# On $\alpha$-valuations of disconnected graphs

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**Abstract**

We introduce the concept of a *free* $\alpha$-valuation of a graph, and prove that the vertex-disjoint union of any collection of graphs with free $\alpha$-valuations has an $\alpha$-valuation. Many bipartite graphs have free $\alpha$-valuations, including the complete bipartite graph $K_{m,n}$ when $m > 1$ and $n > 2$, and the $d$-cube $Q_d$ for $d > 2$.

# 1 Introduction

Let $N$ denote the set of nonnegative integers, and denote the set of integers $\{m, m+1, \ldots, n\}$ by $[m, n]$. For any graph $G$ we call an injective function $h : V(G) \to N$ a *valuation* of $G$. Rosa [16] called such a function $h$ on a graph $G$ with $q$ edges a *$\beta$-valuation* if $h$ is an injection from $V(G)$ into $[0, q]$ such that

$$\{|h(u) - h(v)| : \{u, v\} \in E(G)\} = [1, q].$$

The number $|h(u) - h(v)|$ is called the *label* of the edge $\{u, v\}$. A $\beta$-valuation is now more commonly called a *graceful labeling*. An *$\alpha$-valuation* is a graceful labeling having the additional property that there exists an integer $\lambda$ such that if $\{u, v\} \in E(G)$, then $\{u, v\} = \{a, b\}$, where $h(a) \leq \lambda < h(b)$. The number $\lambda$, which is unique, is called the *critical value* of the $\alpha$-valuation. Let $h(G) = \{h(v) : v \in V(G)\}$. Note that necessarily $0$, $\lambda$, $\lambda+1$, and $|E(G)|$ are in $h(G)$. Moreover, $G$ must be bipartite. If $h$ is an $\alpha$-valuation of $G$, then so is $|E(G)| - h$.

Numerous large classes of bipartite graphs have $\alpha$-valuations; examples include complete bipartite graphs, caterpillars, $d$-cubes and cycles of length $4k$ (see Gallian [10] for a survey). Valuations of graphs are particularly interesting because of their applications to graph decompositions. It is well known that if a graph $G$ with $q$ edges admits an $\alpha$-valuation, then the edge-sets of $K_{2qx+1}$ and $K_{qx,qy}$ can be partitioned into subgraphs isomorphic to $G$ for all positive integers $x$ and $y$ (see [16] and [5]).

According to the latest survey by Joseph Gallian [10], there are over 300 papers on graph labelings and related topics. One section of [10] is dedicated to surveying valuations of disconnected graphs. Some of the investigated disconnected graphs are $K_m \cup K_n$ (in [18]), $K_{m,n} \cup G$ (see [4]), $C_{4t} \cup K_{1,4t-1}$ and $C_{4t+3} \cup K_{1,4t+2}$ (in [3]), $C_s \cup P_n$ (in [9] and [8]), and the graph consisting of unions of cycles (see [11], [13], [14], [1], [2], and [7]). Unfortunately, some authors do not distinguish between $\beta$- and $\alpha$-valuations.

The late Anton Kotzig was the first to consider graphs that are the disjoint union of $r$ cycles of length $s$, denoted by $rC_s$. In 1975 he proved in [11] that when $r = 3$ and $s = 4k > 4$, then $rC_s$ has an $\alpha$-labeling, whereas when $r \geq 2$ and $s = 3$ or 5, $rC_s$ is not graceful. He also proved that $C_{2m} \cup C_{2m}$ has an $\alpha$-valuation for all $m > 1$.

Abrham and Kotzig showed in [1] that $rC_4$ has an $\alpha$-valuation if and only if $r \neq 3$. They also showed that $3C_4$ is graceful. Eshghi [7] proved that a 2-regular bipartite graph with 3 components has an $\alpha$-labeling if and only if the number of edges is a multiple of four except for the case $3C_4$.

In [2] Abrham and Kotzig proved that $C_p \cup C_q$ is graceful if and only if $p + q \equiv 0$ or 3 (mod 4). They also proved that $C_m \cup C_n$ has an $\alpha$-labeling if and only if both $m$ and $n$ are even and $m \equiv n$ (mod 4).

