Graceful labelings of cyclic snakes

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Abstract

In this paper we present graceful and nearly graceful labelings of some graphs. In particular we show, graceful labelings of the kC_4-*snake* (for the general case), kC_8 and kC_{12}-*snakes* (for the even case) and also establish some conditions to obtain graceful labelings of kC_{4n}-*snakes* with some related results. Moreover, of the linear kC_6-*snake*, we show a graceful labeling when k is even and a nearly graceful labeling when k is odd. We also do the connection of these labelings with more restrictive variations of graceful ones.

1 Introduction

A *vertex labeling* of a graph G is an assignment f of labels to the vertices of G that induces for each edge xy a label depending on the vertex labels. In particular, a *difference vertex labeling* is a labeling that induce for each edge xy the label $|f(x) - f(y)|$ named the *weight* of the edge xy. If the set of vertex labels are non-negative integers, f is said a *proper labeling.* In this paper we worked with proper difference vertex labelings. Some of the graph theory notation is taken from [10].

Let $G = (V, E)$ be a simple graph with n vertices and m edges, and $f : V \rightarrow \{0, 1, ..., m\}$ a difference vertex labeling such that the set of weights induced by f is $\{1, 2, ..., m\}$. Such labeling was introduced by Rosa [7] in 1967 under the name of β-*valuation* and popularized by Golomb [5] in 1972 as *graceful labeling.* A graph that accept a graceful labeling is said *graceful.* Several surveys have been written, for instance, Gallian [3] has surveyed graph labelings, including over 200 articles related to graceful graphs.

Let G be a graph of order m and size n, and $f : V(G) \rightarrow S$ be a one-to-one function from the vertices of G to a set of numbers S. The *complementary labeling* of f is the function $f' : V(G) \rightarrow S'$ defined as

$f'(v) = k - f(v)$, where k is any integer greater than the maximum of S. Clearly, f' is also a one-to-one function and $S' = \{k - x : x \in S\}$. Note that both labelings induce the same set of weights.

Rosa in 1988 [8] introduced the following definition. Given a graph G of order m and size n, the one-to-one function $f : V(G) \to \{0, 1, ..., n+1\}$ which weights set could be $\{1, 2, ..., n-1, n\}$ or $\{1, 2, ..., n-1, n+1\}$ is a *nearly graceful labeling* (or *$\hat{\rho}$-labeling*). A graph that accept a nearly graceful labeling is said *nearly graceful.*

It is known that the cycle with n edges C_n is graceful if and only if $n \equiv 0$ or $3(\text{mod}4)$. So, in order to have an overview of the "gracefulness" of C_n we represent the following result that give a nearly graceful labeling of C_n when $n \equiv 1$ or $2(\text{mod}4)$.

Theorem 1. *The cycle with n edges C_n has a nearly graceful labeling with weights* $1, 2, ..., n-1, n+1$ *if and only if* $n \equiv 1$ *or* $2(\text{mod}4)$.

Proof. Since the sum of the weights induced for any labeling of C_n is always even, $1 + 2 + ... + (n-1) + (n+1) \equiv 0(\text{mod}2)$ which is impossible if $n \equiv 1$ or $2(\text{mod}4)$.

Let C_n be a cycle with n edges where $n \equiv 1$ or $2(\text{mod}4)$. Let C_n be described by a circuit $v_1, v_2, ..., v_n, v_1$. The valuation of C_n is described by:

If $n \equiv 1(\text{mod}4)$

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1 \\ (i+1)/2, & \text{if } i = 3, 5, ..., (n-3)/2 \\ (i+3)/2, & \text{if } i = (n+1)/2, (n+5)/2, ..., n \\ n + 2 - i/2, & \text{if } i \text{ is even.} \end{cases}$$

If $n \equiv 2(\text{mod}4)$

$$f(v_i) = \begin{cases} 0, & \text{if } i = 1 \\ (i+1)/2, & \text{if } i = 3, 5, ..., (n-4)/2 \text{ and } n \geqslant 10 \\ (i+3)/2, & \text{if } i = n/2, (n+4)/2, ..., n-1, \text{ for every } n \\ n + 2 - i/2, & \text{if } i \text{ is even.} \end{cases}$$

This complete the proof. □

Frucht in 1992 [2] used the kind of nearly graceful labeling, used in the above theorem, to show that the following graphs are nearly graceful: $P_m \cup P_n$, $S_m \cup S_n$, $S_m \cup P_n$, $G \cup K_2$ where G is graceful, and $C_3 \cup K_2 \cup S_m$ where m is even or $m \equiv 3(\text{mod}14)$.

