Realizability of p-Point, q-Line Graphs with Prescribed Maximum Degree, and Point Connectivity

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Abstract

It is well known that some graph-theoretic extremal questions play a significant role in the investigation of communication network vulnerability. Answering questions concerning the realizability of graph invariants also solves several of these extremal problems. We define a (p, q, κ, Δ) graph as a graph having p points, q lines, point connectivity κ and maximum degree Δ. An arbitrary quadruple of integers (a, b, c, d) is called (p, q, κ, Δ) realizable if there is a (p, q, κ, Δ) graph with $p = a$, $q = b$, $\kappa = c$ and $\Delta = d$. Necessary and sufficient conditions for a quadruple to be (p, q, κ, Δ) realizable are derived. In earlier papers, Boesch and Suffel gave necessary and sufficient conditions for (p, q, κ), (p, q, λ), (p, q, δ), $(p, \Delta, \delta, \lambda)$ and $(p, \Delta, \delta, \kappa)$ realizability, where λ denotes the line connectivity of a graph and δ denotes the minimum degree for all points in a graph.

ARS COMBINATORIA 61(2001), pp. 137-147

Introduction

Here we consider an undirected graph G = (V, X) with a finite point set V and a set X whose elements, called lines, are two point subsets of V. The number of points is denoted by p, and the number of lines $|X|$ is denoted by q(G) or q. This paper uses the notation and terminology of Harary [14]; however a few basic concepts are now reproduced.

The line connectivity of a graph G (denoted by $\lambda(G)$ or λ) is the minimum number of lines whose removal results in a disconnected graph. A graph is called trivial if it has just one point. The point connectivity (denoted by $\kappa(G)$ or κ) is the minimum number of points whose removal results in a disconnected or trivial graph. The number of lines connected to a point v of G is the degree of that point, denoted by $d_v(G)$ or d_v. The minimum degree is denoted by δ or $\delta(G)$ and the maximum degree is denoted by Δ or $\Delta(G)$. If $\delta = \Delta$, the graph is called regular. A p point graph with $\delta = p-1$ is called complete and is denoted by K_p. A set of κ points whose removal disconnects G, or makes G trivial, is called a minimum point disconnecting set. The graph obtained from C_p (the cycle on p points) by adding lines between all pairs of points that are distance at least two but not greater than A apart is denoted by C_p^A.

It is well known that some graph-theoretic extremal questions play a significant role in the investigation of communication network vulnerability [1-13]. Harary [15] found the maximum point connectivity among all graphs with a given number of points and a given number of lines. Answering questions concerning the realizability of graph invariants also solves several of these extremal problems. We define a (p, q, κ, Δ) graph as a graph having p points, q lines, point connectivity κ and maximum degree Δ. An arbitrary quadruple of integers (a, b, c, d) is called (p, q, κ, Δ) realizable if there is a (p, q, κ, Δ) graph with p = a, q = b, κ = c and Δ = d. Necessary and sufficient conditions for a quadruple to be (p, q, κ, Δ) realizable (or, more briefly, realizable) are derived. Boesch and Suffel derived necessary and sufficient conditions for (p, q, κ), (p, q, λ), (p, q, δ), $(p, \Delta, \delta, \lambda)$ and $(p, \Delta, \delta, \kappa)$ realizability in earlier papers [6 - 8].

Preliminaries

First we recall a result given by Harary [15].

Lemma 1: If $2 \le \delta \le p-1$, then there is a graph on p points with $q = \lceil \frac{1}{2} p \delta \rceil$ and $\lambda = \delta = \kappa$. (This graph is a power of cycle and is usually called the Harary graph on p points).

The next lemma contains a complete list of the relevant lower bounds for q.

Lemma 2: The following holds;
(1) $\Delta \le q$,
(2) if $\kappa = 1$, then $p - 1 \le q$,
(3) if $\kappa = 2$, then $p + \Delta - 2 \le q$, and
(4) $\lceil \frac{1}{2}((p-1)\kappa + \Delta) \rceil \le q$.

