Stable and unstable graphs with total irredundance number zero

Teresa W. Haynes
Department of Mathematics
East Tennessee State University
Johnson City, TN 37614 USA

Stephen T. Hedetniemi
Department of Computer Science
Clemson University
Clemson, SC 29634 USA

Michael A. Henning *
University of Natal
Private Bag X01, Scottsville
Pietermaritzburg, 3209 South Africa

Debra J. Knisley
Department of Mathematics
East Tennessee State University
Johnson City, TN 37614 USA

Abstract

For a graph $G = (V, E)$, a set $S \subseteq V$ is *total irredundant* if for every vertex $v \in V$, the set $N[v] - N[S - \{v\}]$ is not empty. The *total irredundance number* $ir_t(G)$ is the minimum cardinality of a maximal total irredundant set of G. We study the structure of the class of graphs which do not have any total irredundant sets; these are called $ir_t(0)$-graphs. Particular attention is given to the subclass of $ir_t(0)$-graphs whose total irredundance number either does not

*Research supported in part by the South African Foundation for Research Development and the University of Natal.

change (*stable*) or always changes (*unstable*) under arbitrary single edge additions. Also studied are $ir_t(0)$-graphs which are either stable or unstable under arbitrary single edge deletions.

1 Introduction

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the *open neighborhood* of v is defined by $N(v) = \{u \in V \mid uv \in E(G)\}$ and its *closed neighborhood* by $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set S is the set $N(S) = \cup_{v \in S} N(v)$ and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The neighbor of an endvertex is called a *support* vertex. Graph theory terminology not presented here can be found in [1] and [2].

The *private neighbor set of a vertex v with respect to a set S* is denoted by $PN[v, S] = N[v] - N[S - \{v\}]$. If $PN[v, S] \neq \emptyset$ for some vertex v and some $S \subseteq V$, then every vertex of $PN[v, S]$ is called a *private neighbor of v with respect to S*, or just an *S-pn*.

A set $S \subseteq V$ of vertices is a *dominating set* if every vertex $v \in V - S$ is adjacent to at least one vertex $u \in S$, and is a *total dominating set* if every vertex $v \in V$ is adjacent to at least one vertex $u \in S$. Dominating sets and total dominating sets, and their many applications have been well-studied; the interested reader is referred to two recent comprehensive books on this subject (cf. [2] and [3]).

Of particular interest in the study of dominating sets in graphs are *minimal dominating sets*. It is easy to see that a set S is a minimal dominating set if and only if the following condition holds:

(1) for every vertex $v \in S$, there exists a vertex $w \in V - (S - \{v\})$ which is not dominated by $S - \{v\}$, i.e., $w \notin N[S - \{v\}]$.

Equivalently, one can say that a dominating set is a minimal dominating set if and only if the following condition holds:

(2) for every vertex $v \in S$, $PN[v, S] \neq \emptyset$.

We say that a set S of vertices is *irredundant* if condition (2) holds. Thus, the minimality condition for a dominating set is the definition of an irredundant set. Irredundant sets have also been well studied in the literature; the reader is again referred to [2] and [3]. However, the concept of a total irredundant set has only recently been defined and studied. A set S is *total irredundant* if and only if the following condition holds:

(3) for every vertex $v \in V$, $PN[v, S] \neq \emptyset$.

Notice that for an irredundant set S, only the vertices in S are required to have a private neighbor with respect to S, while for a total irredundant set every vertex in V must have an S-pn. The *total irredundance number* $ir_t(G)$ of a graph G is the minimum cardinality of a maximal total irredundant set in G, and a total irredundant set with cardinality $ir_t(G)$ is called an ir_t-*set*. If a graph G has no total irredundant set, then $ir_t(G)$ is defined to be zero, and it is convenient to refer to such graphs as $ir_t(0)$-graphs.

Total irredundance in graphs was first defined and algorithmically studied by Hedetniemi, Hedetniemi, and Jacobs in [4], who presented a linear algorithm for computing $ir_t(T)$ for any tree T, and an NP-completeness proof for the UPPER TOTAL IRREDUNDANT SET problem. In this problem one seeks the maximum cardinality of a total irredundant set in a graph G.

