(2 mod 4)-cycles

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Abstract

For two integers $k(>0)$ and $s(\geq 0)$, a cycle of length l is called an (s mod k)-cycle if $l \equiv s \bmod k$. In this paper, the following conjecture of Chen, Dean, and Shreve [5] is proved: Every 2-connected graph with at least six vertices and minimum degree at least three contains a (2 mod 4)-cycle.

1 INTRODUCTION We use [2] for our notation and terminology. Our graphs have no loops or multiple edges and $\delta(G)$ denotes the minimum degree of the graph G. For two integers $k(>0)$ and $s(\geq 0)$, a cycle of length l is called an (s mod k)-cycle if $l \equiv s \bmod k$. Suppose $s \geq 0$ and $k > 0$ are integers such that s is even if k is even. Burr and Erdős [7] proposed the problem: Is there a constant $c(s,k)$ depending only on k and s such that every graph G with $\delta(G) \geq c(s,k)$ contains an (s mod k)-cycle? In 1977 Bollobàs [3] provided an affirmative answer to this problem with $c(s,k) = \frac{(k+1)^k+1}{k}$. In 1983 Thomassen [10] proved that $c(s,k) = 4s(k+1)$ and conjectured that every graph G with $\delta(G) \geq k+1$ contains a ($2s$ mod k)-cycle, where s and k are two positive integers; that is, $c(s,k) = k+1$. In the case $k = 3$, it was conjectured in [1] that every graph G with $\delta(G) \geq 3$ contains a (0 mod 3)-cycle. This conjecture was proved by Chen and Saito [6]. Cai and Shreve [4] obtained the result that every graph G with $\delta(G) \geq 4$ contains a (1 mod 3)-cycle. Then, these two results were combined to imply that Thomassen's conjecture is true for $k = 3$. Dean, Kaneko, Ota, and Toft [8] proved that if G is a 2-connected graph with δ(G)$\geq$ 3, then G contains an (s mod 3)-cycle with some exceptions, when $s = 1$ or 2. Dean, Lesniak, and Saito [9] proved that every graph G with $\delta(G) \geq 3$ contains a (0 mod 4)-cycle which affirmed Thomassen's conjecture for $k = 4$. In 1993, Chen, Dean, and Shreve [5] proved that if G is a 2-connected graph with at least six vertices and $\delta(G) \geq 4$, then G contains a (2 mod 4)-cycle and conjectured that the conclusion was also true if $\delta(G) \geq 3$.

In this paper, we first obtain the following theorem.

Theorem 1. Every hamiltonian graph G of order $p(G) \geq 6$ with $\delta(G) \geq 3$ contains a (2 mod 4)-cycle.

Then, we affirm Chen, Dean and Shreve's conjecture.

Theorem 2. Every 2-connected graph G of order $p(G) \geq 6$ and $\delta(G) \geq 3$ contains a (2 mod 4)-cycle.

Note that neither a cycle C_l of length l, where $l \not\equiv 2 \bmod 4$, nor $G = N \cup M$, where $N \cong K_4 \cong M$ and $|V(N) \cap V(M)| = 1$, contain a (2 mod 4)-cycle. Thus, Theorem 2 is best possible.

2 NOTATION AND LEMMAS We denote the lengths of a cycle C and path P by $l(C)$ and $l(P)$ respectively and the subgraph of G induced by $S \subseteq V(G)$ by $\langle S \rangle$. A cycle of length l is called odd (or even) if $l \equiv 1$ (or $l \equiv 0$) mod 2.

Lemma 3. Let $C = v_1 v_2 \cdots v_k v_1$ be a cycle with $k \equiv 0 \bmod 2$ and $v_i v_j, 2 \leq i+1 < j \leq k$, a chord of C with $j - i \equiv 1 \bmod 2$. Then $C + v_i v_j$ contains a (2 mod 4)-cycle.

Proof. By way of contradiction, suppose that $C + v_i v_j$ contains no (2 mod 4)-cycle. Since C and the cycle $v_i v_{i+1} \cdots v_j v_i$ are even cycles, $k \equiv 0$ and $j-i+1 \equiv 0 \bmod 4$. Thus, G contains the cycle $C^* = v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_k v_1$, and $l(C^*) = k-(j-i-1) \equiv -(j-i+1-2) \equiv 2 \bmod 4$, a contradiction. □

Lemma 4. Let $C = v_1 v_2 \cdots v_k v_1$ be a cycle with $k \equiv 1 \bmod 2$ such that $v_t v_h$ and $v_i v_j$, $1 \leq t \leq i < j \leq h \leq k$, are two distinct chords of C with $j - i \equiv h - t \equiv 0 \bmod 2$. Then $C + \{v_i v_j, v_t v_h\}$ contains a (2 mod 4)-cycle.

Proof. Consider the even cycle $v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_k v_1$. Lemma 4 follows by Lemma 3. □

Lemma 5. Let G consist of the cycle $v_1 v_2 \cdots v_t v_1$ and the two chords $v_i v_j$

and $v_k v_f$, where $1 \le i < j \le k < f \le t$. If $1 \equiv t, i \equiv j$, and $k \not\equiv f$ mod 2, then G contains a (2 mod 4)-cycle.

Proof. Consider the even cycle $v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_t v_1$. Lemma 5 follows by Lemma 3. □

Lemma 6. Let G consist of the cycle $v_1 v_2 \cdots v_p v_1$ and the three chords $v_i v_j, v_x v_y$, and $v_k v_f$, where $1 \le i \le x < y \le k < f \le j \le p$. If $p \equiv 1, x \equiv y$, $k \equiv f$ and $i \not\equiv j$ mod 2, then G contains a (2 mod 4)-cycle.

Proof. Suppose, to the contrary, that G contains no (2 mod 4)-cycle. Since the cycle $C_1 = v_1 v_2 \cdots v_x v_y v_{y+1} \cdots v_p v_1$ is even, $l(C_1) = x + (p - y + 1) \equiv 0$ mod 4. Thus, $y - x \equiv p + 1$ mod 4. Similarly, $f - k \equiv p + 1$ mod 4. Now, $j - i \equiv 3$ mod 4. Thus, for the cycle $C_2 = v_i v_{i+1} \cdots v_x v_y v_{y+1} \cdots v_k v_f v_{f+1} \cdots v_j v_i$, $l(C_2) = (j - i + 1) - (y - x - 1) - (f - k - 1) \equiv 3 + 1 - (p + 1 - 1) - (p + 1 - 1) \equiv 2p \equiv 2$ mod 4, a contradiction. This completes the proof of Lemma 6. □

Lemma 7. Let G consist of the three edges $v_1 v_b, v_c v_t$, and $v_i v_j$, and the path $v_1 v_2 \cdots v_t$ with $1 \le c \le i < j \le b \le t$, and $b \equiv c \equiv t \equiv 1$, and $i \equiv j$ mod 2. Then G contains a (2 mod 4)-cycle.

Proof. By way of contradiction, suppose that G contains no (2 mod 4)-cycle. Then, since the two cycles $C_1 = v_1 v_b v_{b+1} \cdots v_t v_c v_{c-1} \cdots v_1$ and $C_2 = v_1 v_b v_{b-1} \cdots v_j v_i v_{i-1} \cdots v_1$ are even, $l(C_1) = c + (t - b + 1) \equiv 0$, and $l(C_2) = b - (j - i - 1) \equiv 0$ mod 4. Therefore, $c + t + (i - j) + 2 \equiv 0$ mod 4. Hence, the cycle $v_c v_{c+1} \cdots v_i v_j v_{j+1} \cdots v_t v_c$ has length $(t - c + 1) - (j - i - 1) \equiv -c + t + (i - j) + 2 \equiv 2c \equiv 2$ mod 4, a contradiction. □

Let G be a graph with $u, v \in V(G)$ and $u \ne v$. If $n \in \{2, 3\}$, and there are n uv-paths of distinct lengths $i_1, \cdots, i_n$ mod 4 in G, (u, v) is said to have the n–property in G, or, simply, (u, v) has $P(n, G)$.

Lemma 8. Let G be consist of two disjoint cycles $C_1 = v_1 v_2 \cdots v_k v_1$ and $C_2 = u_1 u_2 \cdots u_t u_1$ and three pairwise disjoint paths $P_1 = v_1 w_1 w_2 \cdots w_g u_1$, $P_2 = x_1 x_2 \cdots x_f v_i$, and $P_3 = y_1 y_2 \cdots y_h u_j$, where P_1 connects the two cycles. If $i \ne 1$, $j \ne 1$, $k \equiv t \equiv 1$ mod 2, $V(P_2) \cap V(C_1 \cup C_2) = \{v_i\}$, and $V(P_3) \cap V(C_1 \cup C_2) = \{u_j\}$, then (x_1, y_1) has $P(3, G)$.

Proof. Let $G_1 = C_1 \cup P_1 \cup P_2$ and $G_2 = C_2 \cup P_3$. Since C_1 and C_2 are odd cycles, (x_1, u_1) has $P(2, G_1)$. Similarly, (u_1, y_1) has $P(2, G_2)$. Hence, (x_1, y_1) has $P(3, G)$. □

Lemma 9. Let G consist of the cycle $C = v_1 v_2 \cdots v_{w-1} v_w v_{w+1} \cdots v_t v_1$ and the three pairwise disjoint paths $P_1^* = v_1 u_1 u_2 \cdots u_k v_w$, $P_2^* = x x_1 x_2 \cdots x_g v_i$, and $P_3^* = y y_1 y_2 \cdots y_h v_j$, where $1 < i < w < j \leq t$ with $w \equiv 0 \equiv t$ and $k \equiv 1 \bmod 2$. If $V(P_1^*) \cap V(C) = \{v_1, v_w\}$, $V(P_2^*) \cap V(C) = \{v_i\}$ and $V(P_3^*) \cap V(C) = \{v_j\}$, Then (x, y) has $P(3, G)$.

Proof. The following four $v_i v_j$-paths may be found in G.

$P_1 = v_i v_{i+1} \cdots v_w v_{w+1} \cdots v_j$,

$P_2 = v_i v_{i-1} \cdots v_1 u_1 u_2 \cdots u_k v_w v_{w+1} \cdots v_j$,

$P_3 = v_i v_{i-1} \cdots v_1 v_t v_{t-1} \cdots v_j$, and

$P_4 = v_i v_{i+1} \cdots v_w u_k u_{k-1} \cdots u_1 v_1 v_t v_{t-1} \cdots v_j$,

with

$l(P_1) = j - i$,

$l(P_2) = (i-1) + (k+1) + (j-w+1) - 1 = i + j + k - w$,

$l(P_3) = i + (t-j) = i - j + t$, and

$l(P_4) = (w-i) + (k+2) + (t-j) = -(i+j) + w + k + t + 2$.

Therefore,

$l(P_2) - l(P_1) = 2i + k - w \equiv 1 \bmod 2$,

$l(P_3) - l(P_1) = 2(i-j) + t \equiv 0 \bmod 2$,

$l(P_4) - l(P_1) = -2j + w + k + t + 2 \equiv 1 \bmod 2$, and

$l(P_2) - l(P_4) = 2(i+j) - t - 2w - 2 \equiv 2(i-j) + t + 2 \bmod 4$, since $w \equiv t \equiv 0$, and $k \equiv 1 \bmod 2$.

If $2(i-j) + t \equiv 2 \bmod 4$, then $l(P_3) - l(P_1) \equiv 2 \bmod 4$. Hence, it follows using P_1, P_3, and P_4, that (v_i, v_j) has $P(3, G)$.

If $2(i-j) + t \equiv 0 \bmod 4$, then $l(P_2) - l(P_4) \equiv 2 \bmod 4$. Thus, using P_1, P_2, and P_4, we conclude that (v_i, v_j) has $P(3, G)$.

Hence, (x, y) has $P(3, G)$. □

Lemma 10. Let G consist of an edge $v_i v_j$ and the cycle $C = v_1 v_2 \cdots v_t v_1$, where $1 \leq i \leq x < y \leq j \leq t$ with $1 \equiv t$, and $j \equiv i \bmod 2$, and either $i \neq 1$ or $j \neq t$. If G contains no (2 mod 4)-cycle, then (v_x, v_y) has $P(3, G)$.

