ON (k,l) - KERNELS OF ORIENTATIONS OF SPECIAL GRAPHS

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ABSTRACT. In [1], [2] we can find results concerning kernel-perfect graphs and solvable graphs. These concepts are related to kernels of a digraph. The authors of [2] consider two graph constructions: the join of two graphs and duplication of a vertex. These kinds of graphs preserve kernel-perfectness and solvability of their orientations. In this paper we generalize results from [2] applying them to (k,l) - kernels and two operations: generalized join and duplication of a subset of vertices. The concept of a (k,l) - kernel of a digraph was introduced in [8] and was studied in [6], [7] and [9]. In our considerations we take advantage of the asymmetrical part of digraphs, which was used by H. Galeana-Sanchez in [6] in the proof of a sufficient condition for a digraph to have a (k,l) - kernel.

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1. INTRODUCTION

Let D denote a finite, directed graph (for short: a *digraph*) without loops and multiple arcs, where $V(D)$ is the set of vertices of D and $A(D)$ is the set of arcs of D. An arc $xy \in A(D)$ is called *asymmetrical* (resp. *symmetrical*) if $yx \notin A(D)$ (resp. $yx \in A(D)$). The *asymmetrical part of* D (resp. *symmetrical part of* D), denoted by Asym(D) (resp. Sym(D)) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D. By $D[S]$ we denote the subdigraph of D induced by a nonempty subset $S \subseteq V(D)$.

By a *path* from a vertex x_1 to a vertex x_n in D we mean a sequence of distinct vertices $x_1, \dots, x_n$ from $V(D)$ and arcs $x_i x_{i+1} \in A(D)$ for

$i = 1, 2, \ldots, n-1$ and denote it by $[x_1, \ldots, x_n]$. A *circuit* is a path with $x_1 = x_n$.

Let G be an undirected graph. A digraph D we call an *orientation* of G, when $V(D) = V(G)$ and two vertices are adjacent in G if and only if they are adjacent in D. Symmetrical arcs are admitted in D. Simultaneously, the graph G is called the *underlying undirected graph* of D. By a *clique* Q of a digraph D we mean an induced subdigraph of D such that the underlying undirected graph of Q is isomorphic to a complete graph. An orientation D is *admissible* if every clique of D contains a vertex which is the successor of all its other vertices (such a vertex is called a *sink*). A digraph D is an M*-admissible* orientation if every circuit in D of length three contains at least two symmetrical arcs.

In [1] we can find the following proposition:

Proposition 1.1 (C. Berge and P. Duchet). *An orientation of a complete graph is admissible if and only if every circuit has at least one symmetrical arc.*

From this proposition it follows the next corollary:

Corollary 1.2. *An orientation of a graph is not admissible if and only if it contains a circuit whose all arcs are asymmetrical and vertices induce a clique (such a circuit is called a whirl).*

We denote by $d_D(x,y)$ the length of the shortest path from x to y in D. For any $X, Y \subseteq V(D)$ and $x \in V(D) \setminus X$ we put $d_D(x,X) = \min_{y \in X} d_D(x,y)$, $d_D(X,x) = \min_{y \in X} d_D(y,x)$ and $d_D(X,Y) = \min_{x \in X, y \in Y} d_D(x,y)$. Let k, l be integers, $k \geq 2$ and $l \geq 1$. We say that subset $J \subseteq V(D)$ is a (k,l) - *kernel of* D if

(i) for each $x, y \in J$ and $x \neq y, d_D(x,y) \geq k$ and
(ii) for each $x \in V(D) \setminus J$ there exists $y \in J$ that $d_D(x,y) \leq l$.

If $k = 2$ and $l = 1$, then we obtain the definition of a kernel or in other words a $(2,1)$ - kernel of digraph.

If J satisfies the condition (i), then we say that J is k - *stable* in D (if $k = 2$, then we write that J is *stable* in D). Moreover, we assume that the subset including exactly one vertex also is k - stable in D. We say that J is l - *dominating in* D (for short: *dominating* in D, when $l = 1$), when the condition (ii) is fulfilled. More precisely with respect to the vertex x we say: x is l - *dominated by* J *in* D or J l - *dominates* x *in* D.

