# Lexicographically Optimum Traffic Trees with Maximum Degree Constraints

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**Abstract**

We shall consider a problem of finding an 'optimum' tree which is closely related to the network flow problem proposed by Ford and Fulkerson, and call the solution to this problem a *lexicographically optimum traffic tree* (LOTT). Before examining this problem in detail, we shall review the problem of finding an *optimum requirement spanning tree* (ORST) studied by Hu which is also related to the network flow problem. We can regard the LOTT problem as a min-max problem and the ORST problem as a min-sum problem. It shall be shown that, while LOTTs and ORSTs coincide completely *without* maximum degree constraints, they do not always coincide *with* the constraints. Further, we shall show that LOTTs can be expressed by simple recursion in a special case.

## 1 Introduction

We shall consider a problem of finding an 'optimum' tree which is closely related to the network flow problem proposed by Ford and Fulkerson [5]. As the preliminaries of proposing our problem, we define some notation.

Let $V$ be a set of $n$ vertices and $\binom{V}{2}$ the set of all pairs of distinct vertices in $V$. Also, let $\mathcal{T}_V$ be the set of undirected spanning trees on $V$. A tree $T \in \mathcal{T}_V$ with an edge set $E$ ($|E| = n - 1$) is denoted by $T = (V, E)$, and such $E$ is sometimes denoted by $E^T$ to emphasize that it is the edge set of $T$. Also, the edge $e \in E$ connecting two vertices $v$ and $w$ is denoted by $e = (v, w)$. Assume that a nonnegative value $r_{vw}$ is assigned to each pair $\{v, w\} \in \binom{V}{2}$. For an edge $(v, w) \in E$ of a tree $T = (V, E) \in \mathcal{T}_V$, we define a subtree of $T$ denoted by $T(v) = (V(v), E(v))$ as the connected component of $(V, E \setminus \{(v,w)\})$ containing $v$, while $T(w) = (V(w), E(w))$ is defined as the other connected component. Also, we define the *traffic* of the edge $(v, w)$ by

$$t((v,w),T) = \sum_{x \in V(v), y \in V(w)} r_{xy}.$$

(In terms of the network flow problem, $t((v,w),T)$ is the capacity of the cut dividing $V$ into $V(v)$ and $V(w)$ in the complete graph $K_n$ on $V$ with edge capacities $r_{xy}$ $(x, y \in V)$.) Further, let

$$t^T = [t_1^T, t_2^T, \ldots, t_{n-1}^T]$$

be the sequence of traffics in which the traffics of edges in $T$ are arranged in descending order, that is, $t_1^T \geq t_2^T \geq \cdots \geq t_{n-1}^T$ holds. For mathematical convenience, if $n = 1$ then we set $t^T = [\,]$ (an empty sequence).

The problem we want to solve is to find a tree $T \in \mathcal{T}_V$ which minimizes $t^T$ lexicographically. We call such a tree a *lexicographically optimum traffic tree* (LOTT). If $T$ is a LOTT, then it is obvious that $T$ minimizes

$$t(T) = \begin{cases} \max_{e \in E^T} t(e,T) \ (= t_1^T) & \text{if } n \geq 2 \\ 0 & \text{if } n = 1. \end{cases}$$

Hence, we can regard the LOTT problem as a generalized min-max problem.

Before examining the LOTT problem in detail, we review another problem which is also related to the network flow problem and can be regarded as a min-sum problem. Under the same assumptions stated above, Hu [4] considered a problem of finding a tree $T$ which minimizes

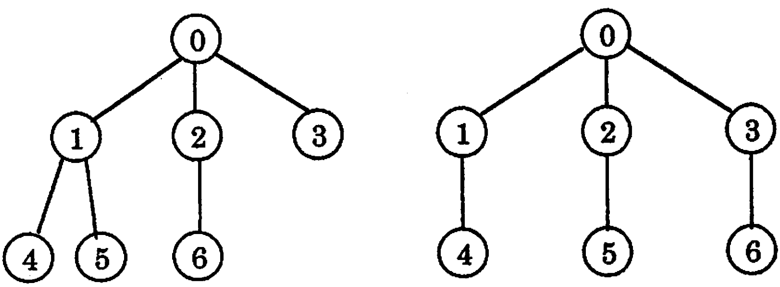
$$f(T) = \begin{cases} \sum_{e \in E^T} t(e,T) = \sum_{\{v,w\} \in \binom{V}{2}} d(v,w;T) r_{vw} & \text{if } n \geq 2 \\ 0 & \text{if } n = 1, \end{cases}$$