Proof. Consider the cycle C_1 and three xy-paths P_1, P_2, and P_3, where

$C_1 = v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_t v_1$,

$P_1 = v_x v_{x+1} \cdots v_y$,

$P_2 = v_x v_{x-1} \cdots v_i v_j v_{j-1} \cdots v_y$, and
$P_3 = v_x v_{x-1} \cdots v_1 v_t v_{t-1} \cdots v_y$.
Then,
$l(C_1) = i + (t - j + 1) = i - j + t + 1 \equiv 0 \bmod 4$,
$l(P_1) = y - x$,
$l(P_2) = (x - i) + (j - y) + 1 = (j - i) + (x - y) + 1$, and
$l(P_3) = x + (t - y) = x - y + t$.
Thus,
$l(P_2) - l(P_1) = j - i + 2(x - y) + 1 \equiv t + 2 + 2(x - y) \bmod 4$, and $l(P_3) - l(P_1) = t + 2(x - y)$. Therefore, $l(P_2) - l(P_3) \equiv 2 \bmod 4$. Since $t \equiv 1 \bmod 2$, $l(P_2) - l(P_1) \equiv l(P_3) - l(P_1) \equiv 1 \bmod 2$. Hence, (v_x, v_y) has $P(3, G)$. □

Lemma 11. Let G consist of the four distinct edges $v_1v_t, v_iv_j, v_xu_1, v_yw_1$ and the three pairwise disjoint paths $v_1v_2...v_t$, $u_1u_2 \cdots u_a$, and $w_1w_2 \cdots w_b$, with $1 \leq x \leq i < j \leq y \leq t$ and $t \equiv 1$ and $i \equiv j \bmod 2$. If G contains no (2 mod 4)-cycles, and $x \equiv y \bmod 2$, then (u_a, w_b) has $P(3, G)$.

Proof. Consider the even cycle C and the three v_xv_y-paths P_1, P_2, and P_3, where
$C = v_1v_2...v_iv_jv_{j+1}...v_tv_1$,
$P_1 = v_xv_{x+1}...v_y$,
$P_2 = v_xv_{x+1}...v_iv_jv_{j+1}...v_y$, and
$P_3 = v_xv_{x-1}...v_1v_tv_{t-1}...v_y$.
Then,
$l(C_1) = i + (t - j + 1) = i - j + t + 1 \equiv 0 \bmod 4$,
$l(P_1) = y - x$,
$l(P_2) = (i - x + 1) + (y - j) = (i - j) + (y - x) + 1$, and
$l(P_3) = x + (t - y) = x - y + t$.
Hence, $l(P_2) - l(P_1) = (i-j)+1 \equiv -t$ and $l(P_3) - l(P_1) = 2(x-y)+t \equiv t \bmod 4$. Since $t \equiv 1 \bmod 2$, (v_x, v_y) has $P(3, G)$. Therefore, (u_a, w_b) has $P(3, G)$. □

Lemma 12. Let G consist of the three edges v_1v_b, v_iv_j, and v_1v_a and the path $v_1v_2 \cdots v_b$, where $1 \leq i < j < a < b$ with $i \equiv j$ and $a \not\equiv b \bmod 2$. If $1 \leq x < y \leq a$ with $x \equiv y \bmod 2$ and G contains no (2 mod 4)-cycles, then (v_x, v_y) has $P(3, G)$.

Proof. Since $a \not\equiv b$ mod 2, either $a \equiv 1$, or $b \equiv 1$ mod 2.

If $i \leq x < y \leq j$, then, by Lemma 10, (v_x, v_y) has $P(3, G)$.

If either $1 \leq x \leq y < i$ or $j \leq x < y \leq a$, then, by Lemma 8, (v_x, v_y) has $P(3, G)$.

If $1 \leq x \leq i$ and $j \leq y \leq a$, then, by Lemma 11, (v_x, v_y) has $P(3, G)$.

If either $i < x \leq j < y \leq a$ or $1 \leq x < i \leq y < j$, then, by Lemma 8, (v_x, v_y) has $P(3, G)$.

Hence the lemma follows. □

Lemma 13. Let G consist of the edge $v_t v_w$ and the cycle $v_1 v_2 ... v_k v_1$, where $1 < t < w \leq k$. If G contains an odd cycle, then (v_x, v_y) has $P(2, G)$ for every pair of distinct vertices v_x and v_y in G.

Proof. If $k \equiv 1$ mod 2, then the result is trivial. Suppose that $k \equiv 0$ mod 2. We assume, without loss of generality, that $x = 1$. Note that the cycle $C = v_1 v_2 \cdots v_t v_w v_{w+1} \cdots v_k v_1$ is odd. If $v_y \in V(C)$, then the result is again trivial. If $v_y \notin V(C)$, then the two paths $v_y v_{y+1} \cdots v_w v_{w+1} \cdots v_k v_1$ and $v_y v_{y+1} \cdots v_w v_t v_{t-1} \cdots v_1$ have different lengths mod 2. Hence, (v_x, v_y) has $P(2, G)$. The proof is complete. □

Lemma 14 (Two-body Lemma). Let G consist of two graphs M and N and two disjoint paths, one a uv-path P_1 and the other an xy-path P_2, such that the two paths both connect M and N, where $u, x \in V(N)$, and

(1) $V(M) \cap V(N) = \{u\}$, if $l(P_1) = 0$, and $l(P_2) \neq 0$,

(2) $V(M) \cap V(N) = \{x\}$, if $l(P_1) \neq 0$, and $l(P_1) = 0$,

(3) $V(M) \cap V(N) = \{x, u\}$, if $l(P_1) = 0$, and $l(P_2) = 0$, and

(4) $V(M) \cap V(N) = \emptyset$, if $l(P_1) \neq 0$, and $l(P_2) \neq 0$.

If (u, x) has $P(3, N)$ and (v, y) has $P(2, M)$, then G contains a (2 mod 4)-cycle.

Proof. Immediate. □

3 PROOF OF THEOREM 1 Assume, to the contrary, that there is a hamiltonian graph G with $p(G) \geq 6$, and $\delta(G) \geq 3$, such that G contains no (2 mod 4)-cycle. First, we orient some hamiltonian cycle C of G. Then, for each $u \in V(C)$, label the $V(C)$ in the direction of the orientation as $u = u_1, u_2, \cdots, u_{p-1}, u_p$, where $p = p(G)$. Let $a(u) = \max\{i : 2 \leq i < p$ and $v_i u_1 \in E(G)\}$. Finally, choose $v \in V(G)$ such that $a(v)$ is as large as possible. Let $a(v) = a$, and, then, set $C = v_1 v_2 \cdots v_{a-1} v_a v_{a+1} \cdots v_p v_1$, where $v = v_1$. Since $p(G) \geq 6$ and $\delta(G) \geq 3$, it follows that $a \geq 4$ and there exists some $w \in \{2, 3, \cdots, p-2\}$ such that $v_w v_p \in E(G)$.

Consider the cycle $C_1 = v_p v_w v_{w+1} \cdots v_p$. Now, $l(C_1) = p - w + 1 \not\equiv 2 \mod 4$. Hence,

$$w \not\equiv p - 1 \mod 4. \qquad (1)$$

Consider the cycle $C_2 = v_p v_1 v_2 \cdots v_w v_p$. Now, $l(C_2) = w + 1 \not\equiv 2 \mod 4$. Hence,

$$w \not\equiv 1 \mod 4. \qquad (2)$$

If $1 < w < a$, then, for the cycle $C_3 = v_p v_{p-1} \cdots v_a v_1 v_2 \cdots v_w v_p$, $l(C_3) = w + (p - a + 1) \not\equiv 2 \mod 4$. Hence,

$$w \not\equiv a - p + 1 \mod 4. \qquad (3)$$

Finally, consider the cycle $C_4 = v_p v_1 v_a v_{a-1} \cdots v_w v_p$. Now, $l(C_4) = a - w + 3 \not\equiv 2 \mod 4$. Hence,

$$w \not\equiv a + 1 \mod 4. \qquad (4)$$

If $a \leq w \leq p - 2$, then, for the cycle $C_5 = v_p v_1 v_a v_{a+1} \cdots v_w v_p$, $l(C_5) = w - a + 3 \not\equiv 2 \mod 4$. Hence,

$$w \not\equiv a - 1 \mod 4. \qquad (5)$$

Since G contains no (2 mod 4)-cycle, $a \not\equiv p, a \not\equiv 2$, and $p \not\equiv 2 \mod 4$.

We have three cases to discuss.

Case 1. $a \equiv 1 \equiv p \mod 2$.

Then, $p - a \equiv 2 \mod 4$.

Subcase 1.1. $a \leq w \leq p - 2$.

Then, by (5), $w \not\equiv p + 1 \mod 4$. By (1), $w \not\equiv 0$ and $w \not\equiv 2 \mod 4$. Again, by (2),

$$w \equiv 3 \mod 4. \qquad (6)$$

Consider v_2; there is a v_x, $3 < x \leq p$, such that $v_x v_2 \in E(G)$, since $a \geq 4$ and $\delta(G) \geq 3$.

If $x \not\equiv 2 \mod 2$, then, since $a \equiv 1 \mod 2$, $x \leq a$ by the maximality of a. By Lemma 5, G contains a (2 mod 4)-cycle, a contradiction. Thus, $x \equiv 2 \mod 2$. Hence, $x = a + 1$ by Lemma 4 and the maximality of a.

Consider v_3; there is a v_y, $y = 1$ or $4 < y \leq p$, such that $v_y v_3 \in E(G)$.

If $y \equiv 3 \mod 2$, then, by Lemma 4, $y = a + 2$. Thus, the cycle $v_1 v_2 v_3 v_{a+2} v_{a+1} v_a v_1$ is a (2 mod 4)-cycle, a contradiction. Therefore, $y \not\equiv 3 \mod 2$. Hence, the cycle $C_1 = v_3 v_4 \cdots v_y v_3$ is even and $l(C_1) = y - 2 \equiv 0 \mod 4$. Thus,

$$y \equiv 2 \mod 4. \tag{7}$$

Hence, since $a \equiv 1 \mod 2$, and a is maximal,

$$y \leq \begin{cases} a+1 & \text{if} \quad a \equiv 1 \mod 4, \quad \text{and} \\ a-1 & \text{if} \quad a \equiv 3 \mod 4, \end{cases}$$

and, by (6),

$$w \geq \begin{cases} a+2 & \text{if} \quad a \equiv 1 \mod 4, \quad \text{and} \\ a & \text{if} \quad a \equiv 3 \mod 4. \end{cases}$$

Therefore, $y < w$. By Lemma 5, G contains a (2 mod 4)-cycle, implying a contradiction.

Subcase 1.2. $1 < w < a$.

Then, by (3), $w \not\equiv a - p + 1 \equiv 3 \mod 4$. Again by (1) and (2),

$$w \equiv p + 1 \mod 4. \tag{8}$$

Since $p - a \equiv 0$ and $w \equiv 0 \mod 2$, it follows that $w \geq 4$ by the maximality of a.

Consider v_2; there is a v_x, $3 < x \leq a + 1$, such that $v_x v_2 \in E(G)$.

If $x \equiv 2 \mod 2$, then, by Lemma 4, $x = a + 1$. Hence, for the cycle $C_1 = v_w v_{w+1} \cdots v_a v_1 v_2 v_{a+1} \cdots v_p v_w$, by (8), $l(C_1) = p - w + 3 \equiv 2 \mod 4$, a contradiction. Thus, $x \not\equiv 2 \mod 2$. Since the cycle $C_2 = v_2 v_3 ... v_x v_2$ is even, $l(C_2) = x - 1 \equiv 0 \mod 4$ and, therefore, $x \geq 5$ and

$$x \equiv 1. \tag{9}$$

If $x > w$, then, for the cycle $C_3 = v_2 v_3 ... v_w v_p v_{p-1} ... v_x v_2$, by (8) and (9), $l(C_3) = (w - 2 + 1) + (p - x + 1) \equiv w + p - x \equiv p + 1 + p - 1 \equiv 2p \equiv 2 \mod 4$, a contradiction. Therefore, $w \geq x$.

Consider v_3; there is a v_y, $y = 1$ or $4 < y \leq a+2$, such that $v_y v_3 \in E(G)$.

If $y \equiv 3 \mod 2$, then by Lemma 4, $y = a + 2$. Thus, the length of the cycle $v_1 v_a v_{a+1} v_{a+2} v_3 v_2 v_1$ is 6, a contradiction. Therefore, $y \not\equiv 3 \mod 2$.

Hence,

$$y \equiv 2 \bmod 4, \text{ and } x < y, \tag{10}$$

by Lemma 3, and, therefore, $y \neq a$.

If $y > a$, then for the cycle $C_4 = v_p v_1 v_a v_{a-1} \cdots v_3 v_y v_{y+1} \cdots v_p$, by (10), $l(C_4) = 1 + (a-3+1) + (p-y+1) = a+p-y = (p-a)+2a-y \equiv 2a \equiv 2 \bmod 4$, a contradiction. Thus, $y < a$.