Proof: We pause to note that (1) and (4) are pertinent when $\kappa = 0$ and $\kappa \ge 3$, respectively. As (1) is obvious and (2) is well known we will now prove (3). Let G be a graph with $\kappa = 2$ and b be a point with $d_b(G) = \Delta$. Since $G - \{b\}$ is connected; it follows that $q(G - \{b\}) \ge p - 2$ and $q(G) \ge p + \Delta - 2$, thus proving (3). Noting that (4) follows from $\kappa \le \delta$, concludes this proof.

The remaining three lemmas find relevant upper bounds for q.

Lemma 3: (1) If $\kappa = 1$ and $p = 2\Delta + 2$, then $q < \frac{1}{2} p \Delta$.
(2) For all graphs $q \le \kappa + \frac{1}{2}(p-1)(p-2)$.

Proof: First we will prove (1). Suppose G is a graph with $\kappa = 1$, $p = 2\Delta + 2$ and $q = \frac{1}{2} p \Delta$. As G is not complete there is a point c whose removal disconnects G into at least two components. Thus the point set of $G - \{c\}$ may be partitioned into two sets T and U such that no lines of $G - \{c\}$ join T and U. Since G is regular of degree Δ, we have $|T \cup \{c\}| \ge \Delta + 1$ and $|T| \ge \Delta$. It is also true that $|U| \ge \Delta$. Because $p = 2\Delta + 2$ either $|T| = \Delta$ or $|U| = \Delta$. Suppose without loss of generality that $|T| = \Delta$. Therefore c is adjacent to every point in T and $d_c(G) \ge \Delta + 1$. As this is impossible, no such G exists. Obviously $q \le \frac{1}{2} p \Delta$ and the result follows. The inequality in (2) was established for all graphs in [8].

Lemma 4: If $p < 2\Delta + 2 - \kappa$, then $q \le \lfloor \frac{1}{2}((p-1)\Delta + \kappa) \rfloor$.

Proof: Let G be a graph with $p < 2\Delta + 2 - \kappa$ and S be a minimum point disconnecting set of G. The point set of $G - S$ may be partitioned into two sets T and U such that no lines of $G - S$ join T and U. Noting that either $|T| \le \Delta - \kappa$ or $|U| \le \Delta - \kappa$ we assume without loss of generality that $|T| \le \Delta - \kappa$. If $|T| = N$ then

$$q \le \frac{1}{2} N (N-1) + \frac{1}{2}(p - N)\Delta + \frac{1}{2} N \kappa.$$

Our goal is to maximize the right side of the inequality on $1 \le N \le \Delta - \kappa$. The fact that this quantity is a quadratic in N with a leading term of $\frac{1}{2} N^2$ tells us the maximum must take place at one of the bounds of the interval. The value of the right side of the inequality at either bound is $\frac{1}{2}(p-1)\Delta + \frac{1}{2}\kappa$ and the result follows.

Lemma 5: For all graphs either $q \le \lfloor \frac{1}{2}(p-1)\Delta + \frac{1}{2}\kappa \rfloor$ or

$$q \le \lfloor \tfrac{1}{2} p \Delta - \tfrac{1}{2} \max[\,(p - \Delta - 1)(2\Delta + 2 - p) - \kappa(\kappa - 1), 0\,] \rfloor .$$

Proof: Since the result is obvious if $(p - \Delta - 1)(2\Delta + 2 - p) - \kappa(\kappa - 1) \le 0$, henceforth we assume $(p - \Delta - 1)(2\Delta + 2 - p) - \kappa(\kappa - 1) > 0$. We pause to note that $p - \Delta - 1 \ge 0$ and $\kappa(\kappa - 1) \ge 0$ implies $p \le 2\Delta + 1$. It also follows that $p \ge 3$ and $0 < \Delta < p - 1$. Our goal is to show that if $q > \lfloor \frac{1}{2}(p-1)\Delta + \frac{1}{2}\kappa \rfloor$ then

$$q \le \lfloor \tfrac{1}{2} p \Delta - \tfrac{1}{2}(p - \Delta - 1)(2\Delta + 2 - p) + \tfrac{1}{2}\kappa(\kappa - 1) \rfloor .$$