A directed graph such that every induced subdigraph has a (k,l) - kernel is called (k,l) - *kernel-perfect*. An undirected graph is (k,l) - *solvable* (resp. (k,l) - M*-solvable)* if every its admissible (resp. M-admissible) orientation has a (k,l) - kernel. Three last concepts are generalizations of kernel-perfect, solvable and M-solvable graph, respectively, which are considered in [1], [2], [3] and [5].

Let $G_0, G_1, \ldots, G_p$, $p \geq 1$, denote $p+1$ pairwise vertex-disjoint undirected graphs (not necessarily with the same cardinalities of vertex sets). The generalized join of graphs $G_0, G_1, \ldots, G_p$ is a graph $J(G_0, G_1, \ldots, G_p)$ obtained by adding a new edge xy, for every $x \in V(G_0)$, $y \in V(G_i)$, where $i = 1, 2, \ldots, p$, to the disjoint union of the graphs $G_0, G_1, \ldots, G_p$. It may be to note that for $p = 1$ we receive the join of two graphs $G_0 \circ G_1$.

Let G be an undirected graph and X be an arbitrary subset of $V(G)$. Let H be a graph isomorphic to $G[X]$. The vertex from $V(H)$ corresponds to $x \in X$ we will denote by x'. *Duplication* of X in G, denoted by G^X, is a graph such that $V(G^X) = V(G) \cup V(H)$ and $E(G^X) = E(G) \cup E(H) \cup E_0 \cup E_1$, where $E_0 = \{x'y : x' \in V(H) \text{ and } y \text{ is a vertex adjacent to } x \in X \text{ in } G\}$ and $E_1 \subseteq \{xy : x \in V(H) \text{ and } y \in V(G)\}$. If $E_1 \neq \emptyset$, then duplication is called *adjacent*. Otherwise duplication is *non-adjacent*. Put $X' = V(H)$. The vertex $x' \in X'$ is called a *duplicate of* $x \in X$ and X' is a *duplicate of* X.

Other more precisely notations will be introduced in the parts of the paper in which they will be used. For concepts not defined here, see [4].

2. Duplication and (k,l) - kernels of its orientations

We will start with some definitions. Let G be an undirected graph, X be an arbitrary subset of $V(G)$ and D be an orientation of duplication G^X. For $x \in X$ by $N(x)$ we denote the subset of vertices not belonging to X and adjacent to x in D (of course they are adjacent to its duplicate $x' \in X'$ in D). Now create a partition of $N(x)$ in the following way:

$N_1(x) = \{y \in N(x) : \text{the arcs } xy,\ x'y \in \text{Asym}(D)\}$,

$N_2(x) = \{y \in N(x) : \text{the arc } yx \in \text{Asym}(D) \text{ or } yx' \in \text{Asym}(D)\}$,

$N_3(x) = N(x) \setminus (N_1(x) \cup N_2(x))$.

It may be to note that if $y \in N_1(x) \cup N_3(x)$, then there exist both of the arcs xy, $x'y$ in D.

By $G - X$ we mean a subgraph of G obtained from G by deleting the subset X. We define an orientation D^* of G as follows: every edge belonging to $A(G - X) \cup A(X)$ is directed as in D and for any $x \in X$

(i) if $y \in N_1(x)$, then the arc $xy \in \text{Asym}(D^*)$ and

(ii) if $y \in N_2(x)$, then the arc $yx \in \text{Asym}(D^*)$ and

(iii) if $y \in N_3(x)$, then the arc $xy \in \text{Sym}(D^*)$.

The idea used in the construction of the orientation D^* is closely related to a construction included in [2].

Let us observe the following result.

Lemma 2.1. *Let G be an undirected graph and $X \subset V(G)$ and D be an admissible orientation of duplication G^X. If either (a) or (b) be satisfied:*

(a) *duplication G^X is non-adjacent and $D[X]$ is isomorphic to $D[X']$,*

(b) *duplication G^X is adjacent, for every $x \in X$ and its duplicate $x' \in X'$ there exists an edge $xx' \in E(G^X)$ and an arc $x'x \in Asym(D)$,*

then the orientation D^* of G is admissible.