Again, consider v_4; there is a v_f, $1 \leq f < 3$ or $5 < f \leq p$, such that $v_4 v_f \in E(G)$.

If $f \equiv 4 \bmod 2$, then by Lemma 4 and the assumption of no (2 mod 4)-cycles in G, $f = a+3$. Thus, cycle $C_5 = v_3 v_2 v_1 v_a v_{a+1} v_{a+2} v_{a+3} v_4 v_5 \cdots v_y v_3$, by (10), satisfies $l(C_5) = y+4 \equiv 2 \bmod 4$, a contradiction. Therefore, $f \not\equiv 4 \bmod 2$.

If $f > 5$, then

$$f \equiv 3 \bmod 4. \tag{11}$$

By Lemma 3, $f > x$. Thus, for the cycle $C_6 = v_2 v_3 v_4 v_f v_{f-1} \ldots v_x v_2$, by (9) and (11), $l(C_6) = (f-x+1)+3 \equiv (3-1+1)+3 \equiv 2 \bmod 4$, a contradiction.

If $f < 3$, then $f = 1$. Thus, for the cycle $C_7 = v_1 v_2 v_x v_{x+1} \cdots v_y v_3 v_4 v_1$, by (10), $l(C_7) = (y-x+1)+4 \equiv (2-1+1)+4 \equiv 2 \bmod 4$, a contradiction.

Case 2. $p \equiv 0 \bmod 2$.

Then $p \equiv 0 \bmod 4$. By Lemma 3, $a \equiv 1 \bmod 2$. By (1), $w \not\equiv 3 \bmod 4$.

Subcase 2.1. $a \leq w \leq p-2$.

By (2) and (5), $w \equiv a+1 \bmod 4$.

Consider v_2; there is a v_x, $3 < x \leq a+1$, such that $v_2 v_x \in E(G)$. By Lemma 3, $x \equiv 2 \bmod 2$.

If $x \leq a$, then by Lemma 4, G contains a (2 mod 4)-cycle, a contradiction. Therefore, $x = a+1$.

Again, consider v_3; there is a v_y, $y = 1$ or $4 < y \leq a+2$, such that $v_y v_3 \in E(G)$. By Lemma 3, $y \equiv 3 \bmod 2$.

If $y = a+2$, then 6 is the length of the cycle $v_1 v_2 v_3 v_{a+2} v_{a+1} v_a v_1$, a contradiction. Therefore, $1 \leq y \leq a$. Thus, G contains a (2 mod 4)-cycle by Lemma 4, a contradiction.

Subcase 2.2. $1 < w < a$.

By (2) and (5),

$$w \equiv a+3 \bmod 4. \tag{12}$$

Then, $p \geq 8$, since $p \geq 6$ and $p \equiv 0 \bmod 4$.

Subcase 2.2.1. $a+1 < p$.

Since $w \equiv a+3 \bmod 4$, $w \geq 4$ by the maximality of a.

Consider v_3; there is a v_x, $x = 1$ or $4 < x \leq a+2$, such that $v_3v_x \in E(G)$. By Lemma 3, $x \equiv 3 \bmod 2$.

If $x > a$, then $x = a+2$. Thus, the cycle $v_1v_2v_3v_{a+2}v_{a+1}v_av_1$, has length 6, a contradiction.

If $w < x \leq a$, then, since the cycle $C_1 = v_1v_2v_3v_xv_{x+1}\cdots v_av_1$ is even, $l(C_1) = 3 + (a - x + 1) \equiv 0 \bmod 4$. Therefore,

$$x \equiv a \bmod 4. \tag{13}$$

Thus, for the cycle $C_2 = v_3v_4\cdots v_wv_pv_{p-1}\cdots v_xv_3$, using (12) and (13), one obtains $l(C_2) = (w-3+1) + (p-x+1) \equiv w - x - 1 \equiv a + 3 - a - 1 \equiv 2 \bmod 4$, a contradiction. Therefore, $x \leq w$.

Consider v_{p-1}; there a v_y, $1 \leq y < p-2$, such that $v_yv_{p-1} \in E(G)$. By Lemma 3, $y \equiv p-1 \bmod 2$.

If $y < w$, then $y < a$ and the cycle $C_3 = v_yv_{y+1}\cdots v_wv_pv_{p-1}v_y$ is even. Thus, $l(C_3) = w - y + 1 + 2 \equiv 0 \bmod 4$, which implies that

$$y \equiv w + 3 \bmod 4. \tag{14}$$

Hence, for the cycle $C_4 = v_1v_2\cdots v_yv_{p-1}v_{p-2}\cdots v_av_1$, by (12) and (14), $l(C_4) = y + ((p-1) - a + 1) \equiv y - a \equiv w + 3 - a \equiv a + 6 - a \equiv 2 \bmod 4$, a contradiction.

If $y > w$, then, since the cycle $C_5 = v_wv_{w+1}\cdots v_yv_{p-1}v_pv_w$ is even, $l(C_5) = y - w + 3 \equiv 0 \bmod 4$, and, hence, by (12),

$$y - a \equiv 0 \bmod 4. \tag{15}$$

Thus, for the cycle $C_6 = v_1v_2v_3v_xv_{x+1}\cdots v_wv_pv_{p-1}v_yv_{y+1}\cdots v_av_1$ (or for the cycle $v_1v_2v_3v_xv_{x+1}\cdots v_wv_pv_{p-1}v_yv_{y-1}\cdots v_av_1$), by (12), (13), and (15), $l(C_6) = 3 + (w - x + 1) + 2 + |y - a| + 1 \equiv w - x + 3 \equiv a + 3 - a + 3 \equiv 2 \bmod 4$, a contradiction.

Subcase 2.2.2. $a + 1 = p$.

Then, $a = p - 1 \geq 7$. Consider v_3; there is a v_x, $x = 1$ or $4 < x \leq p$, such that $v_xv_3 \in E(G)$. By Lemma 3, $x \equiv 3 \bmod 2$. Therefore, $x \not\equiv p$.

If $x = 1$, then for the cycle $C_7 = v_1 v_{p-1} v_{p-2} \cdots v_3 v_1$, $l(C_7) = p - 2 \equiv 2$ mod 4, a contradiction. Thus, $4 < x \leq a$. Since, for the cycle $C_8 = v_1 v_2 v_3 v_x v_{x-1} \cdots v_a v_1$, $l(C_8) = (a - x + 1) + 3 \equiv 0$ mod 4, it follows that

$$x \equiv a \equiv p - 1 \equiv 3 \bmod 4. \tag{16}$$

Thus, $x \geq 7$.

Consider v_5; there is a v_y, $1 \leq y < 4$ or $6 < y \leq p$, such that $v_y v_5 \in E(G)$. By Lemma 3, $y \equiv 5$ mod 2. Thus, $y \neq p$. By Lemma 4, $y = 1$ or $x < y \leq a$.

If $y = 1$, then, for the cycle $C_9 = v_1 v_5 v_6 \cdots v_x v_3 v_2 v_1$, by (16), $l(C_9) = x - 1 \equiv 3 - 1 \equiv 2$ mod 4, a contradiction.

If $x < y \leq a$, then, since the cycle $C_{10} = v_1 v_2 \cdots v_5 v_y v_{y+1} \cdots v_a v_1$ is even, $l(C_{10}) = a - y + 1 + 5 \equiv 0$ mod 4. Hence,

$$y \equiv a + 2 \equiv 1 \bmod 4. \tag{17}$$

Thus, for the cycle $C_{11} = v_3 v_4 v_5 v_y v_{y-1} \cdots v_x v_3$, by (16) and (17), $l(C_{11}) = 3 + (y - x + 1) \equiv 3 + (1 - 3 + 1) \equiv 2$ mod 4, a contradiction.

Case 3. $a \equiv 0$ mod 2.

Then, $a \equiv 0$ mod 4 and, by Lemma 3, $p \equiv 1$ mod 2.

Subcase 3.1. $a \leq w \leq p - 2$.

By (5), $w \not\equiv 3$ mod 4. Again, by (1) and (2), $w = p + 1$ mod 4.

Consider v_3; there is a v_x, $x = 1$ or $4 < x \leq a + 2$, such that $v_3 v_x \in E(G)$.

If $x \not\equiv 3$ mod 2, then $x > a$ by Lemma 3. Therefore, $x = a + 2$. Thus, the length of the cycle $v_1 v_a v_{a+1} v_{a+2} v_3 v_2 v_1$ is 6, a contradiction. Therefore, $x \equiv 3$ mod 2.

If $x \leq w$, then, by Lemma 5, G contains a (2 mod 4)-cycle, which contradicts the initial assumption.

If $x > w$, then $x = a + 1$ and $w = a$. Hence, 6 is the length of the cycle $v_1 v_2 v_3 v_{a+1} v_a v_p v_1$, a contradiction.

Subcase 3.2. $1 < w < a$.

Then, by (1) and (2), either $w \equiv 3$, or $w \equiv p + 1$ mod 4.

Subcase 3.2.1.

$$w \equiv 3 \bmod 4. \tag{18}$$

Subcase 3.2.1.1. $a + 1 \neq p$.

Then, $p \geq a + 3$, since $p \equiv 1 \bmod 2$.

Consider v_{a+2}; there is a v_x, $3 \leq x < a+1$ or $a+3 < x \leq p$, such that $v_x v_{a+2} \in E(G)$.

If $x \equiv a + 2 \bmod 2$, then, $x < w$ by Lemma 4. Therefore, since the cycle $C_1 = v_1 v_2 \cdots v_x v_{a+2} v_{a+3} \cdots v_p v_1$ is even,

$$l(C_1) = x + (p - (a+2) + 1) \equiv 0 \bmod 4. \tag{19}$$

Thus, for the cycle $C_2 = v_1 v_2 \cdots v_x v_{a+2} v_{a+3} \cdots v_p v_w v_{w+1} \cdots v_a v_1$, by (18) and (19), $l(C_2) = x + (a - w + 1) + (p - (a+2) + 1) \equiv x + (p - (a+2) + 1) + (a - w + 1) \equiv 0 + (0 - 3 + 1) \equiv 2 \bmod 4$, a contradiction.

If $x \not\equiv a + 2 \bmod 2$, then $x > a + 2$, clearly.

Consider v_2; there is a v_y, $3 \leq y \leq a + 2$, such that $v_y v_2 \in E(G)$.

If $y \equiv 2 \bmod 2$, then, by Lemma 5, G contains a (2 mod 4)-cycle, contradicting the initial assumption.

If $y \not\equiv 2 \bmod 2$, then $y = a + 1$ by Lemma 3. Thus, for the cycle $C_3 = v_1 v_2 v_{a+1} v_a \cdots v_w v_p v_1$, by (18), $l(C_3) = 2 + 1 + ((a+1) - w + 1) \equiv 1 - w \equiv 1 - 3 \equiv 2 \bmod 4$, a contradiction.

Subcase 3.2.1.2. $a + 1 = p$.

Subcase 3.2.1.2.1. $w = 3$.

Now, $a \geq 8$. We consider v_4. There is a v_x, $1 \leq x < 3$ or $5 < x \leq p$, such that $v_x v_4 \in E(G)$.

If $x \not\equiv 4 \bmod 2$, then $x = p$ by Lemma 3. Hence, 6 is the length of the cycle $v_1 v_2 v_3 v_4 v_p v_a v_1$, a contradiction. Thus $x \equiv 4 \bmod 2$. By Lemma 4, $x = 2$.

Consider v_6; there is a v_y, $1 \leq y < 5$ or $7 < y \leq p$, such that $v_y v_6 \in E(G)$.

If $y \not\equiv 6 \bmod 2$, then, by Lemma 3, $y = a + 1 = p$. Again, by Lemma 5, G contains a (2 mod 4)-cycle, a contradiction.

If $y \equiv 6 \bmod 2$, then, by Lemma 4, G also contains a (2 mod 4)-cycle, a contradiction.

Subcase 3.2.1.2.2. $w \neq 3$.

Therefore,

$$w \geq 7. \tag{20}$$

Consider v_4; there is a v_x, $1 \leq x < 3$ or $5 < x \leq p$, such that $v_4v_x \in E(G)$.

If $x \not\equiv 4 \bmod 2$, then $x = p$ by Lemma 3. Thus, 6 is the length of the cycle $v_1v_2v_3v_4v_pv_av_1$, a contradiction. Therefore, $x \equiv 4 \bmod 2$.

