On the independence number of the Cartesian product of caterpillars

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Abstract

By considering the order of the largest induced bipartite subgraph of G, Hagauer and Klavžar [4] were able to improve the bounds first published by V. G. Vizing [6] for the independence number of the Cartesian product $G \Box H$ for any graph H. In this paper, we study maximum independent sets in $G \Box H$ when G is a caterpillar, and derive bounds for the independence number when H is bipartite. The upper bound we produce is less than or equal to that in [4] when H is also a caterpillar, and is shown to be strictly smaller when H comes from a restricted class of caterpillars.

1 Introduction

The problem of determining the independence number of a graph can be shown to be NP-hard [3]. As there are numerous practical applications for determining the independence number of a graph (for instance, constructive coding problems [5], wire coloring, and processor scheduling), it is of interest to explore the idea further. Recent research seems to support the idea that the independence number of a graph which can be realized as the product of two graphs can be more easily determined if viewed in light of the independence number of the graphs which produce it, since a number of polynomial-time algorithms have been proposed for decomposing a graph with respect to the Cartesian product [1, 2, 7]. This idea is set forward in [4], in which the independence number of products with bipartite graphs, odd paths, and odd cycles is examined in great detail. This paper expands

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upon that idea, examining products with stars and generalizing the results with stars to caterpillars.

We consider finite, undirected, connected, simple graphs $G = (V, E)$ where V is the vertex set of the graph and E is the edge set. For convenience, we let $|G| = |V(G)|$. A set U of vertices is *independent* if no two of them are adjacent. The *independence number*, $\alpha(G)$, is the cardinality of the largest independent set of vertices. We say a set $S \subseteq V(G)$ is an *α-set* of G if S is a maximum independent set of G. Let $\bar{\alpha}(G) = max\{\alpha(G - S) : S$ is independent and $|S| = \alpha(G)\}$. A *2-independent* set in G is the union of two disjoint independent sets, and we denote the cardinality of the largest 2-independent set by $\alpha_2(G)$. One will note that, for a bipartite graph, $\alpha_2(G) = |G|$. A *matching* is an independent set of edges, and we denote the largest matching of G by $\tau(G)$. Given a matching M, we say a vertex x is an *unsaturated* vertex with respect to the matching M if x is incident with no edges in M.

For graphs $G = (V, E)$ and $H = (W, F)$, the Cartesian product $G \Box H$ is the graph with vertex set $V \times W$, and (u, x) is adjacent to (v, y) in $G \Box H$ whenever $uv \in E$ and $x = y$, or $xy \in F$ and $u = v$. Note that $G \Box H$ is connected if and only if both G and H are connected.

A graph is a *star*, denoted $K_{1,k}$, if it consists of one vertex of degree k and k vertices of degree one. A graph is a *caterpillar* if a path remains after the removal of all its vertices of degree one. This path is called the *spine* of the caterpillar. If x_i is a nonnegative integer for $1 \leq i \leq n$, then by $C(x_1, x_2, \ldots, x_n)$, we denote the caterpillar whose spine is the path $u_1, u_2, \ldots, u_n$ such that for each i, u_i is adjacent to a set, L_{u_i}, of degree one vertices, called leaves, where $|L_{u_i}| = x_i$. A *leafless string* of vertices is a set of spine vertices, $V_i = \{u_j, u_{j+1}, \ldots, u_k\}$, which have $x_n = 0$ for $n = j, j+1, \ldots, k$ and are between two vertices, u_{j-1} and u_{k+1}, with $x_{j-1} \geq 1$ and $x_{k+1} \geq 1$. The first set of such vertices (i.e., those vertices with smallest subscripts) is V_1, the second is V_2, and so on. Let $v_i = |V_i|$. When there is no ambiguity, we will refer to $C(x_1, x_2, \ldots, x_n)$ as C. It is also worth noting that a star $K_{1,k}$ may be thought of as a caterpillar $C(k)$.