Proof. Assume on the contrary that D^* is not admissible. Then D^* contains a whirl W in view of Corollary 1.2. It is easy to observe that there exist two vertices from W such that one belongs to X and another one belongs to $V(G) \setminus X$. Otherwise, D also contains the whirl W. Thus, we obtain a contradiction with that D is an admissible orientation of G^X. Hence there exists a path $[y_1, y_2, \dots, y_{n-1}, y_n]$ in W such that $y_1, y_n \in V(G) \setminus X$ and $y_2, \dots, y_{n-1} \in X$. Since the arcs y_1y_2, $y_{n-1}y_n \in \mathrm{Asym}(D^*)$ and $y_2, y_{n-1} \in X$, then $y_1 \in N_2(y_2)$ and $y_n \in N_1(y_{n-1})$. By the definition of $N_1(y_{n-1})$ there exist the arcs $y_{n-1}y_n$, $y'_{n-1}y_n$ in D and both are asymmetrical. By the definition of $N_2(y_2)$ at least one of the arcs y_1y_2, $y_1y'_2$, belongs to $A(D)$ and is asymmetrical in D. If $y_1y_2 \in \mathrm{Asym}(D)$, then there exists the path $[y_1, y_2, \dots, y_{n-1}, y_n]$ in D and it contains only asymmetrical arcs. If $y_1y'_2 \in \mathrm{Asym}(D)$, then two cases are to consider. If the condition (a) holds, then replacing vertices $y_2, \dots, y_{n-1}$ with $y'_2, \dots, y'_{n-1}$ we create a path $[y_1, y'_2, \dots, y'_{n-1}, y_n]$ in D, whose all arcs belong to $\mathrm{Asym}(D)$. If condition (b) holds, then putting the arcs $y_1y'_2$, y'_2y_2 instead the arc y_1y_2 we receive a path $[y_1, y'_2, y_2, \dots, y_{n-1}, y_n]$ in D, whose arcs are included in the asymmetrical part of D. Repeating the same operation for every path $[y_1, y_2, \dots, y_{s-1}, y_s]$ in W such that $y_1, y_s \in G(V) \setminus X$ and $y_2, \dots, y_{s-1} \in X$ we receive a whirl in D, a contradiction, with the assumption. So D^* is admissible orientation of G. ∎

Now we prove the related result with respect to D^* being M-admissible.

Lemma 2.2. *Let G be an undirected graph and $X \subset V(G)$. If D is an M-admissible orientation of duplication G^X such that $D[X]$ is isomorphic to $D[X']$, then the orientation D^* of G is M-admissible.*

Proof. Assume on the contrary that D^* is not M-admissible i.e., there exists circuit $C = [x, y, z, x]$ in D^* having at least two asymmetrical arcs. It is clear that there exist two vertices from C such that one belongs to X and another one belongs to $V(G) \setminus X$. Otherwise, D also contains the circuit C, a contradiction with that D is M-admissible. Without loss of generality we consider two cases:

(a) $x, y \in X$ and $z \in V(G) \setminus X$,
(b) $x \in X$ and $y, z \in V(G) \setminus X$.

At first assume that condition (a) holds. From the definition of D^* and the assumption that $D[X]$ is isomorphic to $D[X']$ it follows that if $xy \in \text{Asym}(D^*)$, then $xy, x'y' \in \text{Asym}(D)$ and if $xy \in \text{Sym}(D^*)$, then $xy, x'y' \in \text{Sym}(D)$. First we assume that $yz \in \text{Asym}(D^*)$. This implies that $z \in N_1(y)$. By the definition of $N_1(y)$ the arcs $yz, y'z \in \text{Asym}(D)$. If $yz \in \text{Sym}(D^*)$, then $z \in N_3(y)$. From the definition of $N_3(y)$ it follows that $yz, y'z \in A(D)$. Now we suppose that $zx \in \text{Asym}(D^*)$. Therefore $z \in N_2(x)$. It follows that $zx \in \text{Asym}(D)$ or $zx' \in \text{Asym}(D)$. If $zx \in \text{Sym}(D^*)$, then $z \in N_3(y)$. For this reason $zx, zx' \in A(D)$. From facts given above it follows that if at least two arcs of C are asymmetrical in D^*, then one of the circuits $[x, y, z, x]$ and $[x', y', z, x']$ contains at least two asymmetrical arcs in D, a contradiction.