If $x \geq w$, then, since the cycle $C_1 = v_1v_2v_3v_4v_xv_{x+1}\cdots v_pv_1$ is even, $l(C_1) = (p - x + 1) + 4 \equiv 0 \bmod 4$. Thus,

$$x \equiv p + 1 \equiv 2 \bmod 4, \tag{21}$$

which implies $x < a$. Hence, for the cycle $C_2 = v_1v_2v_3v_4v_xv_{x-1}\cdots v_zv_pv_1$, by (18) and (21), $l(C_2) = 4 + (x - w + 1) + 2 \equiv (2 - 3 + 1) + 2 \equiv 2 \bmod 4$, a contradiction. Therefore, $x < w$.

If $x > 4$, then, for v_2, there is a v_y, $3 < y \leq p$, such that $v_3v_y \in E(G)$.

If $y \not\equiv 2 \bmod 2$, then $y = p$ by Lemma 3. Again, by Lemma 6, G contains a (2 mod 4)-cycle, a contradiction.

If $y \equiv 2 \bmod 2$, then $y = 4$ by Lemma 4. Again, by Lemma 6, G contains a (2 mod 4)-cycle, a contradiction.

If $x < 4$, then $x = 2$.

Consider v_6; by (20) there is a v_z, $1 \leq 3 < 5$ or $7 < z \leq p$, such that $v_zv_6 \in E(G)$.

If $z \not\equiv 6 \bmod 2$, then, by Lemma 3, $z = p$. By Lemma 5, G contains a (2 mod 4)-cycle, a contradiction. Thus $z \equiv 6 \bmod 2$.

If $z > 6$, then $z \leq a$. By Lemma 6, G contains a (2 mod 4)-cycle, a contradiction.

If $z < 6$, then, by Lemma 4, $z = 4$. Again, by Lemma 6, G contains a (2 mod 4)-cycle, a contradiction.

Subcase 3.2.2.

$$w \equiv p + 1 \bmod 4. \tag{22}$$

Subcase 3.2.2.1. $a + 1 < p$.

Then, the maximality of a implies that $w > 2$. Consider v_2; there is a v_x, $3 < x \leq a + 1$, such that $v_2v_x \in E(G)$.

If $x \not\equiv 2 \bmod 2$, then $x = a + 1$ by Lemma 3. Hence, for the cycle $C_1 = v_2v_3\cdots v_wv_pv_{p-1}\cdots v_{a+1}v_2$, by (22), $l(C_1) = w - 1 + (p - (a+1) + 1) =$

$w+p-a-1 \equiv p+1+p-0-1 \equiv 2p \equiv 2 \bmod 4$, a contradiction. Thus, $x \equiv 2 \bmod 2$. Therefore, $x \leq a$. By Lemma 5, $w < x$. Since the cycle $C_2 = v_1v_2v_xv_{x+1}\cdots v_pv_1$ is even, $l(C_2) = 2+(p-x+1) \equiv 0 \bmod 4$. Hence,

$$x \equiv p+3 \bmod 4. \tag{23}$$

Consider v_{p-1}; as in the symmetric case of v_2, there is a $v_y \in V(G)$ such that $v_yv_{p-1} \in V(G)$ and

$$y \equiv p-1 \bmod 2, \text{ and } y \equiv 2 \bmod 4. \tag{24}$$

If $x \leq y$, then, for the cycle $C_3 = v_2v_3\cdots v_wv_pv_{p-1}v_yv_{y-1}\cdots v_xv_2$, by (22), (23), and (24),$l(C_3) = (w-1)+(y-x+1)+2 = w+y-x+2 \equiv p+1+2-(p+3)+2 \equiv 2 \bmod 4$, a contradiction.

If $y < x$, then, for the cycle $C_4 = v_2v_3\cdots v_yv_{p-1}v_{p-2}\cdots v_xv_2$, by (24), (25), and (26), $l(C_4) = (y-2+1)+((p-1)-x+1) = y+p-x-1 \equiv 2+p-(p+3)-1 \equiv 2 \bmod 4$, a contradiction.

Subcase 3.2.2.2. $a+1=p$.

Then $a \geq 8$.

Subcase 3.2.2.2.1. $w \leq p-7$.

Consider v_{p-4}; there is a v_x, $1 \leq x < p-5$ or $p-3 < x \leq p$, such that $v_{p-4}v_x \in E(G)$. By Lemma 3, $x \equiv p-4 \bmod 2$.

If $x > p-4$, then, since the cycle $C_5 = v_1v_2\cdots v_{p-4}v_xv_{x+1}\cdots v_pv_1$ is even, $l(C_5) = (p-4)+(p-x+1) \equiv 0 \bmod 4$. Thus, $x \equiv 2p+1 \equiv 3 \bmod 4$ and, therefore, $x = a-1$.

For v_{p-6}, there is a v_y, $1 \leq y < p-7$ or $p-5 < y \leq p$, such that $v_yv_{p-6} \in E(G)$. By Lemma 3, $y \equiv p-6 \bmod 2$. Again, by Lemma 6, $y > p-6$. Then, since the cycle $C_6 = v_1v_2\cdots v_{p-6}v_yv_{y+1}\cdots v_pv_1$ is even, $l(C_6) = p-6+(p-y+1) \equiv 0 \bmod 4$. Thus, $y \equiv 2p-1 \equiv 1 \bmod 4$. And, by Lemma 4, $y = p-4$. By Lemma 6, G contains a (2 mod 4)-cycle, providing another contradiction.

If $x < p-4$, then, consider v_{p-2}; there is a v_f, $f = p$ or $1 \leq f < p-3$, such that $v_fv_{p-2} \in E(G)$. By Lemma 3, $f \equiv p-2 \bmod 2$. Again, by Lemmas 4 and 6, either $x < f < p-4$ or $f = p$.

If $f = p$, then, by Lemma 6, $x < w$. Since the cycle $C_7 = v_1v_2\cdots v_xv_{p-4}$ $v_{p-3}\cdots v_pv_1$ is even, it follows that $l(C_7) = x + 5 \equiv 0$ mod 4. Therefore,

$$x \equiv 3 \text{ mod } 4. \tag{25}$$

Thus, for the cycle $C_8 = v_1v_2\cdots v_xv_{p-4}v_{p-5}\cdots v_wv_pv_{p-2}v_av_1$, by (22) and (25), $l(C_8) = x + ((p-4) - w + 1) + 3 = x + p - w \equiv 3 + p - (p+1) \equiv 2$ mod 4, a contradiction.

If $x < f < p-4$, then, since the cycles $C_9 = v_1v_2\cdots v_xv_{p-4}v_{p-3}\cdots v_pv_1$ and $C_{10} = v_1v_2\cdots v_fv_{p-2}v_{p-1}v_pv_1$ are even, $l(C_9) = x+5 \equiv 0$ and $l(C_{10}) = f + 3 \equiv 0$ mod 4. Therefore, $x \equiv 3$ and $f \equiv 1$ mod 4. Thus, for the cycle $C_{11} = v_xv_{x+1}\cdots v_fv_{p-2}v_{p-3}v_{p-4}v_x$, $l(C_{11}) = (f - x + 1) + 3 \equiv f - x \equiv 1 - 3 \equiv 2$ mod 4, a contradiction.

Subcase 3.2.2.2.2. $w > p - 7$.

In this case, $w > a + 1 - 7 \geq 8 - 6 = 2$. On the other hand, by (22), $w \equiv p + 1 \equiv a + 1 + 1 \equiv 2$ mod 4. Hence, $w \geq 6$.

Consider v_4; there is a v_x, $1 \leq x < 3$ or $5 < x \leq p$, such that $v_xv_4 \in E(G)$. By Lemma 3, $x \equiv 4$ mod 2. Again, by Lemma 5, $x > w$.

Consider v_2; there is a v_y, $3 < y \leq p$, such that $v_yv_2 \in E(G)$. By Lemma 3, $y \equiv 2$ mod 2. Again, by Lemmas 4 and 6, $4 < y < x$. Since the two cycles $C_{12} = v_1v_2v_yv_{y+1}\cdots v_pv_1$ and $C_{13} = v_1v_2v_3v_4v_xv_{x+1}v_pv_1$ are even, $l(C_{12}) = 2 + (p - y + 1) \equiv 0$ and $l(C_{13}) = 4 + (p - x + 1) \equiv 0$ mod 4. Therefore, $y \equiv p + 3$ and $x \equiv p + 1$ mod 4. Thus, for the cycle $C_{14} = v_2v_3v_4v_xv_{x-1}\cdots v_yv_2$, $l(C_{14}) = 3 + (x - y + 1) \equiv x - y \equiv p + 1 - (p + 3) \equiv 2$ mod 4, a contradiction.

A contradiction is obtained in every case. Therefore, Theorem 1 is true. □

4 PROOF OF THEOREM 2 To the contrary, assume that there is a 2-connected graph G with $p(G) \geq 6$ and $\delta(G) \geq 3$ such that G contains no (2 mod 4)-cycle. Let $\wp$ be the set of longest paths in G. Note, for $P = v_1v_2\cdots v_t \in \wp$, $N(v_1) \subseteq V(P)$. The following definition is crucial to this proof. Define

$$b(P) = \max\,\{i : v_i \in N(v_1)\}$$

Then, choose $P \in \wp$, say $P = v_1v_2\cdots v_t$, such that $b(P)$ is as large as possible. Let $b(P) = b$ and $N = \langle\{v_1, v_2, \cdots, v_b\}\rangle$. Since $d(v_1) \geq 3$, there is an $a \in \{3, 4, \cdots, t\}$ such that $v_1 v_a \in E(G)$. Clearly,

$$a \not\equiv 2,\ b \not\equiv 2 \text{ and } a \not\equiv b \bmod 4, \tag{26}$$

since, if not, $a \equiv b \bmod 4$. Then, the cycle $v_a v_{a+1} \cdots v_b v_1 v_a$ is a (2 mod 4)-cycle in G, contradicting the assumption.

By the maximality of $b(P)$ and the length of P, Claim 1 follows immediately.

Claim 1. If $v \in V(N)$ and N contains a vv_b-hamiltonian path, then $N_G(v) \subseteq V(N)$. □

Claim 2. If $1 < i < a < j \leq b$, then (v_i, v_j) has $P(3, N)$.

Proof. By way of contradiction, assume that (v_i, v_j) does not have $P(3, N)$. The four following paths are distinct $v_i v_j$-paths in N:

$P_1 = v_i v_{i+1} \cdots v_j$,
$P_2 = v_i v_{i-1} \cdots v_1 v_a v_{a+1} \cdots v_j$,
$P_3 = v_i v_{i-1} \cdots v_1 v_b v_{b-1} \cdots v_j$, and
$P_4 = v_i v_{i+1} \cdots v_a v_1 v_b v_{b-1} \cdots v_j$,
with lengths:
$l(P_1) = j - i$,
$l(P_2) = i + (j - a) = i + j - a$,
$l(P_3) = i + (b - j) = i - j + b$, and
$l(P_4) = (a - i + 1) + (b - j + 1) = (a + b) - (i + j) + 2$.
It follows that,

$$l(P_2) - l(P_1) = 2i - a, \tag{27}$$

$$l(P_3) - l(P_1) = 2(i - j) + b, \tag{28}$$

$$l(P_4) - l(P_1) = a + b - 2j + 2, \tag{29}$$

$$l(P_3) - l(P_2) = a + b - 2j, \tag{30}$$

and

$$l(P_4) - l(P_3) = a - 2i + 2. \tag{31}$$

By (26), there are three cases to be discussed.

Case 1. $a \equiv 1 \equiv b \bmod 2$.

Then, by (26), $a + b \equiv 0 \bmod 4$; and by (27), $l(P_2) - l(P_1) = 2i - a \equiv 1 \bmod 2$.

If $j \equiv 1 \bmod 2$, then, by (30), $l(P_3) - l(P_2) = a + b - 2j \equiv 2 \bmod 4$. It follows using P_1, P_2, and P_3 that (v_i, v_j) has $P(3, N)$, a contradiction.

If $j \equiv 0 \bmod 2$, then, by (29), $l(P_4) - l(P_1) = a + b - 2j + 2 \equiv 2 \bmod 4$. Hence (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_4, a contradiction.

Case 2. $a \equiv 0 \bmod 4$.

Then, by (26), $b \equiv 1 \bmod 2$; and by (28), $l(P_3) - l(P_1) = 2(i-j) + b \equiv 1 \bmod 2$.

If $i \equiv 0 \bmod 2$, then, by (31), $l(P_4) - l(P_3) = a - 2i + 2 \equiv 2 \bmod 4$. Hence, (v_i, v_j) has $P(3, N)$ with P_1, P_3, and P_4, a contradiction.

If $i \equiv 1 \bmod 2$, then by (27), $l(P_2) - l(P_1) = 2i - a \equiv 2 \bmod 4$. Hence, (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction.

Case 3. $b \equiv 0 \bmod 4$.