For $x \in V(H)$, let $G_x = G \Box \{x\}$ and for $u \in V(G)$, let $H_u = \{u\} \Box H$. We call G_x a *layer* of G and H_u a *layer* of H. Note that G_x is isomorphic to G and H_u is isomorphic to H. If $y_1, y_2, \ldots, y_n$ is an established ordering of $V(H)$ and $S \subseteq V(G \Box H)$ with $Y_i = S \cap G_{y_i}$, then we follow the convention established in [4] and write $S = \langle Y_1, Y_2, \ldots, Y_n \rangle$. When $H = K_{1,k}$, we will use the ordering $y_1, y_2, \ldots, y_k, u$ where $y_1, y_2, \ldots, y_k \in L_u$.

2 Preliminary Results

Vizing obtained the following bounds on $\alpha(G \Box H)$,

Theorem 2.1 [6] *For any graphs G and H,*

$$\alpha(G)\alpha(H) + min\{|V(G)| - \alpha(G), |V(H)| - \alpha(H)\}$$
$$\leq \alpha(G\Box H) \leq min\{\alpha(G)|V(H)|, \alpha(H)|V(G)|\}.$$

The lower bound comes from noting that the product of two independent sets is independent. The upper bound arises from the realization that, for $u \in V(G)$ and S an α-set of $G\Box H$, $|S \cap H_u| \leq \alpha(H)$.

Hagauer and Klavžar [4] refined these bounds in the case where one graph is bipartite using the Cartesian product of a bipartition of one graph with a 2-independent set of the other to obtain the lower bound. They obtain the upper bound with the realization that, given a matching M of H and a maximum independent set S of $G\Box H$, for $xy \in M$, $|S\cap(G_x \cup G_y)| \leq \alpha_2(G)$ and for an unsaturated vertex $x \in V(H)$, $|S \cap G_x| \leq \alpha(G)$.

Theorem 2.2 [4] *If H is a bipartite graph, then for any graph G,*

$$\frac{|H|}{2}\alpha_2(G) \leq \alpha(G\Box H) \leq \tau(H)\alpha_2(G) + (|H| - 2\tau(H))\alpha(G).$$

In the case where both graphs are bipartite, the upper bound can be simplified and the lower bound can be improved using an equation found in the proof of the above bounds in [4].

Theorem 2.3 *For any bipartite graphs G, with color classes C_1 and C_2 ($|C_1| \geq |C_2|$), and H, with color classes D_1 and D_2 ($|D_1| \geq |D_2|$),*

$$|C_1||D_1| + |C_2||D_2| \leq \alpha(G\Box H) \leq \tau(H)|G| + (|H| - 2\tau(H))\alpha(G).$$

3 Products with Stars

In this section, we consider the Cartesian product of a star, $K_{1,k}$, with a graph G. We may eliminate the case of $K_{1,k}$ where $k = 1$ as trivial, as $\alpha(G\Box K_{1,1}) = \alpha(G\Box K_2) = \alpha_2(G)$, a result noted in [4].

Theorem 3.1 *For two stars, $K_{1,j}$ and $K_{1,k}$,*

$$\alpha(K_{1,j}\Box K_{1,k}) = jk + 1.$$

Proof. By Theorem 2.2, $\alpha(K_{1,j}\Box K_{1,k}) \leq jk + 1$. To obtain the lower bound, construct the set $T = \langle S, S, \ldots, S, \{x\}\rangle$, where S is a maximum independent set of $K_{1,j}$ and $x \in V(K_{1,j} - S)$. Thus, $|T| = jk + 1$, T is independent and $\alpha(K_{1,j}\Box K_{1,k}) \geq jk + 1$. □

One of the more interesting uses of stars is to serve as "building blocks" for other graphs, especially caterpillars, as we will see later. With that in mind, we provide the following theorem, which characterizes maximum independent sets of Cartesian products when one factor is a star.

Theorem 3.2 *For any graph G and any star $K_{1,k}$, there exists a maximum independent set of $G\Box K_{1,k}$ of the form $\langle A, A, \ldots, A, B\rangle$ where A and B are independent in G.*

Proof. Let $S = \langle X_1, X_2, X_3, \ldots, X_k, X\rangle$ be a maximum independent set of $G\Box K_{1,k}$. Choose i (for $1 \leq i \leq k$), such that $|X_i| \geq |X_j|$ for all $1 \leq j \leq k$. Then $S' = \langle X_i, X_i, \ldots, X_i, X\rangle$ is independent and $|S'| \geq |S|$. □

In the case where the independence number of a bipartite graph is equal to the size of the larger color class, the results from [4] may be used to obtain an exact value for $\alpha(G\Box K_{1,k})$.