Let G be a graph with $(p - \Delta - 1)(2\Delta + 2 - p) - \kappa(\kappa - 1) > 0$ and $q > \lfloor \frac{1}{2}(p-1)\Delta + \frac{1}{2}\kappa \rfloor$. Let S, T and U be defined as they were in the proof of lemma 4. We wish to show that

$$2q = \textstyle\sum_{i\in G} d_i \le p\Delta - (p - \Delta - 1)(2\Delta + 2 - p) + \kappa(\kappa - 1).$$

Let V denote the union of T and U. As $\sum_{i\in G} d_i = \sum_{i\in S} d_i + \sum_{i\in V} d_i$ we seek upper bounds for $\sum_{i\in S} d_i$ and $\sum_{i\in V} d_i$. Note that $\sum_{i\in S} d_i \le \kappa\Delta$. To find an upper bound for $\sum_{i\in V} d_i$ first find bounds for $|T|$ and $|U|$. Suppose without loss of generality that $|T| \le |U|$. We also assume that $|T| \le \Delta - \kappa$. An argument similar to the one used to prove lemma 4 infers $q \le \lfloor \frac{1}{2}((p-1)\Delta + \kappa) \rfloor$, contradicting our assumption that $q > \lfloor \frac{1}{2}((p-1)\Delta + \kappa) \rfloor$. Therefore $|T| \ge \Delta + 1 - \kappa$, $|U| \ge \Delta + 1 - \kappa$ and $|T| \le p - \Delta - 1$. Letting $|T| = N$ gives us

$$\textstyle\sum_{i\in V} d_i \le N(N-1) + (p - N - \kappa)(p - N - \kappa - 1) + \kappa\Delta.$$

We want to maximize the right side of this inequality on $\Delta + 1 - \kappa \le N \le p - \Delta - 1$. This quantity is a quadratic in N with a leading term of $2N^2$, and thus is maximum at one of the bounds of the interval. Substituting for N shows this quadratic has the same value at each bound of the interval, giving us

$$\textstyle\sum_{i\in V} d_i \le \kappa\Delta + (\Delta + 1 - \kappa)(\Delta - \kappa) + (p - \Delta - 1)(p - \Delta - 2).$$

Consequently

$$\textstyle\sum_{i\in G} d_i = \sum_{i\in S} d_i + \sum_{i\in V} d_i \le 2\kappa\Delta + (\Delta + 1 - \kappa)(\Delta - \kappa) + (p - \Delta - 1)(p - \Delta - 2).$$

Showing that

$$2\kappa\Delta+(\Delta + 1 - \kappa)(\Delta - \kappa)+(p - \Delta - 1)(p - \Delta - 2) \leq p\Delta-(p - \Delta - 1)(2 \Delta + 2 - p)+\kappa(\kappa-1)$$

will complete this proof. In fact, simplifying both sides shows they are equal and we are done.

At this point we contemplate whether or not lemma 4 can be used to simplify the statement of lemma 5, and vice versa. To see this is not possible consider the following four cases, each of which satisfies $(p - \Delta - 1)(2 \Delta + 2 - p) - \kappa(\kappa - 1) > 0$. The first two cases are $p = 11, \Delta = 8, \kappa = 4$ and $p = 6, \Delta = 3, \kappa = 1$, each of which satisfies $p < 2 \Delta + 2 - \kappa$. The remaining two cases are $p = 8, \Delta = 4, \kappa = 2$ and $p = 16, \Delta = 8, \kappa = 4$, each of which satisfies $p \geq 2 \Delta + 2 - \kappa$. Substituting each of these four cases into the upper bounds for q given in lemmas 4 and 5 shows the two lemmas cannot be simplified.