Rosa [8] has defined a *triangular snake* (or Δ-*snake*) as a connected graph in which all blocks are triangles and the block-cutpoint graph is a path, and conjectured that Δ_n-*snake* (a Δ-*snake* with n blocks) is graceful for $n \equiv 0$ or $1 (\text{mod} 4)$ and is nearly graceful otherwise. In 1989 Moulton [6] has proved Rosa's conjecture but using instead of nearly graceful labelings an stronger labeling named *almost graceful.*

2 Main results

Consider the following generalization of triangular snakes. For a kC_n-*snake* we understood a connected graph in which the k blocks are isomorphic to the cycle C_n and the block-cutpoint graph is a path. We are interested in study under what conditions are they graceful. Since this kind of graphs are Eulerian, they satisfy the "parity condition" (Lemma 1 on [7]). So, they are graceful if and only if its size $kn \equiv 0$ or $3(\text{mod}4)$. If $kn \equiv 1$ or $2(\text{mod}4)$, we want to know if they are nearly graceful. We study some particular values of n.

The first case, when $n = 3$ was completely solved by Moulton [6]. The second one, when $n = 4$ was partially solved by Gnanajothi [4] in 1991. Ruiz [9] in 1979 constructed families of graceful graphs, inside one of them, appears some kC_4-*snakes*. However, each of these results, presents only one kind of these cyclic snakes. The size of kC_4-*snake* is always congruent with 0 modulo 4, but in contrast with the case $n = 3$, the way to construct kC_4-*snake* from $(k-1)C_4$-*snake* (for $k \geq 3$) is not unique. In next figure we show the two cases that appears now.

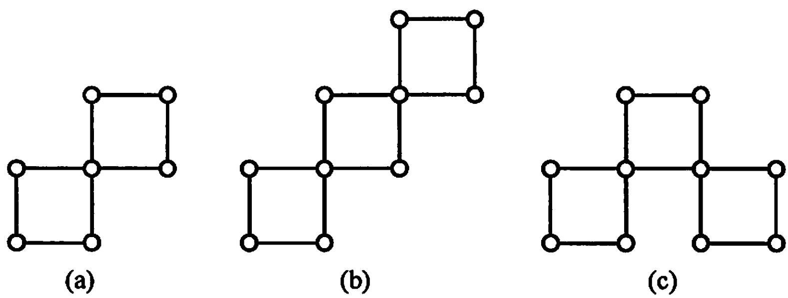


Figure 1

On Figure 1(a) we show the unique $2C_4$-*snake*. Beginning with this graph and using the fact that we only have two vertices where to connect the new copy of C_4, one to distance 1 from the cut vertex and the other at distance 2, we must obtain two non isomorphic $3C_4$-*snakes*, showed in 1(b) and 1(c). As we mentioned before, Gnanajothi partially solved this case, she used the scheme of Figure 1(c). In general, following both schemes we may obtain at most 2^{k-2} different kC_4-*snakes*, some of them are isomorphic.

To construct the kC_4-*snake* we may apply these schemes and since our graph is a bipartite graph (one partite set has black vertices and the other has white vertices) it is possible to embed it, on an square grid as is showed in the next figure.

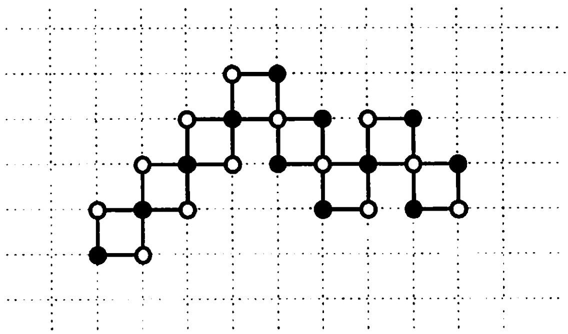


Figure 2

Consider the following numbering of the vertices of C_4 showed in Figure 3. Assuming that x and y are non negative integers and that $x > y$ then, the weights induced are $y-x-1, y-x, y-x+1$, and $y-x+2$.