Recently, Favaron, Haynes, Hedetniemi, Henning, and Knisley [5] initiated the graph theoretic study of total irredundance in graphs. In [5] the parameters $ir_t(G)$ and $IR_t(G)$ are studied, where $IR_t(G)$ equals the maximum cardinality of a total irredundant set in a graph G. Also presented in [5] are:

(i) a characterization of $ir_t(0)$-graphs,
(ii) several results on regular graphs with $ir_t(G) \geq 1$,
(iii) a characterization of trees with $ir_t(T) = 1$, and
(iv) several upper bounds on total irredundance numbers.

This paper is motivated by the somewhat unique properties of the class of graphs G for which $ir_t(G) = 0$. Consider any typical integer-valued parameter of a graph G, like the *independence number* $\beta_0(G)$, the *domination number* $\gamma(G)$, the *vertex covering number* $\alpha_0(G)$, the *irredundance number* $ir(G)$, the *matching number* $\beta_1(G)$ and the *chromatic number* $\chi(G)$ (for definitions the reader is referred to [2]). Let $\zeta(G)$ denote a generic parameter of a graph G and let $\zeta(i)$ denote the class of graphs for which $\zeta(G) = i$. For most parameters $\zeta(G)$, it is almost trivial to characterize the class of $\zeta(0)$ or $\zeta(1)$ graphs. However, the total irredundance number $ir_t(G)$ is one of the few examples where it is anything but trivial to characterize the class of $ir_t(0)$-graphs.

To illustrate the concept of total irredundance, we present some examples. The complete graph K_n is an example of a graph having no total irredundant sets, and hence $ir_t(K_n) = 0$. However the complement of the complete graph $\overline{K}_n$ has only one maximal total irredundant set, i.e., the entire vertex set, and $ir_t(\overline{K}_n) = n$. Note that for a graph G of order n, a total irredundant set is dominating if and only if $ir_t(G) = n$, i.e., $G = \overline{K}_n$.

The central vertex, say v, of a star $K_{1,m}$ with order $n = m+1$ for $m \geq 2$ is a maximal irredundant set, but it is not a maximal total irredundant set, since the endvertices do not have private neighbors with respect to $\{v\}$. On the other hand, the set of all leaves except one is a maximal total irredundant set. It can be seen that $ir_t(K_{1,m}) = m - 1 = n - 2$. In fact, it is shown in [5] that for any graph G of order n without isolated vertices, $ir_t(G) \leq n - 2$ and the stars $K_{1,m}$ for $m \geq 2$ are the only graphs achieving this upper bound.

The complete bipartite graph $K_{r,r}$ has $ir_t(K_{r,r}) = r - 1$, since $r - 1$ vertices from a partite set form an ir_t-set. For n even, let G be K_n minus a perfect matching. Since for every vertex v in G, $\{v\}$ is a total irredundant set of G, $ir_t(G) \geq 1$. However, any two vertices of G form a dominating set of G and therefore cannot be a total irredundant set. Hence, $ir_t(G) \leq 1$, and so $ir_t(G) = 1$.

Also, the graphs G and H shown in Figure 1 have $ir_t(G) = ir_t(H) = 1$ (the set $\{x\}$ is an ir_t-set).

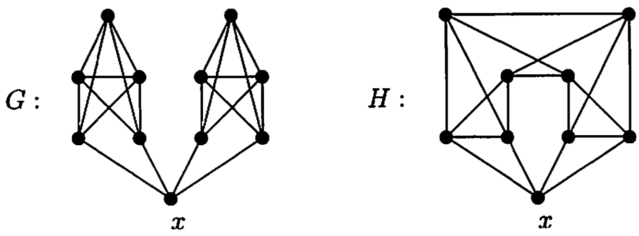

Figure 1: $ir_t(G) = ir_t(H) = 1$.