We note two additional results. Zhou [18] proved that $K_m \cup K_n$ is graceful if and only if $\{m, n\} = \{4, 2\}$ or $\{5, 2\}$. Bu and Cao [4] give some sufficient conditions for the gracefulness of graphs of the form $K_{m,n} \cup G$ and prove that $K_{m,n} \cup P_t$ and the disjoint union of complete bipartite graphs are graceful under some conditions.

In this paper we define what it means for an $\alpha$-valuation to be free, along with the weaker properties of being left-free or right-free. We show that the vertex-disjoint union of a finite collection of graphs with $\alpha$-valuations, all free except that one may be left-free and another right-free, has an $\alpha$-valuation. We prove that various graphs have free $\alpha$-valuations, including most complete bipartite graphs and cubes.

## 2 Free $\alpha$-valuations

Let $G$ be a graph with $\alpha$-valuation $h$ and critical value $\lambda$. We say that $h$ is *left-free* if $\lambda > 0$ and $1 \notin h(G)$, and *right-free* if $\lambda > 0$ and $\lambda - 1 \notin h(G)$. In either case we must have $\lambda > 1$ since $\lambda$ and 0 are in $h(G)$. We call $h$ *free* if it is both left-free and right-free and $\lambda > 2$.

**Theorem 1** *Let $G_i$ be a graph with $\alpha$-valuation $h_i$ and critical value $\lambda_i$ for $i = 1, 2$. Suppose $h_1$ is right-free and $h_2$ is left-free. Then the vertex-disjoint union $G_1 \bigcup G_2$ is a graph with an $\alpha$-valuation $h$ with critical value*

$\lambda_1 + \lambda_2 - 1$. *If $h_1$ is right-free and $h_2$ is free, then $h$ is right-free. If $h_1$ is free and $h_2$ is left-free, then $h$ is left-free. If $h_1$ and $h_2$ are both free, then so is $h$.*

**Proof:** Let $G$ be the vertex-disjoint union of $G_1$ and $G_2$. Let $V(G_i) = X_i \bigcup Y_i$, where if $v \in X_i$, then $h_i(v) \leq \lambda_i$, and if $v \in Y_i$, then $h_i(v) > \lambda_i$, $i = 1, 2$. We define $h$ on $V(G)$ to be $h_1$ on $X_1$, $h_2 + \lambda_1 - 1$ on $X_2 \bigcup Y_2$, and $h_1 + |E(G_2)|$ on $Y_1$.

Notice that the values of $|h(u) - h(v)|$ on edges $uv$ with $u \in X_2$ and $v \in Y_2$ are the same as the values of $|h_2(u) - h_2(v)|$ on the edges $uv$ of $G_2$, namely $1, 2, \ldots, |E(G_2)|$, while the values of $|h(u) - h(v)|$ on the edges between vertices of $X_1$ and $Y_1$ are the same as the values of $|h_1(u) - h_1(v)| + |E(G_2)|$ on edges $\{u, v\}$ of $G_1$, namely $|E(G_2)|+1, |E(G_2)|+2, \ldots, |E(G_2)|+|E(G_1)|$. Thus

$$\{|h(u) - h(v)| : \{u, v\} \in E(G)\} = \{1, 2, \ldots, |E(G)|\}.$$

Now we show that $h$ is one-to-one on $V(G)$. It suffices to show that $h(X_1) \bigcap h(X_2) = h(Y_2) \bigcap h(Y_1) = \emptyset$. The largest element of $h(X_1)$ is $\lambda_1$, and the next smallest is less than $\lambda_1 - 1$ because $G_1$ is right-free. But the smallest element of $h(X_2)$ is $\lambda_1 - 1$, and the next largest is greater than $\lambda_1$ because $G_2$ is left-free. Thus $h(X_1) \bigcap h(X_2) = \emptyset$. Likewise the largest element of $h(Y_2)$ is $|E(G_2)| + \lambda_1 - 1$, and the smallest element of $h(Y_1)$ is $\lambda_1 + 1 + |E(G_2)|$. Thus $h(Y_2) \bigcap h(Y_1) = \emptyset$.

We see that $h$ is an $\alpha$-valuation for $G$ with critical value $\lambda = \lambda_1 + \lambda_2 - 1 > 0$.

Now assume that $h_1$ is right-free and $h_2$ is free. Then $\lambda_2 > 2$ and $\lambda_2 - 1 \notin h_2(G_2)$. Then $\lambda_2 - 1 + \lambda_1 - 1 = \lambda - 1 \notin h(X_2)$. Also the largest element of $h(X_1)$ is $\lambda_1$, and $\lambda_1 < \lambda_1 + (\lambda_2 - 2) = \lambda - 1$. Thus $\lambda - 1 \notin h(X_1)$ also, and so $h$ is right-free.