It remains to analyze the condition (b). Arguing like in case (a) we have:
$yz \in \text{Asym}(D^*)$ if and only if $yz \in \text{Asym}(D)$,
if $zx \in \text{Asym}(D^*)$, then $zx \in \text{Asym}(D)$ or $zx' \in \text{Asym}(D)$,
if $zx \in \text{Sym}(D^*)$, then $zx, zx' \in A(D)$,
if $xy \in \text{Asym}(D^*)$, then $xy, x'y \in \text{Asym}(D)$ and
if $xy \in \text{Sym}(D^*)$, then $xy, x'y \in A(D)$.

From these facts it follows that if C contains at least two asymmetrical arcs in D^*, then there exists one of the circuits $[x, y, z, x]$, $[x', y', z, x']$ in D and has at most one symmetrical arc, a contradiction with the assumption about D. This means that the orientation D^* of G is M-admissible. ∎

We recall that k, l are integers, $k \geq 2$ and $l \geq 1$. Using the above Lemma 2.1 we prove:

Theorem 2.3. *Let G be a (k,l) - solvable graph, X be a subset of $V(G)$ and G^X be non-adjacent duplication of X in G. If D is an admissible orientation of G^X such that $D[X]$ is isomorphic to $D[X']$, $d_D(X, X') \geq k$ and $d_D(X', X) \geq k$, then D has a (k,l) - kernel.*

Proof. At first we prove that D has a (k,l) - kernel, for $k = 2$. It is clear that in this case we need only to show it for $l = 1$. From Lemma 2.1 it follows that D^* is an admissible orientation of G. Hence D^* has a $(2,1)$ - kernel J (i.e., J is a kernel of D^*). It is not difficult to see that if $Y = J \cap X \neq \emptyset$, then $J \cup Y'$ is a kernel of D, where $Y' \subseteq X'$ is duplicate of Y. Indeed, it is easy to observe that the set $J \cup Y'$ is stable in D. So what remains is to show that it is dominating in D. Obviously, every $x \in X \cup X'$ is dominating by $J \cup Y'$ in D. From the definitions of J and the orientation D^* of G it follows that every $y \in V(D)$ such that $d_D(Y, y) \geq 2$ and $d_D(y, Y) \geq 2$ is dominated by $J \cup Y'$ in D. Now, let $y \in V(D) \setminus X$ be the vertex adjacent to some vertex from Y. If there exists $x \in Y$ such that $y \in N_2(x) \cup N_3(x)$, then at least one of the arcs yx, yx' exists in D. Consequently, the subset $J \cup Y'$ dominates the vertex y. If for every $x \in Y$ the vertex y belongs to $N_1(x)$, then the arcs yx and yx' do not exist in D. This means that for every $x \in Y$ the arc $xy \in \mathrm{Asym}(D^*)$ and so y must be dominated in D^* by some vertex of $J \setminus Y$. Then $J \cup Y'$ dominates y in D. Now consider the case, when $J \cap X = \emptyset$. In this case for every $x \in X$ there exists $y \in J$ such that the arc $xy \in A(D^*)$. A consequence of this is that $y \in N_1(x) \cup N_3(x)$. This means that y is a successor of x and x'. Then J is a kernel of D.