Theorem 3.3 *For any star $K_{1,k}$ and any bipartite graph G, with color classes C_1 and C_2 where $\alpha(G) = |C_1|$,*

$$\alpha(G\Box K_{1,k}) = k|C_1| + |C_2|.$$

Proof. Clearly $\langle C_1, C_1, \ldots, C_1, C_2\rangle$ is an independent set. Thus $\alpha(G\Box K_{1,k}) \geq k|C_1| + |C_2|$. By Theorem 2.2,

$$\begin{aligned}\alpha(G\Box K_{1,k}) &\leq |G| + (k-1)\alpha(G)\\ &= |C_1| + |C_2| + (k-1)|C_1| = k|C_1| + |C_2|.\end{aligned}$$

□

Let S_1 and S_2 be disjoint independent sets such that $|S_1 \cup S_2| = \alpha_2(G)$. From among all such 2-independent sets of G, choose one where $|S_2|$ is as small as possible.

Note that for a bipartite graph G with color classes C_1 and C_2 such that $|C_1| \geq |C_2|$, the unique way to find S_1 and S_2 is $S_1 = C_1$ and $S_2 = C_2$. Theorem 3.3 provides a case in which $\alpha(G\Box H) = k|C_1|+|C_2|$. The following theorem, similar in result and proof to a result in [4], can be seen as a natural generalization of Theorem 3.3 to an arbitrary graph G.

Theorem 3.4 *Let $k \geq 2$. If $|S_2| \leq 2$, then $\alpha(G\Box K_{1,k}) = k|S_1| + |S_2|$. Otherwise, $\alpha(G\Box K_{1,k}) \leq k(|S_1| + |S_2|) - 2k + 1$.*

Proof. Let $s_1 = |S_1|$ and $s_2 = |S_2|$. Clearly, $S = \langle S_1, S_1, \ldots, S_1, S_2\rangle$ is an independent set of $G\Box K_{1,k}$. Suppose $|S| < \alpha(G\Box K_{1,k})$ and let S' be a maximum independent set of $G\Box K_{1,k}$. Hence, $|S'| > |S|$. By Theorem 3.2, we may assume $S' = \langle X_1, X_1, \ldots, X_1, X_2\rangle$, for some pair of disjoint independent sets X_1 and X_2 of G. Now let $x_1 = |X_1|$ and $x_2 = |X_2|$. Clearly, $x_1 \geq x_2$. Note that $x_1 + x_2 \leq s_1 + s_2$ since $S_1 \cup S_2$ is a maximum 2-independent set of G. If $x_1 + x_2 = s_1 + s_2$, then by the choice of s_2, $s_1 \geq x_1$, which implies $|S'| \leq |S|$, a contradiction. Hence,

$$x_1 + x_2 \leq s_1 + s_2 - 1$$
$$x_1 \leq s_1 + s_2 - 2.$$

Now,

$$\begin{aligned} |S'| &= kx_1 + x_2 \\ &= (k-1)x_1 + (x_1 + x_2) \\ &\leq (k-1)s_1 + (k-1)s_2 - 2k + 2 + s_1 + s_2 - 1 \\ &= k(s_1 + s_2) - 2k + 1 \\ &= ks_1 + s_2 + (k-1)s_2 - 2k + 1 \\ &= |S| + (k-1)s_2 - 2k + 1. \end{aligned}$$

If $(k-1)s_2 - 2k + 1 \leq 0$, (i.e., $|S_2| \leq 2$), there is a contradiction and S is a maximum independent set of $G\Box K_{1,k}$. Otherwise, $\alpha(G\Box K_{1,k}) \leq k(|S_1| + |S_2|) - 2k + 1$. □