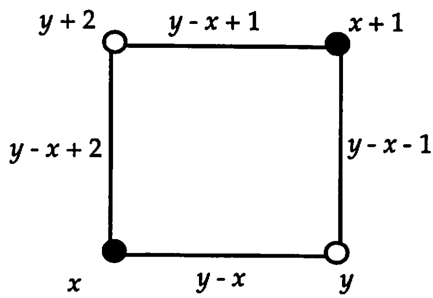


Figure 3

Put black vertices in an string, ordered by diagonals from left to right and inside each diagonal from bottom to top, assign to them consecutive integers from 0 until complete the string. Similarly, put white vertices on an string, ordered for diagonals from right to left and inside each diagonal from bottom to top. Starting with first diagonal assign numbers from an arithmetic progression of difference 2 which first term is one more than the last integer used in the previous assignment, continuing until the last white vertex has been numbered.

The labeling over each copy of C_4 is of the same kind that the used on Figure 3, and is possible to see that the complete labeling is a graceful labeling of kC_4-*snake*. Hence, we have proved the following theorem.

Theorem 2. *The kC_4-snake has a graceful labeling.*

In Figure 4 we show the graceful labeling (obtained from the theorem) of the $8C_4$-*snake* . This result is agree with an unsolved conjecture, posed by Acharya [1] in 1984, about that all connected d-dimensional polyminoes are arbitrarily graceful, for every integer $d \geq 2$.

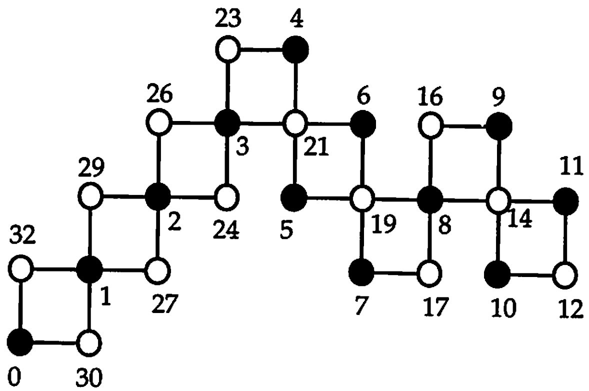


Figure 4

Let $G = 2C_n$-*snake*, G has two blocks and contains only one cut-vertex. Take one of these blocks, there exist $\lceil \frac{n}{2} \rfloor$ different ways to stick a new copy of C_n to G, that depends of the distance that separate the cut-vertex of the block selected and the vertex taken to stick the new block.

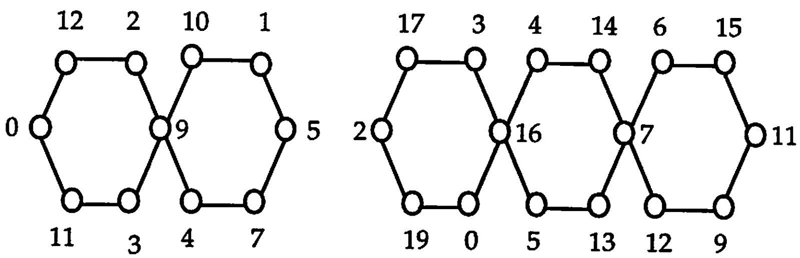
Consider the path P of minimum length that contains all the cut-vertices of G. Beginning in one of its extremes, is possible construct an string of $k-2$ integers such that, any of this integers is the distance between two consecutive cut-vertices of G on the path P. Since P is of minimum length, the integers on the string are taken from $E = \{1, 2, ..., \lceil \frac{n}{2} \rfloor\}$. Hence, any graph $G = kC_n$-*snake*, is represented by an string. Until now, this representation is not unique, because it depends of the extreme of P taken, but considering the strings obtained for both extremes as the same, we avoid the problem.