where $d(v, w; T)$ is the number of edges on the path between two vertices $v$ and $w$ on $T$. He called the solution to this problem an *optimum requirement spanning tree* (ORST), and showed that an ORST is obtained by the Gomory-Hu algorithm [3] when the degrees of vertices are *not* restricted. On the other hand, Anazawa [2] considered a problem of minimizing $f$ in the case when (a) $V = \{0, 1, \ldots, n-1\}$ is assumed, and (b) a positive integer $l_v$ is given to each vertex $v \in V$ and

$$\deg(v) \leq l_v \text{ holds for all } v \in V, \tag{1}$$

where $\deg(v)$ denotes the degree of $v$. Although this problem is not efficiently solvable in general, he showed that if $\{l_v\}$ satisfies

$$l_0 \geq l_1 \geq \cdots \geq l_{n-1} \geq 1 \text{ and } \sum_{v=0}^{n-1} l_v \geq 2(n-1)$$




(a) $T^*$ (b) $\hat{T}$

Figure 1: $T^*$ and $\hat{T}$ for $n = 7$ and $l_0 = l_1 = \cdots = l_6 = 3$.

and $\{r_{vw}\}$ satisfies

$$r_{vw} + r_{v'w'} \geq r_{vw'} + r_{v'w} \tag{2}$$

for all 4-tuple $\{v, v', w, w'\}$ $(v < v', w < w')$ such that $r_{vw}$, $r_{v'w'}$, $r_{vw'}$ and $r_{v'w}$ are all defined, then a particular tree denoted by $T^*$ (which is explicitly definable) is an ORST. Roughly speaking, $T^*$ is obtained by the following 'greedy algorithm': First, to vertex 0, connect the remaining vertices by ascending order of vertex number as many as possible (so that $\deg(0) \leq l_0$ holds); secondly, to vertex 1, connect the remaining vertices by the same order as many as possible (so that $\deg(1) \leq l_1$ holds); and continue to connect the remaining vertices in the same manner until all $n$ vertices are connected. An example of $T^*$ (for $n = 7$ and $l_0 = l_1 = \cdots = l_6 = 3$) is illustrated by Figure 1 (a).

When considering the LOTT problem, we must note whether the degrees of vertices are restricted or not, similarly to the ORST problem. However, the LOTT problem without the maximum degree constraints is trivial, since every LOTTs are ORSTs and vice versa for arbitrarily given $\{r_{vw}\}$, which is indicated by Adolphson and Hu [1]. This means that a LOTT is also obtained by the Gomory-Hu algorithm in the case. On the other hand, under condition (1), there exists a case when a LOTT does not coincide with any ORST for the same $\{r_{vw}\}$. For example, consider a tree $\hat{T}$ illustrated by Figure 1 (b) consisting of $n = 7$ vertices and satisfying $\deg(v) \leq l_v = 3$ for all $v \in V$. Assuming that $r_{vw} = 1$ holds for all $\{v, w\} \in \binom{V}{2}$ which obviously satisfies (2), we find that

$$f(\hat{T}) = 48 > 46 = f(T^*)$$

but

$$t^{T^*} = [12, 10, 6, 6, 6, 6] \succ [10, 10, 10, 6, 6, 6] = t^{\hat{T}}$$

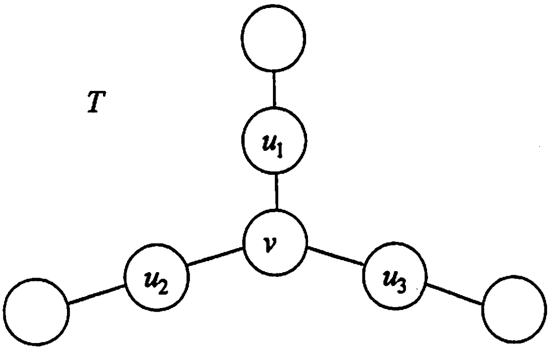


Figure 2: A tree with three maximum traffic edges of traffic 10.

holds, where '$\succ$' means 'lexicographically greater than'. The fact is that $\hat{T}$ is a LOTT among the trees satisfying condition (1) with $l_0 = l_1 = \cdots = l_6 = 3$ if all $r_{vw}$ are equal to each other.

In this paper, we shall devote ourselves to a special case when

$$\text{every vertex } v \in V \text{ satisfies } \deg(v) \leq L \tag{3}$$

for a given integer $L$ ($\geq 2$) and

$$r_{vw} = 1 \text{ holds for all } \{v, w\} \in \binom{V}{2}. \tag{4}$$

Note that, in this case, the solution to the LOTT problem for $n \leq 3$ or $L = 2$ is trivial. We shall give in Section 2 necessary conditions which LOTTs must satisfy and show that any tree satisfying the necessary condition attains the min-max bound $\min_{T \in \mathcal{T}_V} t(T)$. And we shall show in Section 3 that LOTTs can be expressed recursively.