One way of visualizing this upper bound is to note first that, given a largest independent set S of $G\Box K_{1,k}$ it follows that $|S \cap (G_x \cup G_y)| \leq \alpha_2(G)$ for all $y \in L_x$. If we take this sum over all such leaves y, we obtain $k(|S_1| + |S_2|)$ but $|S_2|$ has been counted $(k-1)$ extra times. Recalling that $|S_2| > \frac{2k-1}{k-1}$ since otherwise, we know the exact value for $G\Box K_{1,k}$, we note that $(k-1)|S_2| > (k-1)(\frac{2k-1}{k-1}) = 2k - 1$. So we remove these extra copies to obtain, $k(|S_1| + |S_2|) - (k-1)|S_2| < k(|S_1| + |S_2|) - 2k + 1$.

4 Products with Caterpillars

One might surmise that it would always be beneficial when constructing a maximum independent set of $G\Box K_{1,k}$ of the form $\langle A, A, \ldots, A, B\rangle$ to let A be an α-set of G so as to maximize the number of elements in the set which is counted the most times. However, this is not always the case, as is demonstrated by caterpillars containing an odd number of consecutive degree two vertices on the spine. For example, in the case of $C(2,1,0,0,0,0,0,1,2)\Box K_{1,2}$, one may obtain an independent set of cardinality 22 using an α-set of C, but may obtain an independent set of

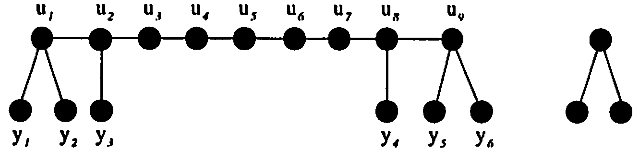

Figure 1: $C(2,1,0,0,0,0,0,1,2)$ and $K_{1,2}$

cardinality 23 using the color classes of C.

The set $I = \{y_1, y_2, y_3, u_3, u_5, u_7, y_4, y_5, y_6\}$ is the unique α-set of the caterpillar C, and $J = \{u_1, u_4, u_6, u_9\}$ is an α-set of $C - I$. These give rise to the independent set $S_1 = \langle I, I, J\rangle$ of $C\Box K_{1,2}$ of cardinality 22. However, if the color classes $C_1 = \{y_1, y_2, u_2, u_4, u_6, u_8, y_5, y_6\}$ and $C_2 = V(C) - C_1$ are used instead, we can produce an independent set $S_2 = \langle C_1, C_1, C_2\rangle$ of cardinality 23.

As a result of this observation, we must restrict the next results to a specific group of caterpillars, those for which $x_i \geq 1$ for all i.

Lemma 4.1 *Let $x_i \geq 1$ for each i, $1 \leq i \leq n$, and let $k \geq 2$. There exists a maximum independent set $S = \langle A, A, \ldots, A, B\rangle$ of $C(x_1, x_2, \ldots, x_n)\Box K_{1,k}$ such that A is an α-set of $C(x_1, x_2, \ldots, x_n)$.*

Proof. Let $S = \langle A, A, \ldots, A, B\rangle$ be a maximum independent set of $C\Box K_{1,k}$. Assume A is not an α-set of C. Suppose first that there does not exist a vertex u_i from the spine of C such that $u_i \in A$ and such that $x_i \geq 2$. Then there exists a vertex u_j on the spine of C such that $x_j = 1$ and such that neither u_j nor the unique leaf $y \in L_{u_j}$ is in A. Now, $u_j \notin B$, for otherwise y could be in A. Thus y must be in B, but this is a contradiction, since putting y in A would create a larger (but still independent) subset of $C\Box K_{1,k}$. Thus, there exists $u_i \in A$ such that $x_i \geq 2$.