For instance, the string (from left to right) of the $8C_4$-*snake* on Figure 4 is $2, 2, 1, 2, 1, 1$. In the case of kC_3-*snakes*, the sequence is $1, 1, ..., 1$. The kind of kC_4-*snake* studied by Gnanajothi also have string $1, 1, ..., 1$. And the labelings gaven by Ruiz considered by kC_4-*snake* with string $2, 2, ..., 2$. Theorem 2 has considered all the possible strings on E. If the string of a given kC_n-*snake* is $\lceil \frac{n}{2} \rfloor, ..., \lceil \frac{n}{2} \rfloor$, we say that kC_n-*snake* is *linear.*

In the next theorem we work with linear kC_6-*snakes*. Before exhibit the method to construct the labelings of these cyclic snakes, we show the labelings of the first cases.

Lemma 1. *The* $2C_6$-*snake is graceful and the linear* $3C_6$-*snake is nearly graceful.*

Proof. It is enough to show the corresponding labelings, that we present in the next figure. □



Graceful 2C6-snake

Nearly graceful 3C6-snake

Figure 5

Consider the labelings of C_6 showed in Figure 6, the labeling in (a) is named *G labeling of* C_6 *of value* t, note that the maximum label used is t and that the weights induced are $t, t-1, t-2, t-3, t-4$, and $t-6$. The labeling in (b) is called *NG labeling of* C_6 *of value* t, here we also have maximum label t but the weights induced are $t, t-2, t-3, t-4, t-5$, and $t-6$. We will use these labelings in the proof of the next theorem.

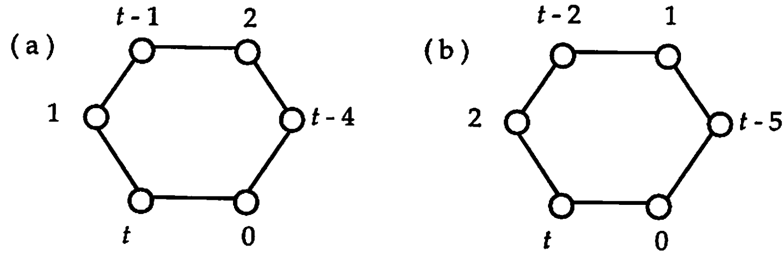


Figure 6

Theorem 3. *The linear kC_6-snake is nearly graceful if k is odd and graceful if k is even.*

Proof. If $k = 1$, C_6-*snake* is C_6 and its nearly graceful labeling was given in Theorem 1. The cases $k = 2$ and $k = 3$ were presented in Lemma 1. Hence, we focus in the cases where $n \geq 4$.

If k is even (odd), consider the nearly graceful (graceful) labeling of $(k-1)C_6$-*snake*. Obtain its complementary labeling by substracting $6k-2$ $(6k-3)$ to each vertex label. Now, the smallest label is 3 and the last three labels are $6k-2$, $6k-4$, and $6k-5$ ($6k-3$, $6k-4$, and $6k-5$). Use the G (NG) labeling of C_6 of value $6k$ $(6k+1)$ that use the labels $6k, 6k-1, 6k-4, 2, 1$, and 0 ($6k+1, 6k-1, 6k-4, 2, 1$, and 0) with induced weights $6k, 6k-1, 6k-2, 6k-3, 6k-4$, and $6k-6$ ($6k+1, 6k-1, 6k-2, 6k-3, 6k-4$, and $6k-5$). Since the vertex with label $6k-4$ on $(k-1)C_6$-*snake* is at distance 3 of the first cut-vertex, we identify the corresponding vertices of C_6 and $(k-1)C_6$-*snake* to obtain a graceful (nearly graceful) labeling of kC_6-*snake*. □

In Figure 7 we present two examples, $4C_6$-*snake* with a graceful labeling and $5C_6$-*snake* with a nearly graceful labeling.