# 2 Necessary conditions of the optimality

First, we add some notation and definitions. For a vertex set $V$ with $|V| = n$ and an integer $L$ ($\geq 2$), let $\mathcal{T}_{V,L}$ be the set of undirected spanning trees on $V$ satisfying condition (3). For a tree $T = (V, E) \in \mathcal{T}_{V,L}$, we call the edges attaining $t(T)$ the *maximum traffic edges* of $T$. In general, a tree may have two or more maximum traffic edges. For example, all $(v, u_i)$ ($i = 1, \ldots, 3$) in Figure 2 are the maximum traffic edges (of traffic 10) of $T$. Also, for a subtree $T(v) = (V(v), E(v))$ of $T$ defined for an edge $(v, \cdot) \in E$, let $\bar{V}(v) = V \setminus V(v)$. When condition (4) is satisfied, we find that $t((v, \cdot), T) = |V(v)| \cdot |\bar{V}(v)| = |V(v)|(n - |V(v)|)$ holds for any edge $(v, \cdot) \in E$.

**Lemma 1** *For two edges $(v,\cdot)$ and $(w,\cdot)$ of a tree $T=(V,E)\in\mathcal{T}_{V,L}$, we define subtrees $T(v)=(V(v),E(v))$ and $T(w)=(V(w),E(w))$, and assume that $|V(v)|\le|\bar{V}(v)|$ and $|V(w)|\le|\bar{V}(w)|$ hold. If $|V(v)|<|V(w)|$, then $t((v,\cdot),T)<t((w,\cdot),T)$ holds.*

**Proof** Note that the function $g(x)=x(n-x)$ $(n>0)$ is monotone increasing on $[0,\frac{n}{2}]$. Since neither $|V(v)|$ nor $|V(w)|$ exceeds $\frac{n}{2}$, we obtain the lemma. □

**Corollary 1** *For a tree $T=(V,E)\in\mathcal{T}_{V,L}$, let $T(v)=(V(v),E(v))$ be a subtree of $T$ defined for an edge $(v,\cdot)\in E$. If $|V(v)|\le|\bar{V}(v)|$, then $t(e,T)<t((v,\cdot),T)$ holds for all $e\in E(v)$.*

**Proof** Let $e=(p,q)\in E(v)$ and assume that $d(p,v;T)>d(q,v;T)$ holds. Defining a subtree $T(p)=(V(p),E(p))$ of $T$ for the edge $(p,q)$, we have $|V(p)|<|V(v)|$. Also, it follows from $|V(v)|\le|\bar{V}(v)|$ that $|V(p)|\le|\bar{V}(p)|$ holds. Hence, we find from Lemma 1 that $t((p,q),T)<t((v,\cdot),T)$ holds. □

**Theorem 1** *For a tree $T=(V,E)\in\mathcal{T}_{V,L}$, let $v\in V$ satisfy* $\deg(v)=m$, *and let $T(u_i)=(V(u_i),E(u_i))$ $(i=1,\ldots,m)$ be subtrees of $T$ defined for $(v,u_i)\in E$ $(i=1,\ldots,m)$. Assume that $|V(u_1)|\le|\bar{V}(u_1)|$ holds. Then $(v,u_1)$ is one of the maximum traffic edges of $T$ if and only if $|V(u_1)|\ge|V(u_i)|$ holds for all $i=2,\ldots,m$.*

**Proof** Since the case of $n=2$ is trivial, suppose that $n\ge3$ holds.
(Necessity) Assume that $|V(u_1)|<|V(u_i)|$ holds for a certain $i$, say $i=2$. Consider a tree $T'$ obtained by deleting $V(u_i)$ $(i=3,\ldots,m)$. Then we find from $|V(u_1)|+1\le|V(u_2)|$ and Corollary 1 that

$$t((v,u_1),T')-t((v,u_2),T')=|V(u_1)|(|V(u_2)|+1)-|V(u_2)|(|V(u_1)|+1)<0$$

holds. Hence, for the original tree $T$, we find from

$$\begin{aligned} t((v,u_1),T) &= |V(u_1)|(\sum_{i=2}^{m}|V(u_i)|+1) \\ &= |V(u_1)|(|V(u_2)|+1)+|V(u_1)|\sum_{i=3}^{m}|V(u_i)| \end{aligned}$$