The trees T with $ir_t(T) = 1$ were characterized in [5] as one of the following three types: type T_1 is a star $K_{1,k}$, $k \geq 1$, with exactly one subdivided edge, type T_2 is a corona of a star $K_{1,k} \circ K_1$ for $k \geq 2$, and type T_3 is a subdivided star $K^*_{1,k}$ for $k \geq 2$. These types are illustrated in Figure 2, where for each type of tree, the set $\{x\}$ is an ir_t-set.

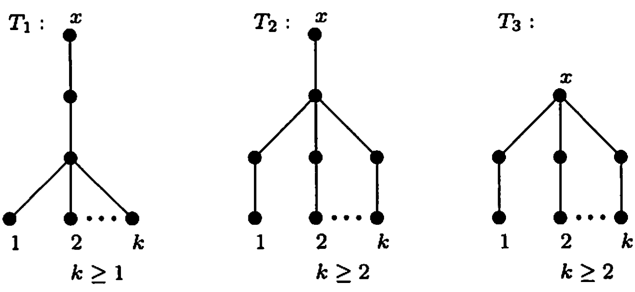

Figure 2: Types of trees T having $ir_t(T) = 1$.

In this paper we examine the effects on the total irredundance number of a graph G when G is modified by deleting or adding an edge. It is interesting that by adding an edge to a graph, it is possible to either increase or decrease the total irredundance number. In fact, the increase (respectively, decrease) can be arbitrarily large. For an example of an arbitrarily large decrease, consider the graph G_p formed from K_{p+2}, where two of its vertices are labelled u and v, by identifying each vertex in $K_{p+2} - \{u, v\}$ with an endvertex of a copy of P_3. See Figure 3 for the graph G_3. The only maximal total irredundant set of G_p is the set of endvertices, so $ir_t(G_p) = p$. Consider $G_p + ux$ where x is an endvertex. Then $\{v\}$ is a maximal total irredundant set of G_p+ux, and hence, $ir_t(G_p+ux) = 1$. On the other hand, for the subdivided star $K^*_{1,m}$, $ir_t(K^*_{1,m}) = 1$ and $ir_t(K^*_{1,m} + e) = m - 1$ for any edge added from the center to an endvertex, demonstrating an arbitrarily large increase. By the same token deleting an edge can also result in an arbitrarily large increase or decrease. Note that adding or deleting an edge need not change the total irredundance number. The graph G in Figure 4 has $0 = ir_t(G) = ir_t(G - uv) = ir_t(G + xv)$. Moreover, $ir_t(G - vy) = 1$ showing that a graph can have both "changing" and "unchanging" edges.

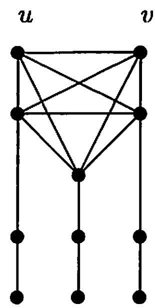

Figure 3: Graph G_3.

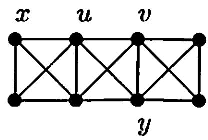

Figure 4: Graph G with "changing" and "unchanging" edges.

We say that a graph G is k^+-*stable* if $ir_t(G) = ir_t(G + e) = k$ for an arbitrary edge $e \in E(\overline{G})$. And similarly, a graph G is k^--*stable* if

$ir_t(G) = ir_t(G - e) = k$ for any edge $e \in E(G)$. A graph G is *k^+-unstable* if $ir_t(G) = k \neq ir_t(G + e)$ and *k^--unstable* if $ir_t(G) = k \neq ir_t(G - e)$ for any arbitrary edge $e \in E(\overline{G})$ and $e \in E(G)$, respectively.

Here we characterize the 0^+-stable, 0^--stable, 0^+-unstable, and 0^--unstable graphs. We begin with some background on $ir_t(0)$-graphs.

2 Graphs G with $ir_t(G) = 0$

Although $ir(G) \geq 1$ for all graphs G, as noted in the introduction, $ir_t(G)$ can equal zero. For example, the complete graph K_n has $ir_t(K_n) = 0$. Moreover, the graphs in Figures 4, 5, and 7 are $ir_t(0)$-graphs.