Let $A' = (A - u_i) \cup L_{u_i}$ and $B' = \alpha(C - A')$. Then $|A'| = |A| + x_i - 1$ and $|B'| \geq |B| - x_i$. Also, A' and B' are independent and disjoint. It now follows that

$$\begin{aligned}
k|A'| + |B'| &\geq k|A| + k(x_i - 1) + |B| - x_i \\
&\geq k|A| + |B| + 2(x_i - 1) - x_i \\
&= k|A| + |B| + x_i - 2 \\
&\geq k|A| + |B|.
\end{aligned}$$

Thus $S' = \langle A', A', \ldots, A', B'\rangle$ is a maximum independent set of $C\Box K_{1,k}$. If A' is an α-set of C, the conclusion follows. If not, there exists $u_i \in A'$ with

$x_i \geq 2$. We may then repeat the above to yield $S'' = \langle A'', A'', \ldots, A'', B'' \rangle$.

There are a finite number of vertices in the spine of C. Therefore the process must terminate, yielding an α-set $S^{(m)} = \langle A^{(m)}, A^{(m)}, \ldots, A^{(m)}, B^{(m)} \rangle$ of $C \Box K_{1,k}$ such that $A^{(m)}$ is an α-set of C. □

Theorem 4.2 *Let $x_i \geq 1$ for each i, $1 \leq i \leq n$, and let $k \geq 2$. If $\langle A, A, \ldots, A, B \rangle$ is a maximum independent set of $C(x_1, x_2, \ldots, x_n) \Box K_{1,k}$, then $k|A| + |B| = k\alpha(C) + \bar{\alpha}(C)$.*

Proof. By Lemma 4.1, there exists a maximum independent set $S = \langle X_1, X_1, \ldots, X_1, X_2 \rangle$ of $C(x_1, x_2, \ldots, x_n) \Box K_{1,k}$ such that $|S| = k|A| + |B|$ and X_1 is an α-set of C. Thus $|X_2| \leq \bar{\alpha}(C)$. And $k|A| + |B| = |S| = k|X_1| + |X_2| \leq k\alpha(C) + \bar{\alpha}(C)$. □

We now develop a procedure to be used in the next theorem (to calculate certain parts of the lower and upper bounds of caterpillar products).

procedure $\xi(G, H, T, U)$;

(* $G = C(x_1, x_2, \ldots, x_n)$ with leafless strings containing $v_1, v_2, \ldots, v_j$ vertices; H is a bipartite graph with color classes C_1 and C_2 such that $|C_1| \geq |C_2|$; T is a real number; U is either 1 or 0. U will be used to distinguish between the value input to designate the calculation of the lower bound (0) and the value input to designate the calculation of the upper bound (1).
*)

```
begin
    T := 0; t := j;
    If U = 0, U := |C1|, Else U := α(H)
    While t ≠ 0 do begin
        If v_t is even, T := T + ⌊v_t/2⌋|H|
        Else if v_t = 1, T := T + α(H)
        Else if v_t is odd, T := T + ⌊v_t/2⌋|H| + U
        t := t − 1
    end (* while *)
end (* ξ *)
```

For convenience, we will use $\xi(G, H)$ to denote the number T obtained from $\xi(G, H, T, 0)$. We will use $\Xi(G, H)$ to denote the number T obtained from $\xi(G, H, T, 1)$.

One should note that

$$\Xi(G, H) = \sum_{v_i \text{ odd}} \left(\lfloor \frac{v_i}{2} \rfloor |H| + \alpha(H) \right) + \sum_{v_i \text{ even}} \left(\lfloor \frac{v_i}{2} \rfloor |H| \right)$$

$$= \sum_{i=1}^{j} \lfloor \frac{v_i}{2} \rfloor |H| + \sum_{v_i \text{ odd}} \alpha(H)$$

and

$$\xi(G,H) = \sum_{v_i \text{ odd}, v_i \neq 1} \left(\lfloor \frac{v_i}{2} \rfloor |H| + |C_1| \right) + \sum_{v_i \text{ even}} \left(\lfloor \frac{v_i}{2} \rfloor |H| \right) + \sum_{v_i = 1} \alpha(H).$$

To illustrate the procedure, we consider the example $G = C(2,0,1,0,0,2,0,0,0,3)$ and $H = C(2,4)$. Note that $\xi(G,H) = 2|H| + \alpha(H) + |C_1| = 27$ and $\Xi(G,H) = 2|H| + 2\alpha(H) = 28$. The independent sets of $G \square H$ with these cardinalities are $M_1 = (\{u_4,u_7,u_9\} \times C_1) \cup (\{u_5,u_8\} \times C_2) \cup (\{u_2\} \times A)$ and $M_2 = (\{u_4,u_7\} \times C_1) \cup (\{u_5,u_8\} \times C_2) \cup (\{u_2,u_9\} \times A)$, where $A = \{c,d,e,f,g,h\}$ is the unique maximum independent set of H.