The proofs of the last two sentences of the theorem are similar. ∎

The following is an immediate consequence of Theorem 1.

**Theorem 2** *Let $G_1, G_2, \ldots, G_n$ be graphs with $\alpha$-valuations $h_1, h_2, \ldots, h_n$. Suppose that $h_1$ is right-free, $h_2, h_3, \ldots, h_{n-1}$, are free, and $h_n$ is left-free. Then the vertex-disjoint union $G_1 \bigcup G_2 \bigcup \cdots \bigcup G_n$ has an $\alpha$-valuation.*

# 3 Applications

In order for the last theorem to be useful we need some classes of graphs with free $\alpha$-valuations. The valuation used in the following theorem is essentially the one given by Rosa in [16].

**Theorem 3** *Let $m$ and $n$ be integers with $2 \le m \le n$. Then $K_{m,n}$ has an $\alpha$-valuation that is left-free and right-free. If also $2 < n$ then it is free.*

**Proof:** Let $V(K_{m,n}) = X \bigcup Y$, where $|X| = m$, $|Y| = n$, and each edge joins a vertex of $X$ to a vertex of $Y$. Let $h(X) = \{0, n, 2n, \ldots, (m-1)n\}$ and $h(Y) = \{(m-1)n+1, (m-1)n+2, \ldots, mn\}$. This is an $\alpha$-valuation with $\lambda = (m-1)n$, and is left- and right-free since $n > 1$. Note that $\lambda \ge n$. Thus if $n > 2$ the $\alpha$-valuation is free. ∎

The *cartesian product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2 = (V, E)$ where $V = V_1 \times V_2$ and $E = \{\{(u_1, u_2), (v_1, v_2)\} : u_1 = v_1 \text{ and } \{u_2, v_2\} \in E_2 \text{ or } u_2 = v_2 \text{ and } \{u_1, v_1\} \in E_1\}$. In particular, the *d-cube*, denoted $Q_d$, is the product of $d$ copies of $K_2$. Kotzig showed that $d$-cubes have $\alpha$-valuations in [12]. In order to show that $Q_d$ has a free $\alpha$-valuation for $d > 2$ we will prove a more general result about a type of valuation introduced by Maheo in [15]. We say that $h$ is a *strong $\alpha$-valuation* for the graph $G$ if $h$ is an $\alpha$-valuation for $G$ with critical value $\lambda$ satisfying the following conditions.

(1) If $X$ is the set of vertices of $G$ with $h(v) \le \lambda$ and $Y = V(G)\backslash X$, then $|X| = |Y| = s$ and $|E(G)| = 2l + s$ for nonnegative integers $l$ and $s$.

(2) There exists an automorphism $\pi$ taking $X$ onto $Y$ that is its own inverse.

(3) If $x \in X$, then $\{x, \pi(x)\} \in E(G)$ and $l + 1 \le h(\pi(x)) - h(x) \le l + s$.

Maheo (who used the term "strongly graceful") proved that if $G$ has a strong $\alpha$-valuation, then so does $G \times K_2$. This useful result yields another proof that cubes have $\alpha$-valuations. It turns out that her proof implies something more.

**Theorem 4** *If the graph $G$ has a free strong $\alpha$-valuation, then so does $G \times K_2$.*

**Proof:** Suppose $G$ has the free strong $\alpha$-valuation $h$, with $\lambda, X, Y, l, s$, and $\pi$ as in the definition above. Since $h$ is free, $2 \le s \le \lambda$. We consider $G \times K_2$ to have vertices $(v, i)$, where $v \in V(G)$ and $i$ is 0 or 1 and edges $\{(x, i), (y, i)\}, i = 0, 1$, where $\{x, y\} \in E(G)$ along with $\{(v, 0), (v, 1)\}, v \in V(G)$.

Maheo [15] proves that the map $h'$ defined as follows is a strong $\alpha$-valuation on $G \times K_2$:

$$h'((v,i)) = \begin{cases} h(v) & \text{if } v \in X \text{ and } i = 0 \\ h(\pi(v)) + l + s & \text{if } v \in X \bigcup Y \text{ and } i = 1 \\ h(v) + 2l + 3s & \text{if } v \in Y \text{ and } i = 0. \end{cases}$$

The critical value of $h'$ is $\lambda' = \lambda + l + s = h'((\pi(x), 1))$, where $h(x) = \lambda$. Thus $\lambda' > 2$.