Since the size of kC_{4n}-*snake* is $4kn \equiv 0 (\text{mod} 4)$, we must hope obtain graceful labelings of these graphs. Before continue in that direction, we introduce some notation.

Let G be a graph and let f be any labeling of G, the labeling f^* of G defined for every $x \in V(G)$ by $f^*(x) = f(x) + t$ (t constant), is a *traslation* of f. Clearly, this translation induces the same weights that f.

Let G be a graph and g a labeling of G. Denote the set of weights induced by g on the edges of G by $W_g = \{w_1, w_2, ..., w_m\}$.

If A is a non-empty set of real numbers, then $A + \xi = \{a + \xi : a \in A\}$, for some constant ξ.

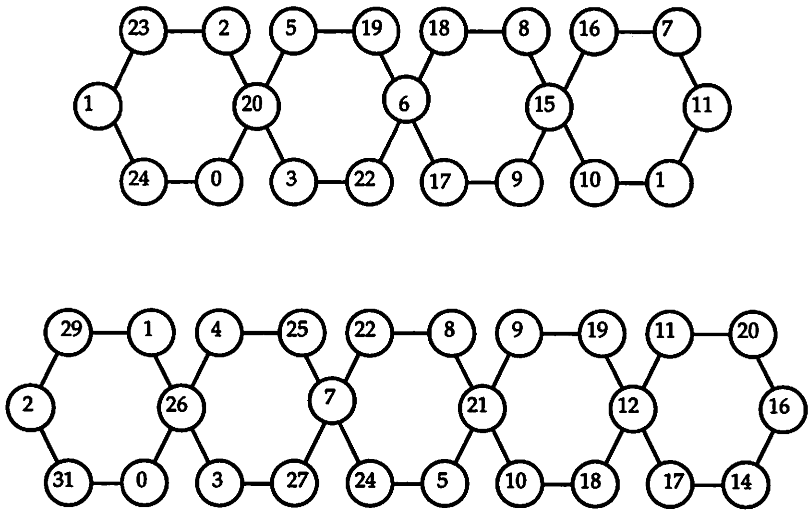


Figure 7

Lemma 2. *Let G be a cycle of size $2n$, with bipartite sets V_1 and V_2. Assume that g is a labeling of G such that $g(V_1) = \{0, 1, ..., n-1\}$ and $g(V_2) = \{x_1, x_2, ..., x_n\}$. Then, there exists a labeling $\overline{g}$ of G such that $W_{\overline{g}} = W_g + \xi$, for any positive constant ξ.*

Proof. We may assume that for any edge $e = uv \in E(G)$, $u \in V_1$ and $v \in V_2$. Using the definition of g we have that $g(u) < g(v)$. Defining $\overline{g}$ as $\overline{g}(V_1) = g(V_1)$ and $\overline{g}(V_2) = g(V_2) + \xi$, we may observe that $\overline{g}$ satisfies $\overline{g}(u) < \overline{g}(v)$. Suppose now, that the weight of e under g is w_i, thus $w_i = g(v) - g(u)$ and its weight under $\overline{g}$ is $\overline{g}(v) - \overline{g}(u) = g(v) + \xi - g(u) = w_i + \xi$. Since this occurs for every edge of G, we may conclude that $W_{\overline{g}} = W_g + \xi$. □

When g is a graceful labeling of a graph G of order p and size q, $W_{\overline{g}} = \{1 + \xi, 2 + \xi, ..., q + \xi\}$.

If G is a cyclic snake whose string contains only even numbers, we say that G is an *even cyclic snake*. In the follow, we only work with even cyclic snakes. The next theorem focus in even kC_8 and kC_{12}-*snakes*. Consider the labelings of C_8 and C_{12} showed in Figure 8.