Then, by (26), $a \equiv 1 \bmod 2$.

Subcase 3.1. $i - j \equiv 1 \bmod 2$.

Then, $l(P_2) - l(P_1) = 2i - a \equiv 1 \bmod 2$, and $l(P_3) - l(P_1) = 2(i-j) + b \equiv 2 \bmod 4$. Hence (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction.

Subcase 3.2. $i - j \equiv 0 \bmod 2$.

Consider v_a; since $v_{a-1}v_{a-2} \cdots v_1 v_a v_{a+1} \cdots v_b$ is a $v_{a-1}v_b$-hamiltonian path in N and $d(v_{a-1}) \geq 3$, by Claim 1, there is a $v_x \in V(N)$, $x < a - 2$ or $x > a + 1$, such that $v_{a-1}v_x \in E(N)$. By Lemma 3, $x \equiv a - 1 \bmod 2$.

If $x < a - 2$, then, since the two odd cycles $v_1 v_a v_{a+1} \cdots v_b v_1$ and $v_{a-1} v_x v_{x+1}$ $\cdots v_{a-1}$ are connected by the path $v_1 v_2 \cdots v_x$, by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction. Therefore, $x > a + 1$.

Subcase 3.2.1. $b > a + 1$.

Consider v_{b-1}; by Claim 1, there is a $v_y \in V(N)$, $y > b - 2$, such that $v_{b-1}v_y \in E(N)$. By Lemma 3, $y \equiv b - 1 \equiv 1 \bmod 2$.

If $y \geq a$, then, since the odd cycles $v_1 v_2 \cdots v_a v_1$ and $v_y v_{y+1} \cdots v_{b-1} v_y$ are connected by the path $v_a v_{a+1} \cdots v_y$, by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction. Therefore, $y < a$.

If $i < y$, then the the two odd cycles $v_1 v_2 \cdots v_a v_1$ and $v_y v_{y+1} \cdots v_{b-1} v_y$ have an odd number of common vertices $v_y, v_{y+1}, \cdots, v_a$ since $y \equiv a \equiv 1 \bmod 2$. By Lemma 9, (v_i, v_j) has $P(3, N)$, a contradiction.

If $i > y$, then the two odd cycles $v_1v_2\cdots v_av_1$ and $v_1v_2\cdots v_yv_{b-1}v_bv_1$ have an odd number of common vertices $v_1, v_2, \cdots, v_y$ since $y \equiv 1 \bmod 2$. Thus, by Lemma 9, (v_i, v_j) has $P(3, N)$, a contradiction. Therefore, $i = y$.

Similarly, by Lemma 9, since $v_1v_av_{a+1}\cdots v_bv_1$ and $v_yv_{y+1}\cdots v_{b-1}v_y$ are two odd cycles with an odd number of common vertices, $v_a, v_{a+1}, \cdots, v_{b-1}$, it follows that $j \leq b-1$.

If $x = b$, then, for the even cycle $C_1 = v_bv_{a-1}v_{a-2}\cdots v_yv_{b-1}v_b$, $l(C_1) = 2 + (a-1-y) + 1 \equiv 0 \bmod 4$. Therefore,

$$y \equiv a + 2 \bmod 4. \tag{32}$$

Thus, for the cycle $C_2 = v_1v_2\cdots v_yv_{b-1}v_{b-2}\cdots v_av_1$, by (32), $l(C_2) = y + (b-1) - a + 1 \equiv y - a \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles. Therefore, $x \leq b-1$.

If $x \leq j \leq b-1$, then, since $i \equiv j \bmod 2$ and the two cycles $v_yv_{y+1}\cdots v_{b-1}v_y$ and $v_{a-1}v_a\cdots v_xv_{a-1}$ are odd, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

Let $j < x$. Then, $v_1v_2\cdots v_yv_{b-1}v_bv_1$ and $v_{a-1}v_a\cdots v_xv_{a-1}$ are two odd, cycles, and by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 3.2.2. $b = a + 1$.

Then, $b = j$.

If $b = 4$, then $N \cong K_4$ and $i = 2$. Therefore, (v_i, v_j) has $P(3, N)$, a contradiction.

If $b \neq 4$, then $b \geq 8$.

Consider v_{a-2}; by Claim 1, there is a $v_z \in V(N)$, $z < a-3$ or $z > a-1$, such that $v_{a-2}v_z \in E(N)$. By Lemma 3, $z \equiv a-2 \bmod 2$. Since the two odd cycles $v_zv_{z+1}\cdots v_{a-2}v_z$ (or $v_{a-2}v_{a-1}v_av_{a-2}$) and $v_av_bv_1v_a$ are connected by one of the paths $v_1v_2\cdots v_z$ or $v_{a-2}v_{a-1}v_a$ (or v_a), by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

This contradiction completes the proof of Claim 2. □

Claim 3. If $1 < i < j \leq a$, then (v_i, v_j) has $P(3, N)$.

Proof. By contradiction, assume that (v_i, v_j) does not have $P(3, N)$. The following three paths are distinct v_iv_j-paths in N:

$P_1 = v_iv_{i+1}\cdots v_j$,

$P_2 = v_iv_{i-1}\cdots v_1v_av_{a-1}\cdots v_j$, and

$P_3 = v_i v_{i-1} \cdots v_1 v_b v_{b-1} \cdots v_j$,
with lengths:
$l(P_1) = j - i$,
$l(P_2) = a - j + i$, and
$l(P_3) = b - j + i$.
Thus,

$$l(P_2) - l(P_1) = a + 2(i - j), \tag{33}$$

and

$$l(P_3) - l(P_1) = b + 2(i - j). \tag{34}$$

If $a \equiv 1 \equiv b \bmod 2$, then the two cycles $v_1 v_2 \cdots v_a v_1$ and $v_1 v_2 \cdots v_b v_1$ are odd. Therefore, by Lemma 10, (v_i, v_j) has $P(3, N)$, a contradiction.

By (26), we have the two following cases to discuss.

Case 1. $b \equiv 0 \bmod 4$.

Then, by (26), $a \equiv 1 \bmod 2$.

If $i \not\equiv j \bmod 2$, then, by (33), $l(P_2) - l(P_1) = a + 2(i - j) \equiv 1 \bmod 2$; and by (34), $l(P_3) - l(P_1) = b + 2(i - j) \equiv 2 \bmod 4$. Thus, (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction. Therefore,

$$i \equiv j \bmod 2, \tag{35}$$

and, therefore,

$$l(P_2) - l(P_1) = a + 2(i - j) \equiv 1 \bmod 2. \tag{36}$$

Consider v_{a-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \leq x < a-2$ or $a < x \leq b$, such that $v_{a-1}v_x \in E(N)$. Thus, by Lemma 3,

$$x \equiv a - 1 \equiv 0 \bmod 2. \tag{37}$$

Subcase 1.1. $j \neq a$.

Subcase 1.1.1. $x < a - 2$.

Then, by (35) and (37), the cycle $C_1 = v_1 v_2 \cdots v_x v_j v_a v_1$ is even. Thus,

$$l(C_1) = x + 2 \equiv 0 \bmod 4. \tag{38}$$

Subcase 1.1.1.1. $i < x$.

If $j \neq a-1$, then, by (35), the two odd cycles $v_x v_{x+1} \cdots v_{a-1} v_x$ and $v_1 v_a$
$v_{a+1} \cdots v_b v_1$ are connected by the path $v_{a-1} v_a$. By Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $j = a-1$, then, by (35), $i \equiv 0 \bmod 2$, and, for a $v_i v_j$-path $P_4 = v_i v_{i+1} \cdots v_x v_i$, $l(P_4) = x - i + 1$. Thus, by (38), $l(P_2) - l(P_4) = (i + a - j) - (x - i + 1) = 2i - x + a - 1 - j = 2i - x \equiv -x \equiv 2 \bmod 4$. Hence, by (36), (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_4, a contradiction.

Subcase 1.1.1.2. $i \geq x$.

Then, for the $v_i v_j$-path $P_5 = v_i v_{i-1} \cdots v_x v_{a-1} v_{a-2} \cdots v_j$, $l(P_5) = a - j + x - i$. Threfore, by (38), $l(P_5) - l(P_2) = (a - j + i - x) - (a - j + i) = -x \equiv 2 \bmod 4$. Hence, by (36), (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_5, a contradiction.

Subcase 1.1.2. $x > a$.

By (37), the cycle $C_2 = v_1 v_2 \cdots v_{a-1} v_x v_{x-1} \cdots v_a v_1$ is even; and, therefore,

$$l(C_2) = x \equiv 0 \bmod 4. \tag{39}$$

Subcase 1.1.2.1. $x > a + 1$.

Consider v_{x-1}; by Claim 1, there is a $v_y \in V(N)$, $1 \leq y < x - 2$ or $x < y \leq b$, such that $v_{x-1} v_y \in E(N)$ since the path $v_{x-1} v_{x-2} \cdots v_a v_1 v_2 \cdots v_{a-1}$ $v_x v_{x+1} \cdots v_b$ is a $v_{x-1} v_b$-hamiltonian path in N. Therefore, by Lemma 3, $y \equiv x - 1 \bmod 2$.

If $y < a - 1$, then, by (37), the cycle $C_3 = v_y v_{y+1} \cdots v_{a-1} v_x v_{x-1} v_y$ is even; and, therefore, $l(C_3) = a - y + 2 \equiv 0 \bmod 4$. Thus, for the cycle $C_4 = v_1 v_2 \cdots v_y v_{x-1} v_{x-2} \cdots v_a v_1$, by (39), $l(C_4) = x - 1 - (a - y - 1) = x + y - a \equiv y - a \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

If $a - 1 < y < x$, then, since the cycle $C_5 = v_{a-1} v_a \cdots v_y v_{x-1} v_x v_{a-1}$ is even, $l(C_5) = 2 + (a - y + 1 + 1) = y - a + 4 \equiv y - a \equiv 0 \bmod 4$. Thus, for the two $v_i v_j$-paths $P_6 = v_i v_{i-1} \cdots v_1 v_b v_{b-1} \cdots v_x v_{a-1} v_{a-2} \cdots v_j$ and $P_7 = v_i v_{i-1} \cdots v_1 v_b \cdots v_{x-1} v_y v_{y-1} \cdots v_j$, $l(P_6) = i + b - x + 1 + a - 1 - j = a + b + i - x - j$ and $l(P_7) = i + b - (x - 1) + 1 + y - j = b + y + i - j - x + 2$, respectively. Therefore, $l(P_6) - l(P_7) = a - y - 2 \equiv 2 \bmod 4$. By (36), (v_i, v_j) has $P(3, N)$ with either P_1, P_6, and P_7, or P_2, P_6, and P_7, a contradiction.

If $y > x$, then, for the even cycle $C_6 = v_{a-1}v_a \cdots v_{x-1}v_yv_{y-1} \cdots v_xv_{a-1}$, $l(C_6) = y - (a-1) + 1 = y - a + 2 \equiv 0 \bmod 4$. On the other hand, for the cycle $C_7 = v_1v_av_{a+1} \cdots v_{x-1}v_yv_{y+1} \cdots v_bv_1$, by (39), $l(C_7) = b - ((a-2) + (y - (x-1) - 1)) = -(a + y - x - 2) \equiv 2a \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

Subcase 1.1.2.2. $x = a + 1$.

Consider v_{a-2}; by Claim 1, there is a $v_y \in V(N)$, $1 \leq y < a - 3$ or $a \leq y \leq b$, such that $v_{a-2}v_y \in E(N)$. Then, by Lemma 3, $y \equiv a - 2 \bmod 2$.

Subcase 1.1.2.2.1. $y \geq a$.

If $i < a - 2$, then $v_{a-2}v_{a-1} \cdots v_yv_{a-2}$ and $v_1v_av_{a+1} \cdots v_bv_1$ are two odd cycles with an odd number of common vertices, $v_a, v_{a+1}, \cdots v_y$. Therefore, by Lemma 9, (v_i, v_j) has $P(3, N)$, a contradiction.

If $i \geq a - 2$, then $i = a - 2$ and $j = a - 1$. Thus, the two odd cycles $v_{a-1}v_av_{a+1}v_{a-1}$ and $v_1v_2 \cdots v_{a-2}v_yv_{y+1} \cdots v_bv_1$ are connected by the path $v_xv_{x+1} \cdots v_y$. By Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 1.1.2.2.2. $y < a - 3$.

If $1 < i < j \leq y$, then the two odd cycles $C_8 = v_yv_{y+1} \cdots v_{a-2}v_y$ and $C_9 = v_{a-1}v_av_{a+1}v_{a-1}$ are connected by the path $v_{a-2}v_{a-1}$. By Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $y \leq i < j \leq a - 2$, then, since $a \equiv 1$ and $y \equiv a - 2 \bmod 2$, by Lemma 10, (v_i, v_j) has $P(3, N)$, a contradiction.