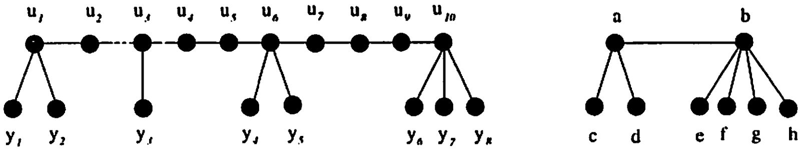

Figure 2: $C(2,0,1,0,0,2,0,0,0,3)$ and $C(2,4)$

Lemma 4.3 *There exists a maximum matching M of $C(x_1,x_2,\ldots,x_n)$ such that M saturates one leaf from each L_{u_i} for which $x_i \geq 1$.*

Proof. Suppose there exists $x_i \geq 1$ and a maximum matching M such that no $y \in L_{u_i}$ is saturated by M. Let $y \in L_{u_i}$. Then u_i is saturated by M. Let u_j be such that $u_iu_j \in M$. Then $M' = (M - \{u_iu_j\}) \cup \{u_iy\}$ is a matching and $|M'| = |M|$. □

We may thus assume that a maximum matching contains a leaf from each L_{u_i} for which $x_i \geq 1$. Using the procedure given above we now find upper and lower bounds for the product of two caterpillars, one arbitrary and one in which all of the spine vertices have at least one leaf.

Theorem 4.4 *Let $u_1,u_2,\ldots,u_n$ denote the n spine vertices of $G = C(x_1,x_2,\ldots,x_n)$. Let $I = \{u_i : x_i \geq 2\}$, $J = \{u_i : x_i = 1\}$, $K = \{u_i : x_i = 0\}$. Then, for $H = C(y_1,y_2,\ldots,y_m)$ with $y_i \geq 1$, for $1 \leq i \leq m$,*

$$\alpha(H) \sum_{i=1}^{n} x_i + \xi(G,H) \leq \alpha(G \square H)$$

and

$$\alpha(G\Box H) \le \alpha(H)\sum_{u_i\in I} x_i + |I|\bar{\alpha}(H) + |J||H| + \Xi(G,H)$$

Proof. To demonstrate the lower bound, let A be an α-set of H and let $X = \{(a,x) : a \in L_{u_i}; 1 \le i \le n; x \in A\}$. Let $V_1, V_2, \ldots, V_j$ be the strings of leafless vertices of G. For each i, $1 \le i \le j$, let X_i be an α-set of $V_i\Box H$. Let $S = X \cup (\cup_{i=1}^{j} X_i)$. Then S is independent and

$$\begin{aligned}
|S| &= \alpha(H)\sum_{i=1}^{n} x_i + \sum_{i=1}^{j}\alpha(V_i\Box H)\\
&= \alpha(H)\sum_{i=1}^{n} x_i + \sum_{v_i \text{ odd}, v_i\ne 1}\alpha(V_i\Box H) + \sum_{v_i \text{ even}}\alpha(V_i\Box H)\\
&\quad + \sum_{v_i=1}\alpha(V_i\Box H)\\
&\ge \alpha(H)\sum_{i=1}^{n} x_i + \sum_{v_i \text{ odd}, v_i\ne 1}\left(\lfloor\tfrac{v_i}{2}\rfloor|H| + |C_1|\right) + \sum_{v_i \text{ even}}\left(\lfloor\tfrac{v_i}{2}\rfloor)|H|\right)\\
&\quad + \sum_{v_i=1}\alpha(H)\\
&= \alpha(H)\sum_{i=1}^{n} x_i + \xi(G,H).
\end{aligned}$$