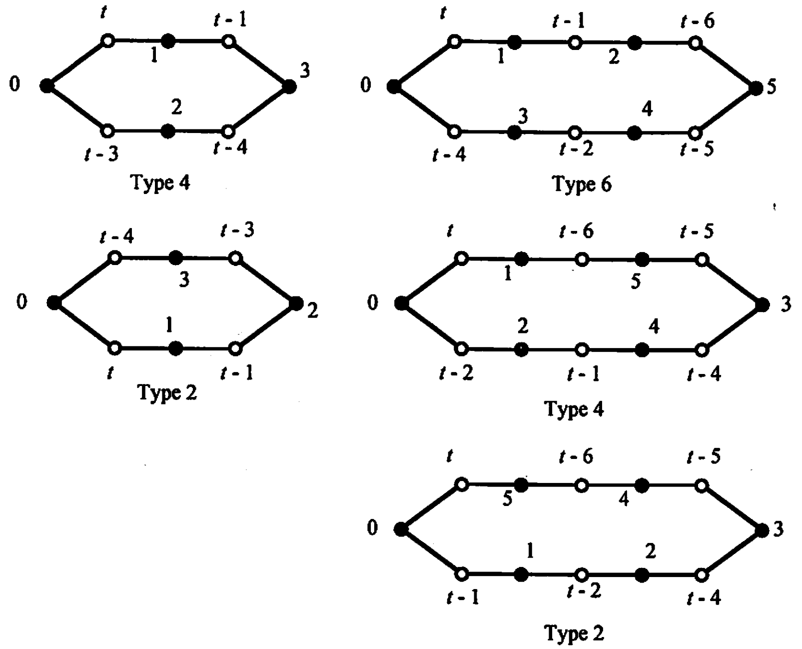


Figure 8

Observe that if $t = 8$ or $t = 12$, the numberings are graceful labelings of C_8 and C_{12}, respectively. In the next theorem we introduce graceful labelings of kC_8 and kC_{12}-*snakes*.

Theorem 4. *The even kC_8 and kC_{12}-snakes are graceful graphs.*

Proof. Let G be any even kC_n-*snake*, where $n = 8, 12$. Then its string is of the form $d_1, d_2, ..., d_{k-2}$, $d_i \in \{2, 4\}$ if $n = 8$ or $d_i \in \{2, 4, 6\}$ if $n = 12$. Denote by $B_{1,n}, B_{2,n}, ..., B_{k,n}$ the consecutive blocks of G. First, we label the blocks of G:

block	labeling used	translation
$B_{1,n}$	Type $\frac{n}{2}$, $t = nk$	
$B_{i+1,n}$	Type d_i, $t = n(k-i)$	$(n-2)i/2$
$B_{k,n}$	Type $\frac{n}{2}$, $t = n$	$(n-2)(k-1)/2$

Second, the weights induced on the block $B_{i,n}$ are $n(k-i)+1, n(k-i)+2, ..., n(k-i)+n$. Since $1 \leq i \leq k$, the weights on G are $1, 2, ..., nk$. Third, the labels used are in the set $\{0, 1, ..., kn\}$. Finally, $B_{i,n}$ and $B_{i+1,n}$ are connected by its vertices of label $(n-2)i/2$, obtaining the graph G with a graceful labeling. □

In Figure 9 we present an example of this labeling for the $4C_{12}$-*snake* with string 4, 2.

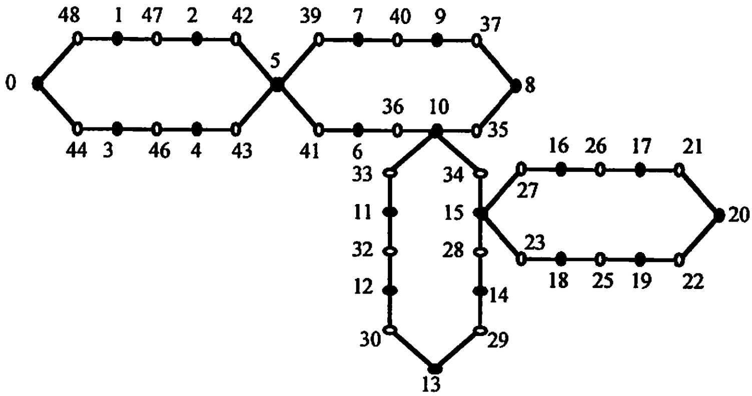


Figure 9

The graceful labeling f of C_{4n} introduced by Rosa in [7] placed the labels 0 and $2n-1$ at distance 2, and the labels $4n$ and $2n+1$ at distance 4. Then, its complementary labeling places the labels 0 and $2n-1$ at distance 4. Hence using these labelings and the method developed in the above theorem, is possible prove the next result.