Now consider $k \geq 3$. In this case for every $x \in X$ and $x' \in X'$ we have that $d_D(x, x') \geq k$ and $d_D(x', x) \geq k$. This means that if $y \in N(x) = N(x')$, then either the arcs yx, $yx' \in \mathrm{Asym}(D)$ or the arcs xy, $x'y \in \mathrm{Asym}(D)$. Let J be a (k, l) - kernel of $D[V(G)]$. If $Y = J \cap X \neq \emptyset$, then the subset $J \cup Y'$ is k - stable. Additionally, from this that $D[X]$ is isomorphic to $D[X']$ it follows that this subset is l - dominating in D. It is not difficult to observe that if $J \cap X = \emptyset$, then J is a (k, l) - kernel of D. This completes the proof. ∎

Note that if $k = 2$, then every orientation D of non-adjacent duplication of G^X satisfies inequalities $d_D(X, X') \geq k$ and $d_D(X', X) \geq k$. Additionally, if the subset X contains only one vertex, then $D[X]$ is isomorphic to $D[X']$. From these facts it follows that Theorem 2.3 is a generalization of the result announced in [2] and concerns duplication of one vertex and a kernel.

Theorem 2.4. [2]*Let G be a solvable graph, $x \in V(G)$ and G' be the graph obtained from G by adjacent duplication of x with a new vertex x'. If D' is an admissible orientation of G' in which the arc $x'x$ is asymmetrical, then D' has a kernel.*

Taking Lemma 2.2 into consideration and proceeding like in the proof of Theorem 2.3 we can prove the following result.

Theorem 2.5. *Let G be a (k, l) - M-solvable graph, X be an arbitrary subset of $V(G)$ and G^X be non-adjacent duplication of X in G. If D is an M-admissible orientation of G^X such that $D[X] = D[X']$, $d_D(X, X') \geq k$ and $d_D(X', X) \geq k$, then D has a (k, l) - kernel.*

The next theorem gives a sufficient condition for an admissible orientation of adjacent duplication to have a (k, l) - kernel.

Theorem 2.6. *Let G be (k,l) - solvable, X be a subset of $V(G)$ and G^X be adjacent duplication of X, such that there exists an edge xx', for every $x \in X$ and its duplicate $x' \in X'$. If D is an admissible orientation of G^X such that the following conditions are satisfied:*

(a) *$D[X]$ is isomorphic to $D[X']$,*
(b) *an arc $x'x \in Asym(D)$ and*
(c) *for every $x \in X$ and $y \in V(G) \setminus X$ if $xy \in Asym(D)$, then $x'y \in Asym(D)$,*

then D has a (k,l) - kernel.

Proof. From Lemma 2.1 it follows that D^* is an admissible orientation of G. As a result of this D^* has a (k,l) - kernel, say J. We shall prove that J is a (k,l) - kernel of D. We need only to show that J is l - dominating in D, because J is k - stable in D. If $x \in V(G) \setminus J$, then there exists a path P from x to some vertex $w \in J$ of length at most l in D^*. If all arcs from P belong to $A(D)$, then x is l - dominated by J in D. Assume that there exists an arc yz belongs to the path P, such that $zy \in$ Asym(D). Observe that $z \in X$ and $y \in N_2(z) \cup N_3(z)$. It is a consequence of the construction of the orientation D^*. This means that at least one of the arcs yz, yz' belongs to $A(D)$. Moreover, by (c) we have $z'y \in$ Asym(D). Combining two facts given above we have that yz, $yz' \in A(D)$, a contradiction that $zy \in$ Asym(D). This means that P also is in D. Hence every $x \in V(G) \setminus J$ is l - dominating by J in D. In that case it remains to prove that all vertices from X' are l - dominating by J in D. Let $x' \in X'$. If x corresponding to x' belongs to J, then $d_D(x', J) = 1 \leq l$. If $x \in X \setminus J$, then from the fact proved earlier it follows that there exists a path $[x, y_1, y_2, \dots, y_n, y]$ from x to some vertex $y \in J$ of length at most l in both D^* and D. Let i be the smallest integer such that $y_i \in V(G) \setminus X$. By (a) there is a path $[x', y'_1, y'_2, \dots, y'_{i-1}]$ in D. In view of the fact that $y_i \in V(G) \setminus X$ we have $y_i \in N_1(y_{i-1}) \cup N_3(y_{i-1})$. In consequence, the arcs $y_{i-1}y_i$ and $y'_{i-1}y_i$ belong to $A(D)$. This means that there exists a path $[y'_{i-1}, y_i, \dots, y]$ in D. So $[x', y'_1, \dots, y'_{i-1}, y_i, \dots, y]$ is a path of length at most l in D, hence $d_D(x', J) \leq l$. All this together shows that J is l - dominating in D and consequently, J is a (k,l) - kernel of D, what completes the proof. ∎