The (p, q, κ, Δ) realizability theorem

Theorem. A quadruple of non-negative integers (p, q, κ, Δ) is realizable if and only if exactly one of the following conditions holds:

(I) $0 = \kappa \leq \Delta < p - 1$, $\Delta \leq q \leq \lfloor \frac{1}{2} p \Delta \rfloor$ and if $p \leq 2 \Delta + 1$, then $q \leq \lfloor \frac{1}{2} (p - 1) \Delta \rfloor$;

(II) $1 = \kappa \leq \Delta$, $p - 1 \leq q$, and if $\Delta = 1$, then $p = 2$:
- (A) $\Delta + 2 \leq p \leq 2 \Delta + 1$ and $q \leq \lfloor \frac{1}{2} ((p - 1) \Delta + 1) \rfloor$;
- (B) $p = \Delta + 1$ and $q \leq 1 + \frac{1}{2} (p - 1)(p - 2)$;
- (C) $p = 2 \Delta + 2$ and $q < \frac{1}{2} p \Delta$;
- (D) $p \geq 2 \Delta + 3$, $q \leq \lfloor \frac{1}{2} p \Delta \rfloor$ and if $\Delta = 2$, then $q < p$;

(III) $2 \leq \kappa = \Delta \leq p - 1$, $p\cdot\Delta$ is even and $q = \frac{1}{2} p \Delta$;

(IV) $2 \leq \kappa < \Delta$, $\lceil \frac{1}{2} ((p - 1) \kappa + \Delta)\rceil \leq q$ and if $\kappa = 2$, then $p + \Delta - 2 \leq q$:
- (A) $\Delta + 1 < p < 2 \Delta + 2 - \kappa$ and $q \leq \lfloor \frac{1}{2} ((p - 1) \Delta + \kappa) \rfloor$;
- (B) $p = \Delta + 1$ and $q \leq \kappa + \frac{1}{2} (p - 1)(p - 2)$;
- (C) $p \geq 2 \Delta + 2 - \kappa$ and either $q \leq \lfloor \frac{1}{2} ((p - 1)\Delta + \kappa)\rfloor$ or $q \leq \lfloor \frac{1}{2} p \Delta - \frac{1}{2} \max [(p - \Delta - 1)(2 \Delta + 2 - p) - \kappa(\kappa - 1), 0] \rfloor$;

(V) $p = 1$ and $q = \kappa = \Delta = 0$.

Proof: The necessity of $\Delta < p - 1$ in (I) follows from the fact that $\Delta = p - 1 \geq 1$ implies $\kappa > 0$. The other conditions in (I) are a consequence of lemmas 2, 4 and 5, and some obvious facts about graphs. We now consider (II). If a connected graph is regular of degree one, then it has exactly two points. Thus $\kappa = \Delta = 1$ implies $p = 2$. Substituting $\kappa = 1$ and $p = 2\Delta + 1$ into lemma 5 gives us $q \leq \lfloor \frac{1}{2}((p-1)\Delta + 1) \rfloor$. The only connected regular graph of degree two is a cycle. Therefore $\kappa = 1$ and $\Delta = 2$ implies $q < p$. The other conditions in (II) follow from lemmas 2, 3, 4 and 5 together with well known facts concerning graphs. The conditions in (III) are a result of properties of a regular graph. The conditions in (IV) are a consequence of lemmas 2, 3, 4 and 5, and basic graph theory. The conditions in (V) are trivial.

We now provide constructions to prove sufficiency.

Case 1. Suppose that $\lceil \frac{1}{2}((p-1)\kappa + \Delta) \rceil \leq q \leq \lfloor \frac{1}{2}((p-1)\Delta + \kappa) \rfloor$, $3 \leq \kappa < \Delta \leq p - 1$ and if $\Delta = p - 1$, then $q \leq \kappa + \frac{1}{2}(p-1)(p-2)$. Let H_1 denote the Harary graph on $p - 1$ points with $\lceil \frac{1}{2}(p-1)(\kappa - 1) \rceil$ lines and $\kappa(H_1) = \kappa - 1$. Form a new graph by taking the union of a single point (which is denoted by b) and H_1. Note that H_1 is not complete. If κ is odd we proceed as follows. Observing that H_1 is a power of cycle which does not include diameters, we now add $\lceil \frac{1}{2}(p - \Delta - 1) \rceil$ independent diameters to H_1. At this point in our construction there are either Δ or $\Delta - 1$ points in H_1 that have degree $\kappa - 1$. Make b adjacent to all of the points of degree $\kappa - 1$ in H_1. If $d_b = \Delta - 1$, then make b adjacent to another point in H_1. Presently our graph contains $\lceil \frac{1}{2}((p-1)\kappa + \Delta) \rceil$ lines. If $\Delta = p - 2$ let c denote the point in H_1 which is not adjacent to b, otherwise let c denote any point in H_1. To complete our construction we add $q - \lceil \frac{1}{2}((p-1)\kappa + \Delta) \rceil$ lines, none of which are incident to c, in such a way that no point has degree exceeding Δ.

