $V(G-a)-V((S+e_1+e_2)-a)(= V(G-S-x-y))$. Let T_3 be a spanning tree which contains $S + e_1 + e_2 + e$. Then since T_3 is not a star, we see that $T_1 \rightarrow T_3 \rightarrow T_2$ by arguing as in the preceding paragraph.

Case 2. $x = y$.

Since $G - x$ is connected and $V(G - x - S) \neq \emptyset$, we see that in $G - x$, there exists an edge e which joins $V(S)$ and $V(G - x - S)$. Arguing as in Case 1, we can find a spanning tree T_3 which is not a star and contains $S + e_1 + e$ as well as a spanning tree T_4 which is not a star and contains $S + e_2 + e$. We see that $T_1 \rightarrow T_3 \rightarrow T_4 \rightarrow T_2$ by arguing as earlier. This completes the proof of the theorem.

3. Proof of Corollary 2

Let G, T_1, T_2 be as in the corollary. It is easy to see that if none of (i) through (v) holds, then T_1 cannot be transformed into T_2 by repeated applications of a transfer of a nonpendant edge. Also it is clear that each of (i), (ii) and (iii) implies $T_1 \rightarrow T_2$, and the statement that (iv) implies $T_1 \rightarrow T_2$ follows immediately from Theorem 1. Thus we need to prove only the statement that (v) implies $T_1 \rightarrow T_2$. Assume that condition (v) holds.

Let $B_1, \cdots, B_k$ be the blocks of G. We proceed by induction on the number l of those blocks B_i for which $T_1|_{B_i} \neq T_2|_{B_i}$. If $l = 0$, then

$T_1 = T_2$, and hence there is nothing to be proved. Thus assume $l \geq 1$. We may assume $T_1|_{B_1} \neq T_2|_{B_1}$ (so $|V(B_1)| \geq 3$). For spanning trees T, T' of B_1, we write $T \rightsquigarrow T'$ to mean $T \cup (T_1|_{B_2} \cup \cdots \cup T_1|_{B_k}) \rightarrow T' \cup (T_1|_{B_2} \cup \cdots \cup T_1|_{B_k})$ (note that a nonpendant edge of T is also a nonpendant edge of $T \cup (T_1|_{B_2} \cup \cdots \cup T_1|_{B_k})$, i.e., $T \rightarrow T'$ implies $T \rightsquigarrow T'$).

In view of the induction hypothesis, it suffices to show that $T_1|_{B_1} \rightsquigarrow T_2|_{B_1}$. Thus we may assume that $l = 1$, i.e., $T_1|_{B_i} = T_2|_{B_i}$ for each $2 \leq i \leq k$. We divide the proof into two cases according to whether $|V(B_1)| \geq 4$ or $|V(B_1)| = 3$.

Case 1. $|V(B_1)| \geq 4$.

Subcase 1.1. $T_1|_{B_1}$ *and* $T_2|_{B_1}$ *are not stars.*

If $|V(B_1)| \geq 5$, we see from the theorem that $T_1|_{B_1} \rightsquigarrow T_2|_{B_1}$ (i.e., $T_1 \rightarrow T_2$). Thus we may assume $|V(B_1)| = 4$. If $\{e \mid e$ is a pendant edge of $T_1|_{B_1}\} = \{f \mid f$ is a pendant edge of $T_2|_{B_1}\}$, then from the case where condition (iii) holds, we see that $T_1|_{B_1} \rightsquigarrow T_2|_{B_1}$. Therefore we may suppose that $\{e \mid e$ is a pendant edge of $T_1|_{B_1}\} \neq \{f \mid f$ is a pendant edge of $T_2|_{B_1}\}$. Then we actually have $\{e \mid e$ is a pendant edge of $T_1|_{B_1}\} \cap \{f \mid f$ is a pendant edge of $T_2|_{B_1}\} = \emptyset$. Write

$$V(B_1) = \{a, b, c, d\}$$

so that

$$\{ab, cd\} = \{e \mid e \text{ is a pendant edge of } T_1|_{B_1}\},$$

$$\{bc, ad\} = \{f \mid f \text{ is a pendant edge of } T_2|_{B_1}\}.$$

Let

e_1 be the unique nonpendant edge of $T_1|_{B_1}$, and

e_2 be the unique nonpendant edge of $T_2|_{B_1}$.

By symmetry, we may assume that a is a cutvertex of G. Since e_1 is a nonpendant edge of $T_1|_{B_1}$, we can transfer e_1 to bc in $T_1|_{B_1}$ (it is possible that $e_1 = bc$). Since ab is a nonpendant edge of $T_1 - e_1 + bc$, we can transfer ab to ad in $T_1 - e_1 + bc$. Moreover since cd is a nonpendant edge of $T_1|_{B_1} - e_1 + bc - ab + ad$, we can transfer cd to e_2 in $T_1|_{B_1} - e_1 + bc - ab + ad$ to obtain $T_2|_{B_1}$ (it is possible that $cd = e_2$). Therefore $T_1 \rightarrow T_2$.

Subcase 1.2. $T_1|_{B_1}$ *or* $T_2|_{B_1}$ *is a star.*

By symmetry(see the remark at the end of Section 1), we may assume that $T_1|_{B_1}$ is a star. Let b be the center of the star. Let a be a cutvertex of G contained in B_1. If B_1 is not an endblock, then B_1 contains at least two cutvertices of G, and hence we may assume $a \neq b$; if B_1 is an endblock, then $a \neq b$ by (v). Thus in either case, we have $a \neq b$.

Hence the edge $ab \in E(T_1|_{B_1})$ is a nonpendant edge of T_1. Since B_1 is a block (with $|V(B_1)| \geq 3$), there exists a vertex $c(\neq b)$ which is adjacent to a in B_1. We can transfer ab to ac in T_1. Since $|V(B_1)| \geq 4$, $T_1|_{B_1} - ab + ac$ is not a star. Consequently if $T_2|_{B_1}$ is not a star, then by Subcase 1.1, $T_1|_{B_1} - ab + ac \rightsquigarrow T_2|_{B_1}$, and hence $T_1|_{B_1} \rightsquigarrow T_2|_{B_1}$; if $T_2|_{B_1}$ is a star, then arguing as above, we see that there exists a spanning tree T of B_1 such that T is not a star and $T_2|_{B_1} \rightsquigarrow T$ (so $T \rightsquigarrow T_2|_{B_1}$), and hence again by Subcase 1.1, we get $T_1|_{B_1} \rightsquigarrow T_2|_{B_1}$.

Case 2. $|V(B_1)| = 3$.

Write

$$V(B_1) = \{a, b, c\}$$

so that

$$E(T_1|_{B_1}) = \{ab, bc\}, \quad E(T_2|_{B_1}) = \{ac, bc\}.$$

Assume for the moment that a is a cutvertex of G. Then ab is a nonpendant edge of T_1, and hence we can transfer ab to ac in T_1. Then we obtain T_2. Therefore $T_1 \rightarrow T_2$. Thus we may assume that a is not a cutvertex. Then by (v), B_1 is not an endblock, and hence both b and c are cutvertices. Consequently, we can transfer bc to ac in T_1, and then ab to bc in $T_1 - bc + ac$. Then we obtain T_2. Therefore $T_1 \rightarrow T_2$. This completes the proof of Corollary 2.

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