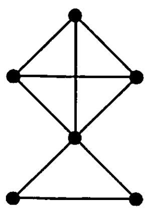

Figure 5: An $ir_t(0)$-graph.

The $ir_t(0)$-graphs were characterized in [5]. Before giving the characterization, we present some more terminology. For distinct vertices u and v, if $N[u] = N[v]$, we say that u and v are *clones* (also called *twins* in the literature). If $N[u] \subset N[v]$, that is, the neighborhood of v properly contains the neighborhood of u, then we say that v is *superior* to u and u is *inferior* to v. If a vertex v is not inferior, superior, or clone to any other vertex, then we say that v is *normal.* If v is superior (inferior) to every vertex in a set S, then we say that v is superior (inferior) to set S.

Notice that the relation "is a clone of" partitions the vertex set V into sets of clones and singleton sets (vertices with no clones). We call the sets of the partition *clone-sets.* Note that each singleton, although not a clone, is referred to as a clone-set. Note also that between any pair of clone-sets, either no edge is present or every possible edge is present. If every possible edge is present between two clone-sets A and B, we say that the vertices of A are *clone-adjacent* to the vertices of B, or just that the clone-sets A and B are clone-adjacent.

We repeat some observations from [5].

Observation 1 *Every clone-set induces a complete subgraph.*

Observation 2 *If $G \neq K_n$, then the complete subgraph induced by a clone-set is not maximal.*

Observation 3 *If G is an $ir_t(0)$-graph, then every vertex of minimum degree must have a clone.*

Note that a singleton set consisting of a normal vertex, or an inferior vertex that is not superior or clone vertex to another vertex, is a total irredundant set (though not necessarily maximal), so any graph G with such a vertex has $ir_t(G) \geq 1$. We now give the characterization of $ir_t(0)$-graphs.

Theorem 4 *A graph G is an $ir_t(0)$-graph if and only if every vertex is either superior or clone to another vertex.*

3 Edge Deletion

For any $ir_t(0)$-graph, deleting an edge cannot decrease the total irredundance number and hence we will be interested only in whether ir_t stays the same or increases. Recall from the introduction that deleting an edge can cause an arbitrarily large increase in the total irredundance number. However, our first result shows that this is not the case for $ir_t(0)$-graphs.

Theorem 5 *If G is an $ir_t(0)$-graph G and $e \in E(G)$, then $ir_t(G-e) \leq ir_t(G) + 2 = 2$.*

Proof. Let G be an $ir_t(0)$-graph. Suppose that for some edge $uv \in E(G)$, $ir_t(G-uv) \geq 3$. Let S be an ir_t-set of $G-uv$. Since $|S| \geq 3$, there exists a vertex, say x, in S such that $x \notin \{u,v\}$. Hence, $N_{G-uv}[x] = N_G[x]$. Since G is an $ir_t(0)$-graph, Theorem 4 implies that x is either a superior or clone vertex in G. If x is superior in G, then x is superior in $G-uv$. Hence, x cannot be in any total irredundant set of $G-uv$, contradicting the fact that $x \in S$. Therefore, x has a clone, say y, in G. If $y \notin \{u,v\}$, then y is a clone of x in $G-uv$, again contradicting the fact that $x \in S$. Hence, $y \in \{u,v\}$. But then x is superior to y in $G-uv$, contradicting the fact that $x \in S$. □

We note that the bound of Theorem 5 is sharp as can be seen with the graph $G = mK_2$, where $ir_t(G-e) = ir_t(G) + 2$ for any edge $e \in E(G)$. Our next result characterizes the edges $e \in E(G)$ for which $ir_t(G-e) = ir_t(G) = 0$.