If $1 < i < y$ and $y < j < a - 2$, then a contradiction follows by Lemma 8 as in the case of $1 < i < j \leq y$.

If $1 < i < y$ and $a - 2 \leq j \leq a - 1$, then, since $a \equiv 1$, $y \equiv a - 2$, and $i \equiv j \bmod 2$, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

If $y \leq i \leq a - 2$ and $j = a - 1$, then $i \neq y$ since $y \equiv a - 2$ and $i \equiv j \equiv a - 1 \bmod 2$. Thus, a contradiction is also obtained as in the case when $a - 2 \leq i < j \leq a - 1$.

Subcase 1.2. $j = a$.

Then, $i \equiv j \equiv 1 \bmod 2$.

If $x < i$, then the two odd cycles $v_1v_jv_{j+1} \cdots v_bv_1$ and $v_xv_{x+1} \cdots v_{a-1}v_x$ are connected by the path $v_1v_2 \cdots v_x$. By Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $i \leq x < j$, then, since $j \equiv 1$, $x \equiv a - 1$, and $i \equiv j$ mod 2, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

If $j < x$, then, for the even cycle $C_{11} = v_1 v_j v_{a-1} v_x v_{x+1} \cdots v_b v_1$, $l(C_{11}) = b - x + 4 \equiv -x \equiv 0$ mod 4. Now, for the $v_i v_j$-path $P_{10} = v_j v_{a-1} v_x v_{x+1} \cdots v_b v_1 v_2$
$\cdots v_i$, $l(P_{10}) = i + b - x + 2$. Thus, $l(P_{10}) - l(P_2) = (i + b - x + 2) - i = b - x + 2 \equiv 2$ mod 4. Hence, by (36), (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_{10}, a contradiction.

Subcase 2. $a \equiv 0$ mod 4.

Then, by (26), $b \equiv 1$ mod 2.

If $i \not\equiv j$ mod 2, then by (33), $l(P_2) - l(P_1) = a + 2(i - j) \equiv 2$ mod 4; and by (34), $l(P_3) - l(P_1) = b + 2(i - j) \equiv 1$ mod 2. Thus, (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction. Hence,

$$i \equiv j \text{ mod } 2 \tag{40}$$

and, therefore,

$$l(P_3) - l(P_1) = b + 2(i - j) \equiv 1 \text{ mod } 2. \tag{41}$$

Consider v_{a-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \leq x < a-2$ or $a < x \leq b$, such that $v_{a-1} v_x \in E(N)$.

Subcase 2.1. $x < a$.

Then, by Lemma 3, $x \equiv a - 1 \equiv 0$ mod 2. Thus, by (40) and Lemma 12, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 2.2. $x > a$.

If $x \not\equiv a - 1$ mod 2, then the cycle $C_{12} = v_{a-1} v_a \cdots v_x v_{a-1}$ is even. Hence, $l(C_{12}) = x - a + 2 \equiv x + 2$ mod 4. Thus, for the cycle $C_{13} = v_1 v_2 \cdots v_{a-1} v_x v_{x-1} \cdots v_a v_1$, $l(C_{13}) = x \equiv 2$ mod 4, which contradicts the hypothesis that G contains no (2 mod 4)-cycles. Therefore,

$$x \equiv a - 1 \equiv 1 \text{ mod } 2. \tag{42}$$

Subcase 2.2.1. $x \neq a + 1$.

Consider v_{x-1}; by Claim 1 there is a $v_y \in V(N)$, $1 \leq y < x - 2$ or $x < y \leq b$, such that $v_{x-1} v_y \in E(N)$.

Subcase 2.2.1.1. $y > x$.

Then, $y \equiv x - 1 \bmod 2$ by Lemma 3 since the cycle $v_1 v_a v_{a+1} \cdots v_b v_1$ is odd. Thus, $C_{14} = v_{a-1} v_x v_{x+1} \cdots v_y v_{x-1} v_{x-2} \cdots v_{a-1}$ is an even. cycle. Therefore, $l(C_{14}) = y - a + 2 \equiv y + 2 \bmod 4$. Hence, for the cycle $C_{15} = v_1 v_2 \cdots v_{a-1} v_x v_{x+1} \cdots v_y v_{x-1} v_{x-2} \cdots v_a v_1$, $l(C_{15}) = y \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

Subcase 2.2.1.2. $y < x - 2$.

Then, since G contains no (2 mod 4)-cycles, $y < a$ by Lemma 4 .

If $y \equiv x - 1 \bmod 2$, then the cycle $C_{16} = v_y v_{y+1} \cdots v_{a-1} v_x v_{x-1} v_y$ is even. Hence, $l(C_{16}) = a - y + 2 \equiv 2 - y \bmod 4$. Thus, for the cycle $C_{17} = v_1 v_2 \cdots v_y v_{x-1} v_x v_{a-1} v_a v_1$, $l(C_{17}) = x + 4 \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

If $y \not\equiv x - 1 \bmod 2$, then, clearly, $x - 1 - y + 1 = x - y \equiv 0 \bmod 4$. Thus, for the cycle $C_{18} = v_1 v_2 \cdots v_y v_{x-1} v_{x-2} \cdots v_a v_1$, by (42), $l(C_{18}) = y + x - a \equiv y + x \equiv 2x \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

Subcase 2.2.2. $x = a + 1$.

Consider v_{a-2}; by Claim 1 there is a $v_y \in V(N)$, $1 \leq y < a - 3$ or $a - 1 < y \leq b$, such that $v_{a-2} v_y \in E(N)$.

Subcase 2.2.2.1. $y > a - 1$.

Subcase 2.2.2.1.1. $y \equiv a - 2 \bmod 2$.

Then, $y = a$ by Lemma 4.

If $j < a$, then the two odd cycles $v_a v_{a-1} v_{a-2} v_a$ and $v_1 v_a v_{a+1} \cdots v_b v_1$ are connected by the path v_a. By Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $j = a$ and $i \geq a - 1$, then, since $a - 2 \equiv a$ and $b \equiv 1 \bmod 2$, by Lemma 10, (v_i, v_j) has $P(3, N)$, a contradiction.

If $j = a$ and $i < a - 1$, then, since $a - 2 \equiv a$ and $b \equiv 1 \bmod 2$, by (40) and Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 2.2.2.1.2. $y \not\equiv a - 2 \bmod 2$.

Then, clearly, $y - a + 3 \equiv 0 \bmod 4$. Therefore,

$$y \equiv 1. \tag{43}$$

If $j < a$, then, by (40), $i < a-2$. Now, by (42) and (43), the two odd cycles $v_{a-2}v_yv_{y-1}\cdots v_xv_{a-1}v_{a-2}$ and $v_1v_av_{a+1}\cdots v_bv_1$ have an odd number of common vertices $v_x, v_{x+1}, \cdots, v_y$. Thus, by Lemma 9, (v_i, v_j) has $P(3, N)$, a contradiction.

If $j = a$, then, by (40), $i \leq a-2$, and $C_{19} = v_{a-1}v_{a+1}v_{a+2}\cdots v_bv_1v_2 \cdots v_{a-1}$ is an even cycle for which $l(C_{19}) = b-1 \equiv 0 \bmod 4$. For the v_iv_j-path $P_{11} = v_jv_{a-1}v_{a+1}v_{a+2}\cdots v_bv_1v_2\cdots v_i$, $l(P_{11}) = b-(a-1-i)$. Now, by (40), $i \equiv j \equiv a \equiv 0 \bmod 2$; and, therefore, $2i \equiv 0 \bmod 4$. Thus, $l(P_{11}) - l(P_1) = (b-a+i+1)-(a-i) = b-2a+2i+1 \equiv 2 \bmod 4$. Hence, by (41), (v_i, v_j) has $P(3, N)$ with P_1, P_3, and P_{11}, a contradiction.

Subcase 2.2.2.2. $y < a-3$.

Then, by Lemma 3, $y \equiv a-2$.

If $i < y$ and $j < a-2$, then the two odd cycles $C_{20} = v_yv_{y+1}\cdots v_{a-2}v_y$, and $C_{21} = v_1v_av_{a+1}\cdots v_bv_1$ are connected by the path $v_{a-2}v_{a-1}v_a$. By Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $i < y$ and $a-2 \leq y \leq a$, then, since $b \equiv 1$, $y \equiv a-2$, and $i \equiv j \bmod 2$, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

If $y \leq i < j \leq a-2$, then, since $b \equiv 1$, and $y \equiv a-2 \bmod 2$, by Lemma 10, (v_i, v_j) has $P(3, N)$, a contradiction.

If $y < i \leq a-2$ and $a-1 \leq j \leq a$, then, C_{20} and C_{21} are connected by the path $v_1v_2\cdots v_y$. Thus, by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $i = y$ and $a-1 \leq j \leq a$, then, since $b \equiv 1$, $y \equiv a-2$, and $i \equiv j \bmod 2$, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

Claim 3 follows. □

Claim 4. If $a \leq i < j \leq b$, then (v_i, v_j) has $P(3, N)$.

Proof. By way of contradiction, assume that (v_i, v_j) does not have $P(3, N)$. The following three paths are distinct v_iv_j-paths in N:

$P_1 = v_iv_{i+1}\cdots v_j$,

$P_2 = v_iv_{i-1}\cdots v_av_1v_bv_{b-1}\cdots v_j$, and

$P_3 = v_iv_{i-1}\cdots v_1v_bv_{b-1}\cdots v_j$,

with lengths:

$l(P_1) = j-i$,

$l(P_2) = b-a-(j-i)+2$, and

$l(P_3) = b - (j - i)$.

Then,

$$l(P_2) - l(P_1) = b - a + 2(i - j) + 2, \tag{44}$$

$$l(P_3) - l(P_1) = b + 2(i - j), \tag{45}$$

and

$$l(P_3) - l(P_2) = a - 2. \tag{46}$$

If $a \equiv 0 \bmod 4$, then by (26), $b \equiv 1 \bmod 2$. Thus, by (45), $l(P_3) - l(P_1) = b + 2(i - j) \equiv 1 \bmod 2$; and by (46), $l(P_3) - l(P_2) = a - 2 \equiv 2 \bmod 4$. Thus, (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction.

It suffices by (26) to discuss the two following cases.

Case 1. $b \equiv 0 \bmod 4$.

Then, by (26), $a \equiv 1 \bmod 2$, and by (44),

$$l(P_2) - l(P_1) \equiv 1 \bmod 2. \tag{47}$$

If $i \not\equiv j \bmod 2$, then by (45), $l(P_3) - l(P_1) \equiv 2 \bmod 4$. Thus, (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction. Therefore, $i \equiv j \bmod 2$.

Consider v_{b-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \leq x < b - 2$, such that $v_{b-1}v_x \in E(N)$. Then, by Lemma 3, $x \equiv 1 \equiv 0 \bmod 2$.

Subcase 1.1. $j \neq b$.

Subcase 1.1.1. $x \geq a$.

Note, $i \equiv j$ and $x \equiv 1 \bmod 2$ and the cycle $v_1 v_a v_{a+1} \cdots v_b v_1$ is odd.

If $a \leq i < j < b$, then, by Lemma 10, (v_i, v_j) has $P(3, N)$, a contradiction.

If $a \leq i < x$, and $j < b - 1$, then the two odd cycles $v_1 v_2 \cdots v_a v_1$ and $v_x v_{x+1} \cdots v_{b-1} v_x$ are connected by the path $v_1 v_b v_{b-1}$. Therefore, by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

If $a \leq i < x$ and $j = b - 1$, then, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 1.1.2. $x < a$.

Now, the cycle $C_1 = v_1 v_a v_{a+1} \cdots v_{b-1} v_x v_{x-1} \cdots v_1$ is even since $a \equiv 1$ and $x \equiv b - 1 \bmod 2$. Therefore, $l(C_1) = b - 1 - (a - x - 1) = b + x - a \equiv x - a \equiv 0 \bmod 4$. Thus, for the $v_i v_j$-path $P_4 = v_i v_{i-1} \cdots v_x v_{b-1} v_{b-2} \cdots v_j$, $l(P_4) = b - x - j + i$. Hence, $l(P_4) - l(P_2) = a - x - 2 \equiv 2 \bmod 4$, where

$P_2 = v_i v_{i-1} \cdots v_a v_1 v_b v_{b-1} \cdots v_j$. Therefore, by (47), (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_4, a contradiction.

Subcase 1.2. $j = b$.

Subcase 1.2.1. $x \geq a$.