and

$$\begin{aligned} t((v,u_2),T) &= |V(u_2)|(\sum_{i\ne2}|V(u_i)|+1) \\ &= |V(u_2)|(|V(u_1)|+1)+|V(u_2)|\sum_{i=3}^{m}|V(u_i)| \end{aligned}$$

that

$$t((v,u_1),T) - t((v,u_2),T) =$$
$$t((v,u_1),T') - t((v,u_2),T') + (|V(u_1)| - |V(u_2)|)\sum_{i=3}^{m}|V(u_i)| < 0$$

holds, which implies that $(v, u_1)$ does not attain $t(T)$.
(Sufficiency) Since $|V(u_1)| \le |\bar{V}(u_1)|$ is assumed, it follows from Corollary 1 that the maximum traffic edges of $T$ must exist in $E \backslash E(u_1)$. Also, for each $i = 2,\ldots,m$, we find from $|V(u_i)| \le |V(u_1)| < \sum_{j\ne i}|V(u_j)| + 1 = |\bar{V}(u_i)|$ and Corollary 1 that the maximum traffic edges of $T$ must exist in $E\backslash E(u_i)$. Hence, all the maximum traffic edges of $T$ must exist in

$$\bigcap_{i=1}^{m}(E \setminus E(u_i)) = \{(v,u_i)|i = 1,\ldots,m\}.$$

Since $|V(u_i)| \le |V(u_1)| \le \frac{n}{2}$ is satisfied for $i = 2,\ldots,m$, we find from Lemma 1 that $(v, u_1)$ is one of the maximum traffic edges of $T$. □

**Remark** It is easy to see from Theorem 1 that if a tree has two or more maximum traffic edges, then all of them are adjacent to each other.

We find from Theorem 1 that any tree $T \in \mathcal{T}_{V,L}$ $(n \ge 2)$ has a vertex $v$ with $\deg(v) = m$ satisfying the following condition: When subtrees $T(u_i) = (V(u_i), E(u_i))$ $(i = 1,\ldots,m)$ of $T$ are defined for edges $(v,u_i)$ $(i = 1,\ldots,m)$, $|V(u_i)| \le |\bar{V}(u_i)|$ holds for all $i = 1,\ldots,m$. We call such $v$ the *maximum traffic vertex* of $T$. Let $V^T$ be the set of all the maximum traffic vertices of $T$. Then $|V^T| \le 2$ holds for all $T \in \mathcal{T}_{V,L}$. If $T$ has two or more maximum traffic edges with the common end vertex $v$, then $V^T = \{v\}$ holds. On the other hand, $|V^T| = 2$ (say $V^T = \{v_1, v_2\}$) holds if and only if $T$ has an edge $(v_1, v_2)$ and $|V(v_1)| = |\bar{V}(v_1)|$ is satisfied for the subtree $T(v_1) = (V(v_1), E(v_1))$ defined for $(v_1, v_2)$. For mathematical convenience, if $T = (\{v\}, \emptyset)$ then we define the maximum traffic vertex of $T$ by $v$.

For a fixed vertex $v \in V$ and a given integer $M$ $(2 \le M \le L)$, let

$$\mathcal{T}_{V,L}(M,v) = \{T \in \mathcal{T}_{V,L} | v \in V^T, \deg(v) \le M\}.$$

Also, for a tree $T \in \mathcal{T}_{V,L}(M,v)$, let

- $m^T = \deg(v)$,
- $T(u_i) = (V(u_i), E(u_i))$ $(i = 1,\ldots,m^T)$ be defined for $(v,u_i)$ $(i = 1,\ldots,m^T)$, and

- $k^T$ be the number of the maximum traffic edges of $T$.

Then it follows from Theorem 1 that

$$|V(u_i)| \begin{cases} = n^T \text{ (say)} & \text{for } i = 1, \ldots, k^T \\ < n^T & \text{for } i = k^T + 1, \ldots, m^T \end{cases}$$

can be assumed without loss of generality. Further, we obtain the following two theorems.

**Theorem 2** *A necessary condition for $T \in \mathcal{T}_{V,L}(M, v)$ to minimize $t^T$ lexicographically in $\mathcal{T}_{V,L}(M, v)$ is that*

(i) *if $n - 1 \leq M$, then $m^T = n - 1$, that is, $|V(u_i)| = 1$ $(i = 1, \ldots, m^T)$ holds,*

(ii) *if $n - 1 > M$, then $m^T = M$ and*

$$|V(u_i)| = \begin{cases} n^T & \text{for } i = 1, \ldots, k^T \\ n^T - 1 & \text{for } i = k^T + 1, \ldots, m^T \end{cases} \tag{5}$$