Theorem 6 *For any $ir_t(0)$-graph G and edge $uv \in E(G)$, $ir_t(G-uv) = ir_t(G) = 0$ if and only if u and v are clone-adjacent, superior vertices in G.*

Proof. Let G be a graph with $ir_t(G) = 0$, and assume that $ir_t(G - uv) = ir_t(G)$ for an edge $uv \in E(G)$. Then either u and v are clone-adjacent or u and v are in the same clone-set in G. First, if u and v are clones, let $S = \{u\}$. Then, in $G - uv$, u is its own S-pn, and every vertex in the $N(u)$ has v as a private neighbor. Thus, S is a total irredundant set of $G - uv$ implying that $ir_t(G - uv) > ir_t(G)$, a contradiction. Therefore, u and v are clone-adjacent in G. If one of u and v, say u, is not a superior vertex, then since $ir_t(G) = 0$, u is in a clone-set of cardinality at least 2. Again let $S = \{u\}$. Then u is its own S-pn, and every vertex in the same clone-set as u has v as an S-pn. Furthermore, since u is not superior, any vertex $x \in N(u)$ that is not in the same clone-set as u, has an S-pn. Thus, S is a total irredundant set of $G - uv$, contradicting the fact that $ir_t(G) = ir_t(G - uv)$. Thus, u and v must be superior vertices in different clone-sets.

Conversely, let u and v be clone-adjacent, superior vertices in G and consider $G - uv$. Since u and v are clone-adjacent and both are superior, they are not superior to each other. Moreover, u (respectively, v) is not superior to a clone of v (respectively, u). Thus, u and v remain superior in $G - uv$ and hence cannot be in any total irredundant set, that is, $ir_t(G) = ir_t(G - uv) = 0$. □

We can now show there are no 0^--stable graphs.

Theorem 7 *No graph G is 0^--stable.*

Proof. If G is 0^--stable, then for every edge $uv \in E(G)$, $ir_t(G) = ir_t(G - uv) = 0$. Hence, by Theorem 6, u and v must be clone-adjacent, superior vertices. But G must have vertices that are not superior and each vertex that is not superior must have a clone (since $ir_t(G) = 0$). As in the proof to Theorem 6, if e is an edge between clones, then $ir_t(G) < ir_t(G - e)$, contradicting the fact that G is 0^--stable. □

The characterization of 0^--unstable graphs follows directly.

Theorem 8 *A connected $ir_t(0)$-graph G is 0^--unstable if and only if no two superior vertices in G are clone-adjacent.*

As previously mentioned, $G = mK_2$ is 0^--unstable. For other examples of 0^--unstable graphs, see Figures 6 and 7.

4 Edge Addition

We now consider the cases 0^+-stable and 0^+-unstable. Again since we are assuming that the graphs G have $ir_t(G) = 0$, the addition of an edge

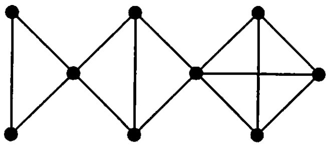

Figure 6: A 0^--unstable graph.

cannot decrease the total irredundance number, so the total irredundance number either stays the same or increases. Although as we showed in the introduction adding an edge can result in an arbitrarily large increase in the total irredundance number, for $ir_t(0)$-graphs this increase is at most 1.

Theorem 9 *If G is an $ir_t(0)$-graph G and $e \in E(\overline{G})$, then $ir_t(G+e) \leq 1$.*

Proof. Let G be an $ir_t(0)$-graph and let $uv \in E(\overline{G})$. Suppose that $ir_t(G+uv) \geq 1$. Let S be an ir_t-set of $G+uv$ and let $x \in S$. We show that $|S| = 1$. By Theorem 4, every vertex of G belongs to a clone-set of cardinality at least 2 or is a superior vertex.

The set S contains no superior vertex of $G+uv$ and no vertex of $G+uv$ that belongs to a clone-set of cardinality 2 or more. In particular, since each of u and v is a superior vertex in $G+uv$, neither u nor v is in S.

Let w be a superior vertex in G. Then there is a vertex w' that is inferior to w, i.e., $N[w'] \subset N[w]$. Since G is an $ir_t(0)$-graph, w' is in a clone-set of cardinality at least 2. Every vertex in this clone-set containing w' is inferior to w, and so w is superior to at least two vertices in G. It follows that w is also a superior vertex in $G+uv$. Hence any superior vertex in G is also a superior vertex in $G+uv$. Thus, no superior vertex of G is in S.