If $a \leq i \leq x$, then, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

If $x < i < b$, then the two odd cycles $v_1 v_2 \cdots v_a v_1$ and $v_x v_{x+1} \cdots v_{b-1} v_x$ are connected by the path $v_a v_{a+1} \cdots v_x$. Thus, by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 1.2.1. $x < a$.

By the argument of Subcase 1.1.2., $x - a \equiv 0 \bmod 4$. Now, for the $v_i v_j$-path $P_5 = v_i v_{i-1} \cdots v_a v_1 v_2 \cdots v_x v_{b-1} v_j$, $l(P_5) = x + 2 + i - a$. Hence, $l(P_5) - l(P_1) = (x+2+i-a) - (b-i) = x - a + 2i + b + 2 \equiv 2 \bmod 4$. Therefore, by (47), (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_5, a contradiction.

Case 2. $a \equiv 1 \equiv b \bmod 2$.

Then, by (26), $a - b \equiv 2 \bmod 4$, and by (45),

$$l(P_3) - l(P_1) = b + 2(i - j) \equiv 1 \bmod 2. \tag{48}$$

If $i \not\equiv j \bmod 2$, then, by (44), $l(P_2) - l(P_1) = b - a + 2(i - j) + 2 \equiv 2 \bmod 2$. Thus, by (48), (v_i, v_j) has $P(3, N)$ with P_1, P_2, and P_3, a contradiction. Therefore, $i \equiv j \bmod 2$.

Consider v_{b-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \leq x < b - 2$, such that $v_{b-1} v_x \in E(N)$.

Subcase 2.1. $j \not\equiv b - 1 \bmod 2$.

Then, clearly, $b - 1 - x + 1 = b - x \equiv 0 \bmod 4$; and by Lemma 5, $x < a$. Thus, for the cycle $C_2 = v_1 v_a v_{a+1} \cdots v_x v_{b-1} v_b v_1$, $l(C_2) = 4 + a - x = 4 + (a - b) + (b - x) \equiv 2 \bmod 4$ since $a - b \equiv 0 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

Subcase 2.2. $x \equiv b - 1 \bmod 2$.

Subcase 2.2. $j \neq b$.

If $i \geq x$, then $x \leq i < j \leq b - 1$; and, therefore, by Lemma 10, (v_i, v_j) has $P(3, N)$, a contradiction.

If $i < x$ and $j \neq b - 1$, then the two odd cycles $v_1 v_2 \cdots v_a v_1$ and $v_x v_{x+1} \cdots v_{b-1} v_x$ are connected by the path $v_{b-1} v_b v_1$. Thus, by Lemma 8,

(v_i, v_j) has $P(3, N)$, a contradiction.

If $i < x$ and $j = b - 1$, then, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 2.2.2. $j = b$.

Subcase 2.2.2.1. $x \geq a$.

If $a \leq i \leq x$, then, by Lemma 11, (v_i, v_j) has $P(3, N)$, a contradiction.

If $x < i$, then the two odd cycles $v_1 v_2 \cdots v_a v_1$ and $v_x v_{x+1} \cdots v_{b-1} v_x$ are connected by the path $v_a v_{a+1} \cdots v_x$. Therefore, by Lemma 8, (v_i, v_j) has $P(3, N)$, a contradiction.

Subcase 2.2.2.2. $x < a$.

Then the cycle $C_3 = v_1 v_2 \cdots v_x v_{b-1} v_b v_1$ is even, and, hence, $l(C_3) = x+2 \equiv 0 \bmod 4$. Thus, for the $v_i v_j$-path $P_5 = v_i v_{i-1} \cdots v_a v_1 v_2 \cdots v_x v_{b-1} v_j$, $l(P_5) = x + 2 + i - a$. Thus, $l(P_5) - l(P_1) = (x + 2 + i - a) - (j - i) = 2(i - j) + b - a + x + 2 \equiv 2 \bmod 4$. Hence, by (48), (v_i, v_j) has $P(3, N)$ with P_1, P_3, and P_5, a contradiction.

This contradiction completes the proof of Claim 4. □

Combining Claims 2, 3, and 4, the following claim is established.

Claim 5. If $1 < i < j \leq b$, then (v_i, v_j) has $P(3, N)$. □

Continuation of the main proof of Theorem 2. Consider v_t, the end vertex with greatest index of the longest path P; if $v_t = v_b$, then $V(G) = V(P)$ and, therefore, G is hamiltonian. Thus, by Theorem 1, G contains a (2 mod 4)-cycle, contradicting the initial assumption. Hence, $t > b$.

By the maximality of P and $\delta(G) \geq 3$, there are z and $w \in \{1, 2, \cdots, t-2\}$ with $z < w$ such that $v_z v_t$, $v_w v_t \in E(G)$. Thus, $z > 1$.

If $z \geq b$, then let $M = \langle \{v_z, v_{z+1}, \cdots, v_t\} \rangle$. By Lemma 3, either $z \equiv t$, or $w \equiv t \bmod 2$; and, therefore, M contains an odd cycle. Since G is 2-connected, there are two disjoint paths, say a uv-path and an xy-path, which connect M and N where u, $x \in V(N)$ and v, $y \in V(M)$. By Claim 5, (u, x) has $P(3, N)$ and, by Lemma 13, (v, y) has $P(2, M)$. By the Two-body Lemma, G again contains a (2 mod 4)-cycle, contradicting the beginning assumption.

If $z < b$ and $w \geq b$, then set $M = \langle\{v_w, v_{w+1}, \cdots, v_t\}\rangle$. Now, the cycle $v_w v_{w+1} \cdots v_t v_w$ is not a (2 mod 4)-cycle. Hence, (v_w, v_t) has $P(2, M)$. By Claim 5, (b, z) has $P(3, N)$. Thus, by the Two-body Lemma, G contains a (2 mod 4)-cycle, leading to a contradiction.

Finally, we discuss the last case in which $1 < z < w < b$.

We assume that:

$$v_\alpha v_t \in E(G) \text{ with } 1 < \alpha < b \text{ and } \alpha \equiv t - 1 \bmod 4, \tag{49}$$

$$v_\beta v_t \in E(G) \text{ with } 1 < \beta < b \text{ and } \beta \equiv b - t + 1 \bmod 4, \tag{50}$$

$$v_\gamma v_t \in E(G) \text{ with } 1 < \gamma < a \text{ and } \gamma \equiv a - t + 1 \bmod 4, \tag{51}$$

$$v_\eta v_t \in E(G) \text{ with } 1 < \eta \leq a \text{ and } \eta \equiv a - b + t + 1 \bmod 4, \tag{52}$$

and

$$v_\theta v_t \in E(G) \text{ with } a \leq \theta < b \text{ and } \theta \equiv a + b - t - 1 \bmod 4. \tag{53}$$

Then,

$$\beta - \alpha \equiv b + 2(1 - t) \bmod 4, \tag{54}$$

$$\gamma - \alpha \equiv a + 2(1 - t) \bmod 4, \tag{55}$$

$$\eta - \alpha \equiv a - b + 2 \bmod 4, \tag{56}$$

and

$$\theta - \alpha \equiv a + b + 2(1 - t) + 2 \bmod 4. \tag{57}$$

Five cycles are now given together with their lengths:

$l(C_\alpha) = t - \alpha + 1 \equiv 2 \bmod 4$, where $C_\alpha = v_\alpha v_{\alpha+1} \cdots v_t v_\alpha$,

$l(C_\beta) = \beta + t - b + 1 \equiv 2 \bmod 4$, where $C_\beta = v_\beta v_{\beta-1} \cdots v_1 v_b v_{b+1} \cdots v_t v_\beta$,

$l(C_\gamma) = \gamma + t - a + 1 \equiv 2 \bmod 4$, where $C_\alpha = v_\gamma v_{\gamma-1} \cdots v_1 v_a v_{a+1} \cdots v_t v_\gamma$,

$l(C_\eta) = a - \eta + 2 + t - b + 1 \equiv 2 \bmod 4$, where $C_\eta = v_\eta v_{\eta+1} \cdots v_a v_1 v_b v_{b+1} \cdots v_t v_\eta$,

and

$l(C_\theta) = (t - b + 1) + (\theta - a) + 2 \equiv 2 \bmod 4$, where $C_\theta = v_\theta v_{\theta-1} \cdots v_a v_1 v_b v_{b+1} \cdots v_t v_\theta$.

Since G contains no (2 mod 4)-cycles, the following claim will now be established.

Claim 6. Both $z \not\equiv f$ and $w \not\equiv f$ mod 4 for any $f \in \{\alpha, \beta, \gamma, \eta, \theta\}$.

Case 1. $a \equiv 0 \bmod 4$.

Then, by (26), $b \equiv 1 \bmod 2$.

Subcase 1.1. $t \equiv 0 \bmod 2$.

Then, by (54) and (56), $\beta - \eta = (\beta - \alpha) - (\eta - \alpha) \equiv 2b + 2(1-t) - a - 2 \equiv 2 \bmod 4$. On the other hand, by (54), $\beta - \alpha \equiv b + 2(1-t) \equiv b + 2 \bmod 4 \equiv 1 \bmod 2$. By (55), $\gamma - \alpha \equiv a + 2(1-t) \equiv 2 \bmod 4$. Thus, by Claim 6, $a \leq z < w < b$.

Now, by (54) and (57), $\beta - \theta = (\beta - \alpha) - (\theta - \alpha) \equiv b + 2(1-t) - (a + b + 2(1-t) + 2) \equiv 2 \bmod 4$. On the other hand, by (54), $\beta - \alpha \equiv 1 \bmod 2$. Therefore, by (49), $z \equiv w \equiv \alpha + 2 \equiv t + 1 \bmod 4$. Thus, the cycle $v_z v_{z+1} \cdots v_w v_t v_z$ is a (2 mod 4)-cycle, giving a contradiction.

Subcase 1.2. $t \equiv 1 \bmod 2$.

Then, by (54), $\beta - \alpha \equiv b + 2(1-t) \equiv b \bmod 4$. Hence, by Claim 6, (49), and (50), either $z \equiv b - t + 3$ and $w \equiv t + 1$ or $w \equiv b - t + 3$ and $z \equiv t + 1 \bmod 4$.

We assume, without loss of generality, that $z \equiv b - t + 3 \bmod 4$. By (54) and (57), $\beta - \theta = (\beta - \alpha) - (\theta - \alpha) \equiv a + 2 \equiv 2 \bmod 4$. Hence, if $a \leq z < b$, then $z \not\equiv \theta \equiv b - t + 3 \bmod 4$. Therefore, $z < a$. By the maximality of b, $z < a - 1$.

Consider v_{a-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \leq x < a - 2$ or $a < x \leq b$, such that $v_{a-1} v_x \in E(N)$.

If $x \equiv a - 1 \bmod 2$, then, since $b \equiv t \equiv z \equiv 1 \bmod 2$, by Lemma 7, $x < z$; and for the even cycle $C_1 = v_1 v_2 \cdots v_x v_{a-1} v_a \cdots v_b v_1$, $l(C_1) = x + (b - (a-1) + 1) \equiv x + b + 2 \equiv 0 \bmod 4$. Hence, for the cycle $C_2 = v_x v_{x+1} \cdots v_z v_t v_{t-1} \cdots v_{a-1} v_x$, $l(C_2) = (t - (a-1) + 1) + (z - x + 1) \equiv t + 2 + (b - t + 3) + (b + 2) + 1 \equiv 2b \equiv 2 \bmod 4$, which contradicts the assumption that G contains no (2 mod 4)-cycles.

If $x \not\equiv a - 1 \bmod 2$, then, by Lemma 3, $x > a$ and $x - (a-1) + 1 \equiv x + 2 \equiv 0 \bmod 4$. Hence, for the cycle $C_3 = v_1 v_2 \cdots v_{a-1} v_x v_{x-1} \cdots v_a v_1$, $l(C_3) = x \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles.

Case 2. $a \equiv 1 \equiv b \bmod 2$.

Then, by (26), $a - b \equiv 2 \bmod 4$.

Subcase 2.1. $t \equiv 0 \bmod 2$.

Then, by (54), $\beta - \alpha \equiv b + 2(1-t) \equiv b + 2 \bmod 4 \equiv 1 \bmod 2$. Therefore, by (49) and (50), either $z \equiv t+1$ and $w \equiv b-t+3$ or $w \equiv t+1$ and $z \equiv b-t+3 \bmod 4$.