A minimum point disconnecting set of a power of cycle with no diameters is composed of two separate consecutive strings of points. As a result, it can easily be shown that our graph has the desired point connectivity. The graph also fulfills the other requirements of this case.

We now consider the case when κ is even. The construction is essentially the same as when κ was odd, except for the following differences. Here H_1 includes diameters, however if $p - 1$ is odd we may again add independent diameters to H_1. If $p - 1$ is even we may add lines of the form $(i, i + p/2 - 1)$ to H_1, where i is an arbitrary point in H_1. Since $\Delta \geq 5$ a sufficient number of lines can be added to H_1. It is easily verified that our graph has the desired properties. Therefore any quadruple satisfying this case is realizable.

Case 2. Suppose that $\lfloor \frac{1}{2}((p-1)\Delta + \kappa) \rfloor + 1 \leq q \leq \lfloor \frac{1}{2} p \Delta \rfloor$, $2 \leq \kappa < \Delta$ and $p \geq 2\Delta + 2$. Let H_2 denote the Harary graph on $p - \Delta - 1$ points with $\lceil \frac{1}{2}(p - \Delta - 1)(\Delta - 1) \rceil$ lines and $\kappa(H_2) = \Delta - 1$. Take the union of H_2 and $K_{\Delta+1}$ to form a single graph. Observe that H_2 is not complete. If κ is even we proceed as follows. If $p - \Delta - 1$ is even and $\Delta - 1$ is odd add $\lfloor \frac{1}{2}(p - \Delta - 1 - \kappa) \rfloor$ lines of the form $(i, i + p/2 - 1)$ to H_2, where i is an arbitrary point in H_2. Otherwise add $\lfloor \frac{1}{2}(p - \Delta - 1 - \kappa) \rfloor$ independent diameters to H_2. We pause to note that H_2 has either κ or $\kappa + 1$ points of degree $\Delta - 1$. Add κ independent lines joining points in $K_{\Delta+1}$ to points of degree $\Delta - 1$ in H_2. Next, partition the points of degree $\Delta + 1$ in $K_{\Delta+1}$ into pairs. For each of these pairs, delete the line which joins the points in that pair. Presently our graph contains $\lfloor \frac{1}{2} p \Delta \rfloor$ lines. We finish our construction by deleting $\lfloor \frac{1}{2} p \Delta \rfloor - q$ independent lines in $K_{\Delta+1}$, none of which are adjacent to lines previously deleted in $K_{\Delta+1}$. Since an independent set of lines was deleted from $K_{\Delta+1}$ and $\kappa < \Delta$, our graph has the desired point connectivity. It can easily be proven that our graph has the other desired properties.

On the other hand, if κ is odd we alter the previous construction as follows. Instead of adding κ lines joining points in $K_{\Delta+1}$ to points in H_2, add $\kappa + 1$ such lines in the following manner. Two lines are made incident to the same point in $K_{\Delta+1}$ (denote this point by e) while the remaining $\kappa - 1$ lines are independent. Since one more line joins points in $K_{\Delta+1}$ to points in H_2, we change the number of diameters we add to H_2 accordingly. Next, delete lines in $K_{\Delta+1}$ in such a way that all points in $K_{\Delta+1}$ have degree Δ. Finally, complete this construction by deleting $\lfloor \frac{1}{2} p \Delta \rfloor - q$ lines in the manner of the previous construction. Note that the set of lines removed from $K_{\Delta+1}$ is not independent. Let F denote one of the deleted lines which was incident to e. As F can be replaced by a path containing points in H_2 and the remaining deleted lines form an independent set, our graph has point connectivity κ. The graph also fulfills the other requirements of this case.