Replacing Lemma 2.1 in the proof of Theorem 2.6 to Lemma 2.2 we can formulate the following result.

Theorem 2.7. *Let G be (k,l) - M-solvable, X be an arbitrary subset of $V(G)$, and G^X be adjacent duplication of X, such that there exists an edge xx', for every $x \in X$ and its duplicate $x' \in X'$. If D is an M-admissible orientation of G^X such that the following conditions are satisfied:*

(a) *$D[X]$ is isomorphic to $D[X']$,*

(b) *an arc* $x'x \in Asym(D)$ *and*
(c) *for every* $x \in X$ *and* $y \in V(G) \setminus X$ *if* $xy \in Asym(D)$, *then* $x'y \in Asym(D)$,

then D *has a* (k, l) *- kernel.*

3. On $(k, k-1)$ - kernels of an orientation of the generalized join

At first we give some propositions about the generalized join, which are necessary to prove the main result of this section.

Let G_0, $G_1, \ldots, G_p$ be undirected graphs and $k \geq 2$. We say that an orientation D of the generalized join $J(G_0, G_1, \ldots, G_p)$ has a property α with respect to G_i for a fixed integer $1 \leq i \leq p$, if every circuit C of length $3 \leq m \leq k+1$ in D containing at least one vertex from $V(G_0)$ and from $V(G_i)$ has at least $m-1$ symmetrical arcs.

The next result is obvious.

Proposition 3.1. *Every induced subgraph of the generalized join also is the generalized join or an induced subgraph of disjoint union of some components of the generalized join.*

Proposition 3.2. *Let* G_0, $G_1, \ldots, G_p$ *be undirected graphs and* D *be an orientation of the generalized join* $J(G_0, G_1, \ldots, G_p)$ *having a property* α *with respect to* G_i *for a fixed integer* $1 \leq i \leq p$. *Let* S *be a* $(k-1)$ *- dominating subset of* $D[V(G_0)]$ *and* $x \in V(G_i)$ *be a vertex such that* $d_D(x, S) \geq 2$. *Then there exists an arc* yx *for every* $y \in V(G_0)$.

Proof. Let $y \in V(G_0)$. By the definition of the generalized join there exists at least one of the arcs xy, yx in D. If $y \in S$, then the asymmetrical arc $yx \in A(D)$ because of $d_D(x, S) \geq 2$. Now consider $y \in V(G_0) \setminus S$. Seeing as S is $(k-1)$ - dominating in $D[V(G_0)]$, we have that there exists $z \in S$ such that $d_D(y, z) \leq k-1$. Moreover, from the hypothesis it follows that $d_D(y, z) \geq 2$ (i.e., the arc zy is asymmetrical in D). We state that the arc $yx \in V(D)$. Otherwise, $xy \in \text{Asym}(D)$. But this means that there exists a circuit of length $m \leq k+1$ containing the vertices x, y, z and at least two asymmetrical arcs, a contradiction. ∎

Proposition 3.3. *Let G_0, $G_1, \dots, G_p$ be the undirected graphs and D be an orientation of the generalized join $J(G_0, G_1, \dots, G_p)$ having a property α with respect to G_i for a fixed integer $1 \le i \le p$. Let $x \in D[V(G_i)]$, $y \in D[V(G_j)]$ with $1 \le j \le p$ and for every $w \in D[V(G_0)]$ $wx \in A(D)$. If $d_D(x,y) \le k-1$, then $d_D(y,x) \le k-1$ or for every $w \in D[V(G_0)]$ $wy \in Asym(D)$.*