Let z belong to a clone-set of G not containing u or v. Suppose z is not a superior vertex of G. Then, z belongs to a clone-set of cardinality at least 2 in G. Let z' be a clone of z. Then, in $G+uv$, $N[z] = N[z']$ and so z and z' are clones in $G+uv$. Thus, $z \notin S$. Hence, S contains no vertex that is in a clone-set of G not containing u or v.

Let X and Y denote the clone-sets containing u and v, respectively. Since $u, v \notin S$, and since each vertex of S belongs to X or Y, we may assume that $x \in X$. If X contains a vertex different from u and x, then such a vertex would have no S-pn, a contradiction. Hence, $|X| = 2$. Furthermore, in $G+uv$, $N[u] = N[x] \cup \{v\}$. Thus, v is the unique S-pn of u. This implies that S contains no vertex of Y, and so x is only vertex of S, i.e., $|S| = 1$. □

Theorem 10 *For any $ir_t(0)$-graph G and edge $uv \in E(\overline{G})$, $ir_t(G) = ir_t(G+uv)$ if and only if each of u and v is either a superior vertex of G or in a clone-set of G with cardinality at least 3.*

Proof. Let $ir_t(G) = ir_t(G+uv) = 0$ for $uv \in E(\overline{G})$. Then u and v are in different clone-sets and these clone-sets are not clone-adjacent. Also, by Theorem 4 every vertex in $G+uv$ belongs to a clone-set of cardinality at least 2 or is a superior vertex. Moreover, since $ir_t(G) = 0$ implies that every vertex of G belongs to a clone-set of cardinality at least 2 or is a superior vertex, each of u and v is a superior vertex in $G+uv$. If u and v are superior in G, then they are still superior in $G+uv$. Suppose that u is not superior in G and the clone-set of G containing u has cardinality 2. Let $S = \{u'\}$, where u' is the clone of u. Then in $G+uv$, u' is its own S-pn and u has v as its S-pn. But since u is not superior in G, its clone u' is not superior, and hence every vertex in $N[u'] - \{u\}$ has an S-pn. Thus, S is a total irredundant set of $G+uv$ implying that $ir_t(G+uv) \geq 1$, which is a contradiction. Therefore, in G each of u and v is either superior or in a clone-set of cardinality at least 3.

Conversely, let each of u and v be a superior vertex or be in a clone-set of cardinality at least 3 in G. Then in $G+uv$, u and v become superior vertices; while the other vertices belong to a clone-set of cardinality at least 2 or are superior vertices. Hence, $ir_t(G+uv) = 0 = ir_t(G)$. □

We can now characterize the connected 0^+-stable graphs.

Theorem 11 *A connected $ir_t(0)$-graph G is 0^+-stable if and only if one of the following holds*

(i) *G is a complete graph,*

(ii) *G contains no clone-set of cardinality 2,*

(iii) *G has exactly one clone-set of cardinality 2 and this set dominates G.*

Proof. First we consider the sufficiency. If G is a complete graph, then vacuously G is 0^+-stable. Since $ir_t(G) = 0$, Theorem 4 states that every vertex belongs to a clone-set of cardinality at least 2 or is a superior vertex. Suppose G has no clone-set of cardinality 2 and let $e \in E(\overline{G})$. Then in $G+e$ each endvertex of e becomes a superior vertex, while all other vertices belong to a clone-set of cardinality at least 2 or are superior vertices. Hence, $ir_t(G+e) = 0$. Suppose G has exactly one clone-set $\{x,y\}$ of cardinality 2 and that $\{x,y\}$ dominates G. Since $ir_t(G) = 0$, every vertex different from x and y is a superior vertex or belongs to a clone-set of cardinality at least 3. Now let $e \in E(\overline{G})$, and again each endvertex of e is a superior vertex in

$G+e$, while all other vertices belong to a clone-set of cardinality at least 2 or are superior vertices in $G+e$. Hence, $ir_t(G+e)=0$. This proves the sufficiency.