Case 3. Suppose that $\lfloor \frac{1}{2}((p - 1)\Delta + \kappa) \rfloor + 1 \leq q \leq \lfloor \frac{1}{2} p \Delta - \frac{1}{2} \max[(p - \Delta - 1)(2\Delta + 2 - p) - \kappa(\kappa - 1), 0] \rfloor$, $2 \leq \kappa < \Delta$ and $2\Delta + 2 - \kappa \leq p \leq 2\Delta + 1$. Since $\lfloor \frac{1}{2}((p - 1)\Delta + \kappa) \rfloor + 1 \leq q$ lemma 4 results in the above lower bound of p and lemma 5 yields the above upper bound of q. First, we consider the possibility that $(p - \Delta - 1)(2\Delta + 2 - p) \geq \kappa(\kappa - 1)$. If this is the case, let C denote the graph consisting of κ isolated points and form the union of C and $K_{\Delta+1-\kappa}$. Next, join every point of C to every point in $K_{\Delta+1-\kappa}$. We now form the union of this graph and $K_{p-\Delta-1}$. Denote the points in $K_{\Delta+1-\kappa}$ by A and the points in $K_{p-\Delta-1}$ by B. At this juncture all of the points of A have degree Δ, all of the points of B have degree $p - \Delta - 2$ and all of the points of C have degree $\Delta + 1 - \kappa$.

We wish to prove that $p \geq \kappa + \Delta$. Assuming this inequality is false gives us $p - \Delta - 1 < \kappa - 1$. As a result of $(p - \Delta - 1)(2\Delta + 2 - p) \geq \kappa(\kappa - 1)$ it follows that $2\Delta + 2 - p > \kappa$, contradicting $2\Delta + 2 - \kappa \leq p$. Therefore $p \geq \kappa + \Delta$ and B contains at least $\kappa - 1$ points. If $|B| > \kappa - 1$ join κ points of B in a one-to-one fashion to the points of C. On the other hand, if $|B| = \kappa - 1$ then join the points of B in a one-to-one fashion to $\kappa - 1$ points of C. Note that $|B| = p - \Delta - 1 \geq \kappa - 1$ and $|C| = \kappa$. Before lines joining points of B to points of C were added, we had

$$\Delta(p - \Delta - 1) - \sum_{i \in B} d_i = (p - \Delta - 1)(2\Delta + 2 - p)$$

and $\Delta\kappa - \sum_{i \in C} d_i = \kappa(\kappa - 1)$. Consequently, we can now add lines joining points of B to points of C so that every point of C will have degree Δ and no point of B will have degree exceeding Δ. This is done in such a way that the degrees of the points of B differ by at most one. At this point, our graph contains

$$\lfloor \tfrac{1}{2} p\Delta - \tfrac{1}{2}(p - \Delta - 1)(2\Delta + 2 - p) + \tfrac{1}{2}\kappa(\kappa - 1) \rfloor$$

lines. Deleting

$$\lfloor \tfrac{1}{2} p\Delta - \tfrac{1}{2}(p - \Delta - 1)(2\Delta + 2 - p) + \tfrac{1}{2}\kappa(\kappa - 1) \rfloor - q$$

independent lines in $K_{\Delta + 1 - \kappa}$ finishes the construction. As a result of the lower bound of q, every point has degree of at least $\kappa + 1$. It can easily be shown that the final graph has the desired properties.