Proof. Assume on the contrary that $d_D(x,y) \le k-1$, but $d_D(y,x) \ge k$ and there exists $w \in D[V(G_0)]$ such that $yw \in A(D)$. At first we consider the case, when $i = j$. Since $d_D(x,y) \le k-1$ and $d_D(y,x) \ge k$,then at least one arc from the shortest path from x to y is asymmetrical. This means that for every vertex $w \in D[V(G_0)]$ there exists a circuit of length $m \le k+1$ containing the vertices x, y, w and at least two asymmetrical arcs, a contradiction. Now we take into consideration $i \ne j$. It is not difficult to observe that for $k = 2$ the proposition is true, since $d_D(x,y) \ge 2 = k$. Let $k \ge 3$. From the assumptions it follows that there exists $w \in D[V(G_0)]$ such that $yw \in A(D)$ and $wx \in A(D)$. This means that $d_D(y,x) = 2 \le k-1$, a contradiction with the assumption that $d_D(y,x) \ge k$. ∎

The following theorem is the main result of this section. It gives a sufficient condition for an orientation of the generalized join of graphs to be $(k,k-1)$ - kernel-perfect.

Theorem 3.4. *Let D_0, $D_1, \dots, D_p$ be $p+1$ vertex - disjoint $(k,k-1)$ - kernel-perfect digraphs and let G_0, $G_1, \dots, G_p$ be the underlying undirected graphs. Let D be an orientation of the generalized join $J(G_0, G_1, \dots, G_p)$ such that for every $0 \le i \le p$ we have $D[V(D_i)] = D_i$. If the orientation D has a property α with respect to every $0 \le i \le p$ and $D[V(D_i)]$ is isomorphic to D_i, then D is $(k,k-1)$ - kernel-perfect.*

Proof. From Proposition 3.1 it follows that we need only to prove that D has a $(k, k-1)$ - kernel. For better clarity we split the proof into two parts.

I. Assume that $k \ge 3$ and there exist an integer s with $1 \le s \le p$ and $x \in V(D_s)$ such that for every $y \in V(D_0)$ an arc $yx \in A(D)$ is asymmetrical in D. For every $1 \le i \le p$, we denote by B_i the subset of $V(D_i)$ such that $x \in B_i$ if and only if for every $y \in V(D_0)$ $yx \in$ Asym(D). Let $I = \{i : B_i \ne \emptyset\}$. It follows from the assumption that I is not empty, since $s \in I$. Further, from the assumption that D_i is $(k, k-1)$ - kernel-perfect and $D[V_i]$ is isomorphic to D_i it follows that for $i \in I$ the subdigraph $D[B_i]$ has a $(k, k-1)$ - kernel, say J_i. Note that $\bigcup_{i \in I} J_i$ is k - stable in D, by the definitions of the subsets B_i, J_i and of the generalized join. Now we show that $\bigcup_{i \in I} J_i$ is $(k-1)$ - dominating in D. Let $y \in V(D) \setminus \bigcup_{i \in I} J_i$. If $y \in V(D_0)$, then $d_D(y, J_i) = 1 \le k-1$ for every $i \in I$ by the definition of B_i. If $y \in \bigcup_{1\le i\le p} (V(D_i) \setminus B_i)$, then there exists an arc yw in D, where $w \in V(D_0)$. Combining two facts given above we have that $d_D(y, \bigcup_{i\in I} J_i) = 2 \le k-1$. If $y \in \bigcup_{i\in I} (B_i \setminus J_i)$, then $d_D(y, \bigcup_{i\in I} J_i) \le k-1$, by the definition of J_i. This means that $\bigcup_{i\in I} J_i$ is a $(k, k-1)$ - kernel of D.