Now suppose $h'((v, i)) = 1$. Then $v \in X$, $i = 0$, and $h(x) = 1$, which contradicts the assumption that $h$ is free. Likewise suppose $h'((v, i)) = \lambda' - 1 = \lambda + l + s - 1$. This implies $i = 1$ and $h(\pi(v)) = \lambda - 1$, again a contradiction. Thus $h'$ is free. ∎

In [6] it is proved that if $m$ is any positive integer, then $K_{m,2} \times K_2$ has a strong $\alpha$-valuation. It turns out that if $m > 1$ then this valuation is also free, and so we get the following result.

**Theorem 5** *Let $m$ and $n$ be positive integers, $m > 1$. Then $K_{m,2} \times Q_n$ has a free $\alpha$-valuation.*

**Proof:** In the proof in [6] mentioned above the graph $K_{m,2} \times K_2$ is represented as having vertices $y_0, y_1, \ldots, y_{m+1}, z_0, z_1, \ldots, z_{m+1}$ and edges $\{y_i, y_j\}, \{z_i, z_j\}$ for $i = 0, m+1, 1 \le j \le m$, and $\{y_i, z_i\}$ for $0 \le i \le m+1$. It is proved that a strong $\alpha$-valuation $h$ is defined by

$$h(y_i) = \begin{cases} m & \text{if } i = 0 \\ 5m + 3 - i & \text{if } 1 \le i \le m \\ 0 & \text{if } i = m + 1, \end{cases}$$

$$h(z_i) = \begin{cases} 3m + 1 & \text{if } i = 0 \\ 3m - 2i + 2 & \text{if } 1 \le i \le m \\ 3m + 2 & \text{if } i = m + 1. \end{cases}$$

(The case $m = 4$ is shown in Figure 1.)

Note that $h(z_0) - h(z_1) = 1$, so the critical value of $h$ is $\lambda = h(z_1) = 3m > 2$. Since for $1 \le i \le m$ we have $h(y_i) \ge 4m + 3$ and $h(z_i) \ge m + 2$, if $m > 1$ then no vertex has value 1. Now suppose for some vertex $v$ we have $h(v) = \lambda - 1 = 3m - 1$. Clearly $v = z_i$ for $1 \le i \le m$. But then $3m - 1 = 3m - 2i + 2$, which is impossible. Thus $h$ is a free strong $\alpha$-valuation, and the present theorem follows from Theorem 4. ∎

Notice that $K_{2,2} = K_2 \times K_2 = Q_2$. The $\alpha$-valuation given for $K_{2,2}$ in Theorem 3 is left- and right-free, but not free. However, by taking $m = 2$ in the last theorem we get the following.

**Theorem 6** *The cube $Q_d$ has an $\alpha$-valuation that is left-free and right-free if $d = 2$ and free if $d > 2$.*

# 4 A generalization

The method of proof in Theorem 1 can be generalized as follows.

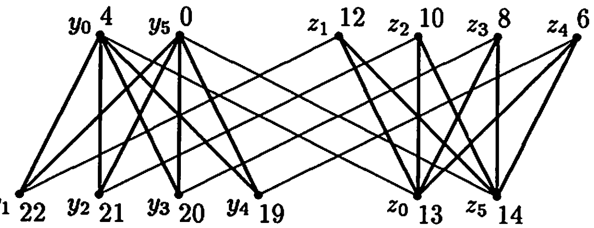


Figure 1: A free strong $\alpha$-valuation for $K_{2,4} \times K_2$

**Theorem 7** *Let $G_i$ be a graph with $\alpha$-valuation $h_i$ and critical value $\lambda_i$ for $i = 1, 2$. Suppose $k$ is an integer with $0 < k < \lambda_1$ and let $X_1$ be the set of vertices $x$ of $G_1$ with $h_1(x) \leq \lambda_1$. Suppose that whenever $x \in X_1$, then $h_1(x) - k \notin h_2(G_2)$. Then the vertex-disjoint union of $G_1$ and $G_2$ has an $\alpha$-valuation.*

**Proof:** Let $G$ be the vertex-disjoint union of $G_1$ and $G_2$, and assume $V(G_i) = X_i \bigcup Y_i$, where $h_i(v) \leq \lambda_i$ for $v \in X_i$ and $h_i(v) > \lambda_i$ for $v \in Y_i$, $i = 1, 2$. We define $h$ on $G$ to be $h_1$ on $X_1$, $h_2 + k$ on $X_2 \bigcup Y_2$, and $h_1 + |E(G_2)|$ on $Y_1$.