We now consider the case when $(p - \Delta - 1)(2\Delta + 2 - p) < \kappa(\kappa - 1)$ and $p > \kappa + \Delta$. First, repeat the previous construction excluding the step where lines were deleted. However, in this case, every point of B has degree Δ and C contains at least one point which has degree less than Δ. Add the appropriate Harary graph to the points of C and/or join pairs of points of C, so that our graph has $\lceil \tfrac{1}{2} p\Delta \rceil$ lines. Deleting $\lceil \tfrac{1}{2} p\Delta \rceil - q$ independent lines in $K_{\Delta + 1 - \kappa}$ completes this construction.

If $(p - \Delta - 1)(2\Delta + 2 - p) < \kappa(\kappa - 1)$ and $p \leq \kappa + \Delta$ then form the union of $K_{\Delta - \kappa + 1}$ and the graph composed of $p - \Delta - 1 + \kappa$ isolated points. Denote κ of the isolated points by C, the remaining $p - \Delta - 1$ isolated points by B and the points of $K_{\Delta - \kappa + 1}$ by A. Join each point of C to each point of A and B. We pause to note that every point of A has degree Δ, every point of B has degree κ and every point of C has degree $p - \kappa \leq \Delta$. Next, add the appropriate Harary graph to the points of B and/or join pairs of points of B and, if necessary, do the same to the points of C. It may also be necessary to delete a line which joins a point of B to a point of C. Presently our graph contains $\lceil \tfrac{1}{2} p\Delta \rceil$ lines. Delete lines as in the previous construction and we are finished with case 3.

Case 4. Suppose that $p + \Delta - 2 \leq q \leq \lfloor \tfrac{1}{2}((p - 1)\Delta + \kappa) \rfloor$, $2 = \kappa < \Delta \leq p - 1$ and if $\Delta = p - 1$, then $q \leq \kappa + \tfrac{1}{2}(p - 1)(p - 2)$. Consider the cycle on p points and let f and g denote two adjacent points of the cycle. Add $q - p \geq \Delta - 2$ lines so that the following holds. Exactly $\Delta - 2$ of these lines are incident to g, none of these lines are incident to f and no point has degree exceeding Δ. This graph fulfills the requirements of this case.

Case 5. Suppose that $1 = \kappa \leq \Delta$, $p - 1 \leq q \leq \lfloor \frac{1}{2} p \Delta \rfloor$, and if $\Delta = 1$, then $p = 2$. We also assume that the following conditions hold: (1) if $\Delta + 2 \leq p \leq 2\Delta + 1$, then $q \leq \lfloor \frac{1}{2}((p - 1)\Delta + 1) \rfloor$, (2) if $p = \Delta + 1$, then $q \leq 1 + \frac{1}{2}(p - 1)(p - 2)$, and (3) if $\Delta = 2$ or $p = 2\Delta + 2$, then $q < \frac{1}{2} p \Delta$.

First let us consider when $q \leq \lfloor \frac{1}{2}((p - 1)\Delta + 1) \rfloor$. Let T be any tree on p points with $\Delta(T) = \Delta$ and let h be a point of degree one in T. Adding $q - p + 1$ lines, none of which are incident to h, in such a way that all points in T have degree at most Δ yields a graph satisfying our present assumptions.

Next we entertain the possibility that $\lfloor \frac{1}{2}((p - 1)\Delta + 1) \rfloor + 1 \leq q$, $p \geq 2\Delta + 2$, and that one of the following holds ; Δ and $p - \Delta - 1$ are both odd or $q < \lfloor \frac{1}{2} p \Delta \rfloor$. Note that $\Delta \geq 2$. Form the union of $K_{\Delta+1}$ and a path containing $p - \Delta - 1 \geq \Delta + 1$ points. Denote the points of $K_{\Delta+1}$ by A and the points of the path by B. Join a point of B with degree one to one of the points of A. Delete a line of $K_{\Delta+1}$ which is incident to the point of degree $\Delta + 1$. Observe that our graph now has $\frac{1}{2}\Delta(\Delta + 1) + p - \Delta - 2$ lines. To finish this construction, add $q - \frac{1}{2}\Delta(\Delta + 1) - p + \Delta + 2$ lines joining points of B, without generating any points of degree greater than Δ. Notice that if $q = \lfloor \frac{1}{2} p \Delta \rfloor$, then Δ and $p - \Delta - 1$ are both odd and every point of B has degree Δ.