Next we consider the necessity. Suppose $G \neq K_n$ is 0^+-stable. Then for any edge $e \in E(\overline{G})$, $ir_t(G) = ir_t(G+e) = 0$. From Theorem 10, we know that for any edge $e = uv$ such that $ir_t(G) = ir_t(G+uv)$, each of u and v is either a superior vertex or in a clone-set of cardinality at least 3 in G. In particular, if G has a clone-set $\{x,y\}$ of cardinality 2, then Theorem 10 implies that $\{x,y\}$ dominates G. Thus, $\{x,y\}$ is the only clone-set of cardinality 2 in G. □

Theorem 12 *A connected $ir_t(0)$-graph $G \neq K_n$ is 0^+-unstable if and only if for each pair of non-adjacent clone-sets, at least one of the clone-sets has cardinality* 2 *and contains no superior vertex.*

Proof. Let G be a connected 0^+-unstable graph and let $uv \in E(\overline{G})$. Then, $ir_t(G+uv) \geq 1$ and both u and v are superior in $G+uv$. In fact, by Theorem 9, $ir_t(G+uv) = 1$. Let $S = \{x\}$ be an ir_t-set of G. Let A and B be the clone-sets containing u and v, respectively. It follows from the proof of Theorem 9, that $x \in A$ or $x \in B$. Furthermore, the clone-set that contains x has cardinality 2 and has no superior vertex.

Conversely, suppose that G is not complete and for each pair of non-adjacent clone-sets, at least one has cardinality 2 and contains no superior vertex. Let $uv \in E(\overline{G})$ and let A and B be the clone-sets containing u and v, respectively. We may assume that $|A| = 2$ and that A contains no superior vertex. Let $A = \{u, u'\}$. Then, $\{u'\}$ is a total irredundant set of $G+uv$, and so $ir_t(G+uv) \geq 1$. By Theorem 9, $ir_t(G+uv) = 1$. Since uv is an arbitrary edge of $\overline{G}$, this shows that G is 0^+-unstable. □

For an example of a 0^+-unstable graph, see the graph in Figure 7.

5 Concluding Remarks

In this section we consider the classes of induced subgraphs of $ir_t(0)$-graphs, and of 0^--stable, 0^+-stable and 0^+-unstable graphs. In particular, we show that any graph G is an induced subgraph of some graph in each of these classes. All of these results are based on the following operations on graphs.

The *composition* of graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, in which vertices (u,v) and (u',v') are adjacent if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. In other words, the composition $G[H]$ is the graph obtained by replacing each vertex in G with a copy of H and joining every vertex in the copy associated with a vertex u to every vertex in the copy associated with a vertex v, if vertices

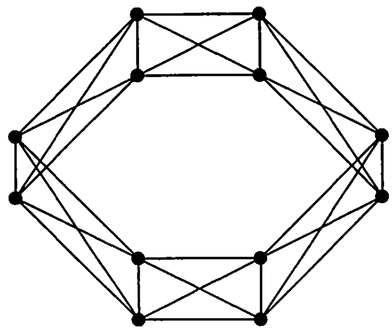

Figure 7: A graph with clone-sets of cardinality 2.

u and v are adjacent in G. For an illustration, the graph in Figure 7 is the composition $C_6[K_2]$.

The *corona* $G \circ K_1$ is the graph obtained from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and the pendant edge vv' are added. The generalized 2-*corona* is obtained from a copy of G, where for each vertex $v \in V(G)$, two new vertices v' and v'', and the edges vv' and $v'v''$ are added (that is, for each vertex in $V(G)$, a pendant path of length 2 is added by identifying an endvertex of the new path P_3 with v).

Obviously, the graph G is an induced subgraph of each of the corona $G \circ K_1$, the 2-corona of G, and the composition graph $G[H]$. Also, we note that every vertex in $V(G)$ is superior in $G \circ K_1$, and no vertex of G is superior in the 2-corona of G. Moreover, if a vertex v is superior in G, then v is also a superior vertex in $G[H]$.