II. Now assume that $k = 2$ or for every $1 \le i \le p$ and every $x \in V(D_i)$ there exists $y \in V(D_0)$ such that $xy \in A(D)$. Let J_0 be a $(k, k-1)$ - kernel of D_0. If every vertex $x \in V(D_i)$ with $1 \le i \le p$ is $(k-1)$ - dominated by J_0, then J_0 is a $(k, k-1)$ - kernel of D and the theorem follows. So assume that there exist integer j and $x \in V(D_j)$ such that $d_D(x, J_0) \ge k$. Then from Proposition 3.2 it follows that $R = \{y \in V(D_j) : \text{for every } z \in V(D_0) \text{ we have } d_D(z, y) = 1\}$ is not empty. Let J_R be a $(k, k-1)$ - kernel of $D[R]$. Note that

(3.1) if $y \in V(D_0) \cup R \setminus J_R$, then $d_D(y, J_R) \le k-1$.

This follows from the definition of R and J_R. Now we show that

(3.2) for $y \in V(D) \setminus J_R$ if $d_D(J_R, y) \le k-1$, then $d_D(y, J_R) \le k-1$.

Owing to (3.1), it is enough to consider $y \in \bigcup_{1\le i\le p} V(D_i) \setminus R$. By the definition of $J_R \subseteq R$, for every $x \in J_R$ and $w \in V(D_0)$ we have $wx \in V(D)$. From this fact it follows that if $d_D(x, y) \le k-1$, then $d_D(y, x) \le k-1$ or for every $w \in D[V(G_0)]$, $wy \in \mathrm{Asym}(D)$, in view of Proposition 3.3 At the same time taking the assumption from the beginning of this part of the proof we have that if $d_D(x, y) \le k-1$, then $d_D(y, x) \le k-1$. So (3.2) is fulfilled. Let S_i be a subset of $V(D_i)$ such that $x \in S_i$ if and only if $d_D(J_R, x) \ge k$ and $d_D(x, J_R) \ge k$, for every $1 \le i \le p$. Notice that for $i \ne j$ and $k = 2$ we have that $S_i = V(D_i)$. Since every digraph D_i is $(k, k-1)$ - kernel-perfect, then the induced subdigraph $D[S_i]$ has a $(k, k-1)$ - kernel, say J_i, for $1 \le i \le p$. We show that $J_R \cup \bigcup_{1\le i\le p} J_i$ is a $(k, k-1)$ - kernel of D. At first we show k - stability of this subset. It is easy to observe that for $k = 2$ the subset $J_R \cup \bigcup_{1\le i\le p} J_i$ is stable in D. Let $k \ge 3$. From the definitions of J_R, S_i and J_i if follows that we need only to prove that $d_D(J_s, J_t) \ge k$ for every integers $s \ne t$. Assume on the contrary that there exist $x \in J_s$ and $y \in J_t$ such that $d_D(x, y) \le k-1$. Therefore, there exists $w \in V(D_0)$ such that $d_D(x, w) = 1$. Moreover, $d_D(x, J_R) = 1$ (from the definition of R). This means that $d_D(x, J_R) = 2 < k$, a contradiction with fact that $x \in J_p \subseteq S_p$. To complete the proof we need to show that $J_R \cup \bigcup_{1\le i\le p} J_i$ is $(k-1)$-dominating in D. Let $y \in V(D) \setminus \left(J_R \cup \bigcup_{1\le i\le p} J_i \right)$ be the vertex such that $d_D(y, J_R) \geq k$. From (3.2) it follows that $d_D(J_R, y) \geq k$. So $y \in \bigcup_{1 \leq i \leq p} S_i$. Hence y is $(k-1)$ - dominated by $\bigcup_{1 \leq i \leq p} J_i$ and the theorem is proved. ∎

It may by to note that for $p = 1$ and $k = 2$ we receive the result announced in [2] considering the join two graphs and a kernel.

Theorem 3.5. *([2])Let D_1, D_2 be two vertex - disjoint kernel-perfect digraphs and let G_1, G_2 be the underlying undirected graphs. Let D be an orientation of the join $J(G_1, G_2)$ such that $D[V(D_1)] = D_1$, $D[V(D_2)] = D_2$, and every circuit C in D of length three such that $V(C) \cap V(D_1) \neq \emptyset$ and $V(C) \cap V(D_2) \neq \emptyset$ has at least two symmetrical arcs. Then D is kernel-perfect.*

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