On the other hand, it may be that $q = \frac{1}{2} p \Delta$, $p \geq 2\Delta + 3$, $\Delta \geq 3$ and either Δ or $p - \Delta - 1$ is even. Let H_3 denote the Harary graph on $p - \Delta - 1$ points with $\frac{1}{2}(p - \Delta - 1)\Delta$ lines and $\kappa(H_3) = \Delta$. Take the union of H_3 and $K_{\Delta+1}$ to form a single graph. Let j be a point of H_3. Note that H_3 is both regular of degree Δ and not complete. As H_3 is a power of cycle, it contains two nonadjacent points both of which are adjacent to j. Denote two such points by k and l. Add line {k, l} and delete lines {j, k} and {j, l}. Now join j to two points of $K_{\Delta+1}$ and delete the line joining the two points of degree $\Delta + 1$. This graph has the desired qualities and we are finished with case 5.

Case 6. Suppose that $0 = \kappa \leq \Delta \leq p - 1$, $\Delta \leq q \leq \lfloor \frac{1}{2} p \Delta \rfloor$ and if $p \leq 2\Delta + 1$, then $q \leq \lfloor \frac{1}{2}(p - 1)\Delta \rfloor$. Furthermore, assume $\Delta = p - 1$ if and only if $p = 1$. If $p = 1$, then use the graph K_1. Henceforth, we consider only $p \geq 2$. If $q \leq \lfloor \frac{1}{2}(p - 1)\Delta \rfloor$, then we proceed as follows. Take p isolated points and denote one of them by m. Join m to $\Delta < p - 1$ points and let n denote one of the remaining isolated points. Adding $q - \Delta$ lines, none of which are incident to n, so that no point has degree greater than Δ completes our construction.

However, if $q \geq \lfloor \frac{1}{2}(p - 1)\Delta \rfloor + 1$ form the union of $K_{\Delta+1}$ and the graph composed of $p - \Delta - 1 \geq \Delta + 1$ isolated points. We end this construction by adding $q - \frac{1}{2}\Delta(\Delta + 1)$ lines, making sure that every point has degree at most Δ.

Case 7. Suppose that $2 \leq \kappa = \Delta \leq p - 1$, $q = \frac{1}{2} p \Delta$ and $p \cdot \Delta$ is even. In this case the Harary graph on p points with $\frac{1}{2} p \Delta$ lines and point connectivity Δ is sufficient.

We are finished with our constructions and will now show sufficiency. If we assume $2 \le \kappa < \Delta$ and $q \le \lfloor \frac{1}{2} ((p-1)\Delta + \kappa) \rfloor$ then cases 1 and 4 show the sufficiency of the conditions of the theorem. Cases 2 and 3 show the sufficiency of the conditions of the theorem for $2 \le \kappa < \Delta$ and $q \ge \lfloor \frac{1}{2} ((p-1)\Delta + \kappa) \rfloor + 1$. Cases 5,6 and 7 show the conditions of the theorem are sufficient if we also have $\kappa = \Delta$ or $\kappa < 2$ and our proof is finished.

Conclusion

The (p, q, κ, Δ) realizability theorem in this paper solves several extremal problems. If any three of the parameters p, q, κ and Δ are given we can find the range of values for the unknown parameter. Here we will look at the problem of finding the maximum value of κ among all (p, q, Δ) graphs, which is denoted by $\max(\kappa \mid p, q, \Delta)$. The solution is given below (the proof is straightforward).

$$\max(\kappa \mid p, q, \Delta) = \begin{cases} 0 & \text{, if } q < p-1 \text{ or } p = 1 \\ \min(\lfloor (2q - \Delta)/(p-1) \rfloor, \Delta - 1), & \text{if } q \ge p-1 \ge 1 \text{ and } p\Delta \text{ is odd} \\ \lfloor (2q - \Delta)/(p-1) \rfloor & \text{, otherwise.} \end{cases}$$

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