To prove the upper bound, let S be a largest independent set of $G\Box H$.
For $u_i \in J$ and $y \in L_{u_i}$, $|S \cap (H_{u_i} \cup H_y)| \le \alpha_2(H)$
For $u_i \in I$ and $w_1, w_2, \ldots, w_{x_i} \in L_{u_i}$, Lemma 4.1 implies
$|S \cap (H_{u_i} \cup (\cup_{i=1}^{x_i} H_{w_i})| \le x_i\alpha(H) + \bar{\alpha}(H)$
For a string of v_i leafless vertices $(u_k, u_{k+1}, \ldots, u_{k+v_i-1})$,
 if v_i is even, $|S \cap (\cup_{r=k}^{k+v_i-1} H_{u_r})| \le \lfloor\frac{v_i}{2}\rfloor|H|$, while
 if v_i is odd, $|S \cap (\cup_{r=k}^{k+v_i-1} H_{u_r})| \le \lfloor\frac{v_i}{2}\rfloor|H| + \alpha(H)$
Thus, $\alpha(G\Box H) \le \alpha(H)\sum_{u_i\in I} x_i + |I|\bar{\alpha}(H) + |J||H| + \Xi(G,H)$. □

The inequality in the previous theorem is strict whenever $|H| - \alpha(H) > \bar{\alpha}(H)$ (in other words, if an α-set of H is also a color class of H).

Continuing with the example mentioned previously ($G = C(2,0,1,0,0,2,0,0,0,3)$, $H = C(2,4)$), this theorem implies that $75 \le \alpha(G\Box H) \le 81$. Theorem 2.2 states that $72 \le \alpha(G\Box H) \le 84$ and Theorem 2.3 improves the lower bound to $74 \le \alpha(G\Box H)$.

One important note is that the proof of the lower bound in Theorem 2.2 does not rely on H being a caterpillar. One may conclude in general that, for $G = C(x_1, x_2, \ldots, x_n)$ and H a bipartite graph, $\alpha(H)\sum_{i=1}^{n} x_i + \xi(G,H) \leq \alpha(G\Box H)$ using the same proof as in Theorem 4.4. The strength of the upper bound lies in the realization that, for G and H as in the statement of the theorem, $|S \cap (H_{u_i} \cup (\cup_{i=1}^{x_i} H_{w_i})| \leq x_i\alpha(H) + \bar{\alpha}(H)$. If H bipartite, we would instead have $|S \cap (H_{u_i} \cup (\cup_{i=1}^{x_i} H_{w_i})| \leq |H| + (x_i - 1)\alpha(H)$, which in turn would lead to an upper bound equivalent to the one in Theorem 2.2. In the case of Theorem 4.4, however, the upper bound is no larger than the upper bound in Theorem 2.2, as the next theorem shows.

Theorem 4.5 *Let $y_i \geq 1$ for $1 \leq i \leq m$, $G = C(x_1, x_2, \ldots, x_n)$, and $H = C(y_1, y_2, \ldots, y_m)$, and let I, J, and K be defined as in the statement of Theorem 4.4. Then,*

$$\alpha(H)\sum_{u_i \in I} x_i + |I|\bar{\alpha}(H) + |J||H| + \Xi(G,H) \leq \tau(G)\alpha_2(H) + (|G| - 2\tau(G))\alpha(H).$$

Proof. Let G and H be as in the statement of the theorem and let M be a maximum matching of G. By Lemma 4.3, for each vertex $x \in I$ there exists an edge in M that joins x to one of its leaves. For each vertex $y \in J$ there exists an edge in M that joins y to one of its leaves. Thus

$$|M| = |I| + |J| + \sum_{i=1}^{j} \lfloor \frac{v_i}{2} \rfloor.$$

The unsaturated vertices are (1) the leaves in L_{u_i} for $u_i \in I$ that are not in M ($x_i - 1$ of them) and (2) one vertex for each odd v_i.
Thus the number of unsaturated vertices is