Theorem 13 *Every graph G is an induced subgraph of an $ir_t(0)$-graph.*

Proof. Any arbitrary graph G is an induced subgraph of the composition $G[K_2]$. But every vertex in $G[K_2]$ is clone to another vertex. In particular, let the two vertices of K_2 be labelled a and b, and let $u \in V(G)$. Then vertices (u, a) and (u, b) are clones in the graph $G[K_2]$. Therefore, $G[K_2]$ is an $ir_t(0)$-graph. □

Corollary 14 *There does not exist a forbidden subgraph characterization of the class of $ir_t(0)$-graphs.*

We note that the graph $G[K_2]$ is not necessarily the smallest $ir_t(0)$-graph which contains G as an induced subgraph. A smaller embedding of

G as an induced subgraph of an $ir_t(0)$-graph can be obtained by *cloning* only the vertices of G which are neither superior or clone to another vertex in G.

Theorem 15 *Every connected graph G is an induced subgraph of a 0^+-unstable graph.*

Proof. Let G be an arbitrary connected graph of order n. We construct a 0^+-unstable graph H as follows. Begin with the graph $G \cup (K_n \circ K_1)$, where $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(K_n) = \{u_1, u_2, ..., u_n\}$, and add edges $v_i u_i$ for $1 \leq i \leq n$. Call this graph G'. Obviously, G is an induced subgraph of G'. Note that if two vertices are clones in G, they are not clones in G'. Furthermore, no new clones sets are formed in G', that is, no vertex of G' has a clone. Moreover, the only superior vertices in G' are the vertices of $V(K_n)$.

Now let $H = G'[K_2]$. Then every clone-set of H has cardinality 2 and the only superior vertices of H are clone-adjacent. Thus, $ir_t(H) = 0$. According to Theorem 12, a graph is 0^+-unstable if and only if for each pair of non-adjacent clone-sets, at least one of the clone-sets has cardinality 2 and contains no superior vertex. Hence, H is a 0^+-unstable graph with G as an induced subgraph. □

Corollary 16 *There does not exist a forbidden subgraph characterization of the class of 0^+-unstable graphs.*

Theorem 17 *Every connected graph G is an induced subgraph of a 0^+-stable graph.*

Proof. According to Theorem 11, a connected $ir_t(0)$-graph is 0^+-stable if and only if one of the following holds:

(i) G is a complete graph,

(ii) G contains no clone-set of cardinality 2,

(iii) G contains exactly one clone-set of cardinality 2 and this set dominates G.

Condition (ii) is satisfied by every connected graph of the form $G[K_3]$. In particular, graphs of the form $G[K_3]$ are $ir_t(0)$-graphs, since every vertex in such a graph is clone to two other vertices. □

Corollary 18 *There does not exist a forbidden subgraph characterization of the class of 0^+-stable graphs.*

Theorem 19 *Every graph G is an induced subgraph of a 0^--unstable graph.*

Proof. Let G be a connected graph. We construct a 0^--unstable graph H as follows. Begin with the 2-corona of G and call this graph G'. Note that the only superior vertices in G' are the vertices superior to an endvertex, and no two of these are clone-adjacent. Let $H = G'[K_2]$. Then every vertex of H is in a clone-set of cardinality 2, so H is an $ir_t(0)$-graph. Furthermore, no two superior vertices of H are clone-adjacent. According to Theorem 8, a graph is 0^--unstable if and only if no two superior vertices are clone-adjacent. Hence, H is a 0^--unstable graph with G as an induced subgraph. □

Corollary 20 *There does not exist a forbidden subgraph characterization of the class of 0^--unstable graphs.*

We conclude by mentioning two potential areas for future research.

1. Characterize $ir_t(k)$-graphs for $k \geq 1$.
2. Investigate the k^--stable, k^--unstable, k^+-stable, and k^+-unstable graphs for $k \geq 1$.

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