Theorem 5. *The even kC_{4n}-snake with string $d_1, d_2, ...d_{k-2}$, where $d_i \in \{2, 4\}$, has a graceful labeling.*

The labelings of C_{16} and C_{20} showed in Figure 10 (which are graceful if t=16 and t=20, respectively), located the labels 0 and 7 at distance 8, and the labels 0 and 9 at distance 10, respectively. So, using this and the previous theorem we may state the following.

Theorem 6. *The even kC_{4n}-snake ($n = 4, 5$) with string $d_1, d_2, ...d_{k-2}$, where $d_i \in \{2, 4, 2n\}$ has a graceful labeling.*

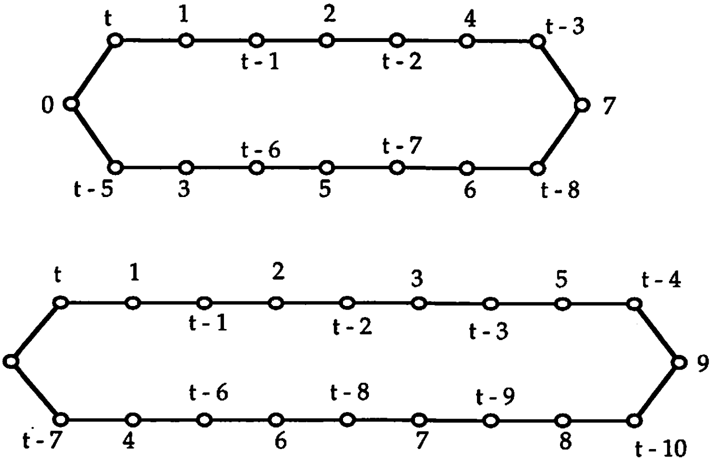


Figure 10

3 Connections with other labelings

In this section, we connect the graceful labelings of kC_{4n}-*snakes*, obtained in the previous section, with some variations of graceful labelings. All the information about this labelings was taken from Gallian's survey [3].

An α-*labeling* of a graph G is a graceful labeling with the additional property that there exists an integer λ so that for each edge xy of G, either $f(x) \leq \lambda < f(y)$ or $f(y) \leq \lambda < f(x)$. This definition was introduced in [7] by Rosa. He observed that if there exists an α-labeling of the graph G, then G is a bipartite graph. It follows from definition, that λ must be the smaller of the two vertex labels that yield the weight 1.

A graph G of order q is *k-graceful* if there is a labeling f from $V(G)$ to $\{0, 1, ..., q + k - 1\}$ such that the weights induced by f are $k, k + 1, ..., q + k - 1$. This labeling was introduced independently by Slater and, Maheo and Thuiller in 1982. Graphs that are k-graceful for all $k \geq 1$ are named *arbitrarily graceful.*

A generalization of these labelings are the *(k, d)-graceful labelings*, where the labels are taken from $\{0, 1, ..., k + (q - 1)d\}$ and the set of weights is $\{k, k + d, ..., k + (q - 1)d\}$. Acharya and Hegde did this generalization in 1990.

The kC_{4n}-*snakes* are bipartite graphs, so we must hope that their graceful labelings, presented here, will also be α-labelings. When $n = 1$ (Theorem 2), λ equals the number of diagonals where the black points lie. The cyclic snakes in theorems 6, 7, and 8 are evens, in these cases $\lambda = k(n-2)/2$. Then, all of them are α-labelings.

Gallian [3] has mentioned that Bu and Zhang, have shown the following results:

If G has an α-labeling, then G is $(k,\ d)$-graceful for all k and d.

Let G be a connected graph. G is a $(kd,\ d)$-graceful graph if and only if G is a k-graceful graph.

Therefore, we may write again these theorems, changing graceful labelings for the more restrictive α-labelings. Moreover, it is possible to find its k-labelings, numbering adequately the bipartite set containing the biggest labels.

Acknowledges

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