$$\sum_{u_i \in I}(x_i - 1) + \sum_{v_i \text{ odd}} 1.$$

Therefore,

$$\begin{aligned}
&\tau(G)\alpha_2(H) + (|G| - 2\tau(G))\alpha(H) \\
&= \left(|I| + |J| + \sum_{i=1}^{j}\lfloor \frac{v_i}{2} \rfloor\right)\alpha_2(H) + \left(\sum_{u_i \in I}(x_i - 1) + \sum_{v_i \text{ odd}} 1\right)\alpha(H) \\
&= |J||H| + |I||H| + \left(\sum_{u_i \in I} x_i\right)\alpha(H) - |I|\alpha(H) + \left(\sum_{i=1}^{j}\lfloor \frac{v_i}{2} \rfloor\right)|H| \\
&\quad + \left(\sum_{v_i \text{ odd}} 1\right)\alpha(H)
\end{aligned}$$

$$= |J||H| + \Xi(G,H) + |I|(|H| - \alpha(H)) + \alpha(H) \sum_{u_i \in I} x_i$$

$$\geq |J||H| + \Xi(G,H) + |I|\bar{\alpha}(H) + \alpha(H) \sum_{u_i \in I} x_i.$$

□

The bounds in Theorem 4.4 are especially good when the independence number of one of the graphs is much larger than the cardinality of the larger color class of that graph. For instance, $C(5,4,5,5)\Box C(3,0,0,3,2,0,0,1)$, where Theorem 2.2 implies $196 \leq \alpha(G\Box H) \leq 233$ but Theorem 4.4 implies $217 \leq \alpha(G\Box H) \leq 227$.

In fact, the lower bound can be made arbitrarily better than the lower bound in Theorem 2.2. However, the lower bound is not always better, as can be seen in cases where the difference between the size of the largest color class of a graph and the independence number of that graph is small. For example, for $G = C(2,1,0,0,0,0,2,1,2,0,0,0,1,1)$ and $H = C(2,1,1,3,1)$, Theorem 2.2 implies $156 \leq \alpha(G\Box H)$ while Theorem 4.4 implies $126 \leq \alpha(G\Box H)$.

In order to obtain good bounds on the product of two caterpillars regardless of how the size of the largest color class of one graph differs from the independence number of that graph, it would be best to take the maximum of the lower bound in Theorem 2.2 and Theorem 4.4 along with the upper bound in Theorem 4.4.

In conclusion, it is interesting to note yet another counter-intuitive aspect of the Cartesian product. By [4], an α-set of $C(x_1, x_2, \ldots, x_n)\Box P_{2n+1}$ has the form $\langle A, B, \ldots, A, B, A\rangle$. In many examples, an α-set can be obtained with $|A| = \alpha(C)$ or $|A| = |C_1|$ where C_1 is the largest color class of C. However, this is not always the case.

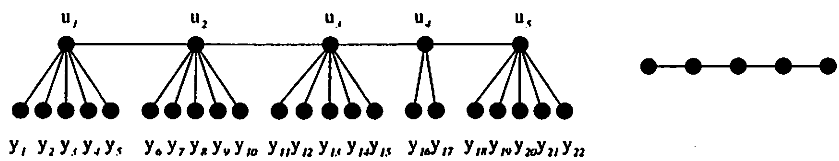

Figure 3: $C(5,5,5,2,5)$ and P_5

For $C(5,5,5,2,5)\Box P_5$, let $M_1 = \{y_i : 1 \leq i \leq 22\}$, $M_2 = \{u_1, u_3, u_5\}$. Then $S_1 = \langle M_1, M_2, M_1, M_2, M_1\rangle$ has cardinality 72. Let C_1 and C_2 be the color classes of C. Then $|C_1| = 17$ and $|C_2| = 10$ and $S_2 = \langle C_1, C_2, C_1, C_2, C_1\rangle$ has cardinality 71. But if we let $M_3 = (M_1 - \{y_{16}, y_{17}\}) \cup \{u_4\}$ and $M_4 = \{u_1, u_3, u_5, y_{16}, y_{17}\}$, we get an independent set $S_3 =$

$\langle M_3, M_4, M_3, M_4, M_3 \rangle$ with cardinality 73. This once again provides evidence that calculating the independence number of a graph is indeed a difficult task.

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