The proof that the edge labels of $G$ with respect to $h$ are exactly $\{1, 2, \ldots, |E(G)|\}$ is the same as in the proof of Theorem 1. Thus to complete the proof that $h$ is an $\alpha$-valuation it suffices to show that the sets $h(X_1), h(Y_1), h(X_2)$ and $h(Y_2)$ are pairwise disjoint, since $0 \leq h(v) \leq |E(G_1)| + |E(G_2)|$ for all $v \in V(G)$.

It is clear that $h(X_1) \bigcap h(Y_1) = h(X_2) \bigcap h(Y_2) = \emptyset$. Suppose that $h(X_1) \bigcap h(X_2 \bigcup Y_2) \neq \emptyset$. Then there exist $x \in X_1$ and $v \in X_2 \bigcup Y_2$ such that $h_1(x) = h_2(v) + k$. But this contradicts the hypothesis of this theorem. Finally, suppose $y \in Y_1$ and $v \in X_2 \bigcup Y_2$. Then $h(y) = h_1(y) + |E(G_2)| \geq \lambda_1 + 1 + |E(G_2)|$, while $h(v) = h_2(v) + k \leq |E(G_2)| + \lambda_1 - 1$. Thus $h(Y_1) \bigcap h(X_2 \bigcup Y_2) = \emptyset$, and the proof is completed. ∎

We offer the following as an application of Theorem 7.

**Theorem 8** *If $p$ is any positive integer, then the graph consisting of two vertex-disjoint copies of $C_{4p} \times K_2$ has an $\alpha$-valuation.*

**Proof:** In [17] Snevily gives an $\alpha$-valuation $h_1$ for $G_1 = C_{4p} \times K_2$ as part of a proof that $C_{4p} \times Q_n$ has an $\alpha$-valuation for all $p$ and $n$. The only information we need about this valuation is that it has critical value $\lambda_1 = 6p - 1$ and that if $X_1$ and $Y_1$ are the sets of vertices with values $\leq \lambda_1$ and $> \lambda_1$, respectively, then $h_1(X_1) \subseteq [0, 2p-1] \bigcup [4p, 6p-1]$ and $h_1(Y_1) \subseteq [6p, 12p]$. The case $p = 2$ is shown in Figure 2.

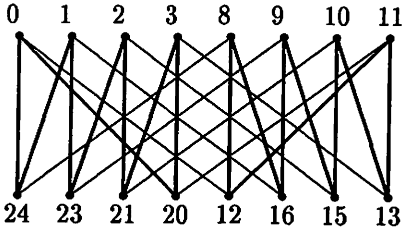


Figure 2: An $\alpha$-valuation for $C_8 \times K_2$

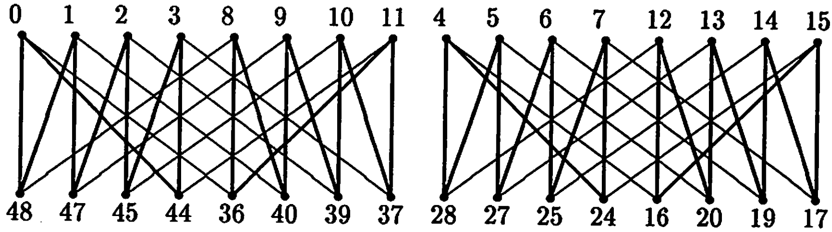


Figure 3: An $\alpha$-valuation for two copies of $C_8 \times K_2$

Now let $G_2$ be a disjoint copy of $G_1$, with $h_2$ defined in the same way. We take $k = 2p$ in Theorem 7. Suppose $x \in X_1$. Then $h_1(x) - k \in [-2p, -1] \bigcup [2p, 4p - 1]$, which does not intersect $h_2(G_2)$. The resulting $\alpha$-valuation when $p = 2$ is shown in Figure 3. ∎

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