If $w \equiv t+1$ and $z \equiv b-t+3 \bmod 4$, then, for the cycle $C_4 = v_1v_2\cdots v_zv_tv_w$
$v_{w+1}\cdots v_bv_1$, $l(C_4) = z+b-w+2 \equiv (b-t+3)+b-(t+1)+2 \equiv 2b \equiv 2 \bmod 4$, which contradicts the hypothesis that G contains no (2 mod 4)-cycles. Therefore,

$$z \equiv t+1 \text{ and } w \equiv b-t+3 \bmod 4. \tag{58}$$

By the maximality of b, $w < b-1$.

Consider v_{b-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \le x < b-2$, such that $v_{b-1}v_x \in E(N)$.

If $x \not\equiv b-1 \bmod 2$, then $(b-1)-x+1 = b-x \equiv 0 \bmod 4$; and by Lemma 3, $x < z$. Hence, for the cycle $C_5 = v_xv_{x+1}\cdots v_zv_tv_{t-1}\cdots v_{b-1}v_x$, by (58), $l(C_5) = (z-x+1)+(t-(b-1)+1) = (t+1)-x+t-b+3 \equiv 2t-x-b \equiv -2b \equiv 2 \bmod 4$, a contradiction. Therefore, $x \equiv b-1$.

Now, by (54) and (55), $\beta - \gamma \equiv (\beta-\alpha)-(\gamma-\alpha) \equiv b-a \equiv 2 \bmod 4$. On the other hand, by (54), $\beta - \alpha \equiv 1 \bmod 2$. Hence, by (49), $w \equiv t+1$ if $1 < w < a$. Thus, $w \ge a$. Since the cycle $C_6 = v_1v_2\cdots v_xv_{b-1}v_bv_1$ is even,

$$l(C_6) = x+2 \equiv 0 \bmod 4. \tag{59}$$

If $x \ge w$, then the cycle $C_7 = v_1v_av_{a-1}\cdots v_zv_tv_wv_{w+1}\cdots v_xv_{b-1}v_bv_1$ (or $v_1v_av_{a+1}\cdots v_zv_tv_wv_{w+1}\cdots v_xv_{b-1}v_bv_1$) is even, by (58) and (59), $l(C_7) = (x-w+1)+4+(a-z+1) = x+a-w-z+6 \equiv 2+a-(b-t+3)-(t+1)+2 \equiv a-b \equiv 2$ (or $l(C_7) = (x-w+1)+4+(z-a+1) = x+z-w-a+6 \equiv 2+(t+1)-(b-t+3)-a+2 \equiv 2t+(b-a)-2b-2 \equiv 2b \equiv 2 \bmod 4$), a contradiction. Therefore, $x < w$.

If $x \le z$, then, for the cycle $C_8 = v_xv_{x+1}\cdots v_zv_tv_wv_{w+1}\cdots v_{b-1}v_x$, by (58), $l(C_8) = (z-x+1)+((b-1)-w+1)+1 = z+b-x-w+2 \equiv (t+1)+b-2-(b-t+3)+2 \equiv 2 \bmod 4$, a contradiction. Therefore, $z < x$.

Since $w \equiv b-t+3 \equiv 0 \bmod 2$, by (59), $w > x+1$. Consider v_{x+1}; by Claim 1, there is a $v_y \in V(N)$, $1 \le y < x$ or $x+2 < y \le b$, such that $v_{x+1}v_y \in E(N)$.

If $y \not\equiv x-1 \bmod 2$, then, since $z \not\equiv t \bmod 2$, by Lemma 3, $y < z$ and $(x+1)-y+1 = x-y+2 \equiv 0 \bmod 4$. Hence, for the cycle $C_9 =$

$v_1v_2\cdots v_yv_{x+1}v_{x+2}\cdots v_wv_tv_{t-1}\cdots v_bv_1$, $l(C_9)=y+(w-(x+1)+1)+(t-b+1)=y+w+t-x-b+1\equiv(x-2)+(b-t+3)+t-x-b+1\equiv 2$ mod 4, a contradiction. Therefore, $y\equiv x-1$ mod 2.

If $y>x+2$, then, by Lemma 4, $y=b$; and, hence, for the cycle $C_{10}=v_{x+1}v_bv_{b+1}\cdots v_tv_wv_{w-1}\cdots v_{x+1}$, by (58) and (59), $l(C_{10})=(w-(x+1)+1)+(t-b+1)=w-x+t-b+1\equiv(b-t+3)-2+t-b+1\equiv 2$ mod 4, a contradiction. Therefore, $y<x$. Since the cycle $C_{11}=v_1v_2\cdots v_yv_{x+1}v_{x+2}\cdots v_bv_1$ is even, $l(C_{11})=y+(b-(x+1)+1)+1=y+b-x\equiv 2$ mod 4, a contradiction.

If $y\geq a$, then, $C_{12}=v_1v_av_{a+1}\cdots v_yv_{x+1}v_{x+2}\cdots v_wv_tv_{t-1}\cdots v_bv_1$, a cycle, satisfies by (58), $l(C_{12})=1+(y-a+1)+(w-(x+1)+1)+(t-b+1)=y+w+t-a-x-b+3\equiv(x-b)+(b-t+3)+t-a-x-b+3\equiv -2b+(b-a)+2\equiv 2$ mod 4, a contradiction. Therefore, $y<a$. By Lemma 4, $a<x+1$. Thus, for the cycle $C_{13}=v_1v_2\cdots v_yv_{x+1}v_x\cdots v_av_1$, by (59), $l(C_{13})=y+(x+1-a+1)\equiv(x-b)+x-a+2\equiv 2x-2b+(b-a)+2\equiv 2$ mod 4, a contradiction.

Subcase 2.2. $t\equiv 1$ mod 2.

Then by (54), $\beta-\alpha\equiv b+2(1-t)\equiv b\equiv 1$ mod 2; and by (57), $\theta-\alpha\equiv a+b+2(1-t)+2\equiv 2$ mod 4. Hence, if either $a\leq z<b$ or $a\leq w<b$, then, by (50), either $z\equiv\beta+2\equiv b-t+3$ or $w\equiv b-t+3$ mod 4. On the other hand, by (54) and (55), $\beta-\gamma\equiv(\beta-\alpha)-(\gamma-\alpha)\equiv b-a\equiv 2$ mod 4. Thus, if either $1<z<a$ or $1<w<a$, then, by (49), either $z\equiv\alpha+2\equiv t+1$ or $w\equiv t+1$ mod 4. Hence, $z\equiv t+1$ and $w\equiv b-t+3$ mod 4. For the cycle $C_{14}=v_1v_2\cdots v_zv_tv_wv_{w+1}\cdots v_bv_1$, $l(C_{14})=z+(b-w+1)+1=t+1+b-(b-t+3)+2\equiv 2t\equiv 2$ mod 4, a contradiction.

Case 3. $b\equiv 0$ mod 4.

Then, by (26), $a\equiv 1$ mod 2.

Subcase 3.1. $t\equiv 0$ mod 2.

Then, by (54), $\beta-\alpha\equiv b+2(1-t)\equiv 2$ mod 4; and by (57), $\theta-\alpha\equiv a+b+2(1-t)+2\equiv a$ mod $4\equiv 1$ mod 2. Hence, if either $a\leq z<b$ or $a\leq w<b$, then, by (53), either $z\equiv\theta+2\equiv a-t+1$ or $w\equiv a-t+1$ mod 4. On the other hand, by (55), $\gamma-\alpha\equiv a+2(1-t)\equiv a+2$ mod $4\equiv 1$ mod 2. Therefore, if either $1<z<a$ or $1<w<a$, then, by (51), either

$z \equiv \gamma + 2 \equiv a - t + 3$ or $w \equiv a - t + 3$ mod 4. Hence, $z \equiv a - t + 3$ mod 4 with $1 < z < a - 1$ and $w \equiv a - t + 1$ mod 4 with $a \leq w \leq b$.

Consider v_{a-1}; by Claim 1, there is a $v_x \in V(N)$, $1 \leq x < a - 2$ or $a < x \leq b$, such that $v_{a-1}v_x \in E(N)$. By Lemma 3, $x \equiv a - 1$ mod 2.

If $x > a$, then the cycle $C_{15} = v_1v_2 \cdots v_{a-1}v_xv_{x-1} \cdots v_av_1$ is even. Therefore, $l(C_{15}) = x \equiv 0$ mod 4. Thus, $C_{16} = v_zv_{z+1} \cdots v_av_xv_{x+1} \cdots v_tv_z$, a cycle, satisfies $l(C_{16}) = (t - z + 1) - (x - (a - 1) - 1) = t + a - z - x + 1 \equiv t + a - (a - t + 3) + 1 \equiv 2t - 2 \equiv 2$ mod 4, a contradiction.

If $x < a$, then the cycle $C_{17} = v_1v_2 \cdots v_xv_{a-1}v_av_1$ is even. Hence, $l(C_{17}) = x + 2 \equiv 0$ mod 4. Thus, $C_{18} = v_1v_2 \cdots v_xv_{a-1}v_a \cdots v_wv_tv_{t-1} \cdots v_bv_1$, a cycle, satisfies $l(C_{18}) = (t - b + 1) + x + (w - (a - 1) + 1) = t + x + w - b - a + 3 \equiv t + 2 + (a - t + 1) - a + 3 \equiv 2$ mod 4, a contradiction.

Subcase 3.2. $t \equiv 1$ mod 2.

Then, by (57), $\theta - \alpha \equiv a + b + 2(1 - t) + 2 \equiv a + 2$ mod 4 $\equiv 1$ mod 2. Hence, if either $a \leq z < b$ or $a \leq w < b$, then, by (49) and (53), either $z \equiv t + 1$ or $a - t + 1$ mod 4, or $w \equiv a - t + 1$ or $t + 1$ mod 4. On the other hand, by (55), $\gamma - \alpha \equiv a + 2(1 - t) \equiv a \equiv 1$ mod 2; and, by (55) and (56), $\eta - \gamma \equiv (\eta - \alpha) - (\gamma - \alpha) \equiv 2$ mod 4. Thus, if either $1 < z < a$ or $1 < w < a$, then, by (49), either $z \equiv \alpha + 2 \equiv t + 1$ or $w \equiv t + 1$ mod 4. Hence, the two following subcases are left to discuss.

Subcase 3.2.1. $w \equiv t + 1$ and $z \equiv a - t + 1$ mod 4 with $a \leq z < w \leq b - 1$.

Then, for the cycle $C_{19} = v_1v_av_{a+1} \cdots v_zv_tv_wv_{w+1} \cdots v_bv_1$, $l(C_{19}) = (z - a + 1) + (b - w + 1) + 2 = z - a - w \equiv (a - t + 1) - a - t - 1 \equiv -2t \equiv 2$ mod 4, a contradiction.

Subcase 3.2.2. $z \equiv t + 1$ and $w \equiv a - t + 1$ mod 4 with $a \leq w \leq b - 1$.

If $z \leq a$, then, for the cycle $C_{20} = v_1v_av_{a-1} \cdots v_zv_tv_wv_{w+1} \cdots v_bv_1$, $l(C_{20}) = (a - z + 1) + (b - w + 1) + 2 = a - z - w \equiv a - (t + 1) - (a - t + 1) \equiv 2$ mod 4, a contradiction. Therefore, $z > a$.

Consider v_{b-1}; by Claim 1 there is a $v_x \in V(N)$, $1 \leq x < b - 2$, such that $v_{b-1}v_x \in E(N)$. By Lemma 3, $x \equiv b - 1$ mod 2.

If $w \leq x$, then the cycle $C_{21} = v_1v_av_{a+1} \cdots v_xv_{b-1}v_bv_1$ is even; and, hence, $l(C_{21}) = x - a + 4 \equiv x - a \equiv 0$ mod 4. Thus, for the cycle $C_{22} = v_1v_2 \cdots v_zv_tv_wv_{w+1} \cdots v_xv_{b-1}v_bv_1$, $l(C_{22}) = z + (x - w + 1) + 3 = z + x - w \equiv$

$(t+1)+a-(a-t+1) \equiv 2t \equiv 2 \bmod 4$, a contradiction. Therefore, $x < w$. Thus, the cycle $C_{23} = v_1 v_a v_{a-1} \cdots v_x v_{b-1} v_b v_1$ (or $v_1 v_a v_{a+1} \cdots v_x v_{b-1} v_b v_1$) is even. Therefore, $l(C_{23}) = a-x+4 \equiv a-x \equiv 0$ (or $x-a+4 \equiv x-a \equiv 0$) mod 4. Hence, for the cycle $C_{24} = v_1 v_2 \cdots v_x v_{b-1} v_{b-2} \cdots v_w v_t v_{t-1} \cdots v_b v_1$, $l(C_{24}) = x + (t-w+1) = a+t-(a-t+1)+1 \equiv 2t \equiv 2 \bmod 4$, a contradiction.

This contradiction completes the proof of Theorem 2. □

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