*hold.*

**Proof** Since the case of $n \leq 3$ is trivial, suppose that $n \geq 4$ holds. In the case of $n - 1 \leq M$, let $\hat{T} \in \mathcal{T}_{V,L}(M, v)$ be a tree satisfying $m^{\hat{T}} = n - 1$ and $T \in \mathcal{T}_{V,L}(M, v)$ a tree with $m^T < n - 1$. Then $\hat{T}$ satisfies $t(\hat{T}) = n - 1$ obviously, while $T$ satisfies $t(T) \geq 2(n - 2)$ since $n^T \geq 2$ holds. Hence, we find that

$$t(T) - t(\hat{T}) \geq 2(n - 2) - (n - 1) = n - 3 > 0$$

holds, which implies $t^T \succ t^{\hat{T}}$. In the case of $n - 1 > M$, let $T = (V, E) \in \mathcal{T}_{V,L}(M, v)$ be a tree not satisfying the above condition. Also, let $w_1 \in V(u_{k^T})$ be a vertex with $\deg(w_1) = 1$ and $w_0$ the vertex with $(w_0, w_1) \in E$. Next, we construct another tree $T' = (V, E')$ as follows: If $T$ satisfies $m^T < M$, then let

$$E' = E \setminus \{(w_0, w_1)\} \cup \{(v, w_1)\};$$

else, if $T$ satisfies $m^T = M$ and $|V(u_j)| \leq n^T - 2$ for some $j \geq k^T + 1$, then choose a vertex $w_2 \in V(u_j)$ with $\deg(w_2) < L$ and let

$$E' = E \setminus \{(w_0, w_1)\} \cup \{(w_2, w_1)\}.$$

Then it is obvious that $T' \in \mathcal{T}_{V,L}(M, v)$ holds. Also, we find that

- if $T$ satisfies $k^T = 1$, then $t(T') < t(T)$ holds,
- otherwise, $t(T') = t(T)$ and $k^{T'} < k^T$ hold,

which implies that $t^T \succ t^{T'}$ holds. □

**Theorem 3** *The necessary condition in Theorem 2 is equivalent to the condition that*

$$m^T = \begin{cases} n-1 & \text{if } n-1 \le M \\ M & \text{if } n-1 > M \end{cases}$$

*and*

$$|V(u_i)| = \begin{cases} n'+1 & \text{for } i = 1,\ldots,r \\ n' & \text{for } i = r+1,\ldots,m^T \end{cases} \tag{6}$$

*hold for nonnegative integers $n'$ and $r$ satisfying $n-1 = n'm^T + r$ $(0 \le r < m^T)$.*

**Proof** In the case of $n-1 \le M$, $m^T = n-1$ holds if and only if $n' = 1$ and $r = 0$ are satisfied. Then equation (6) is equivalent to $|V(u_i)| = 1$ $(i = 1,\ldots,m^T)$. Consider the case of $n-1 > M$, where $m^T = M$ holds. If $k^T = m^T$, then equation (5) means $|V(u_i)| = n^T = (n-1)/k^T$ $(i = 1,\ldots,m^T)$, which is equivalent to equation (6) for $n' = (n-1)/k^T$ and $r = 0$. If $k^T < m^T$, then it follows from equation (5) that $n-1 = n^T k^T + (n^T-1)(m^T-k^T) = (n^T-1)m^T + k^T$ holds. Hence, we find that equation (5) is equivalent to equation (6) for $n' = n^T - 1$ and $r = k^T$. □

**Corollary 2** *For a given set $V$ with $|V| = n$ and a given integer $L$ $(\ge 2)$, a necessary condition for $T \in \mathcal{T}_{V,L}$ to be a LOTT is that*

$$m^T = \deg(v) = \begin{cases} n-1 & \text{if } n-1 \le L \\ L & \text{if } n-1 > L \end{cases}$$

*holds for a vertex $v \in V^T$ and equation (6) holds for nonnegative integers $n'$ and $r$ satisfying $n-1 = n'm^T + r$ $(0 \le r < m^T)$.*

**Proof** This is a special case of Theorem 3 where $M = L$ is set. □

**Corollary 3** *If $T \in \mathcal{T}_{V,L}$ satisfies the condition in Corollary 2, then $T$ minimizes $t(T)$ $(= t_1^T)$.*

**Proof** For a fixed vertex $v \in V$ and a fixed $M$ $(2 \le M \le L)$, we easily find that $T \in \mathcal{T}_{V,L}(M, v)$ minimizes $t(T)$ if $T$ satisfies the condition in Theorem 3. Since $n'$ in Theorem 3 is nonincreasing as $M$ is increasing, we obtain the corollary. □

## 3 Recursive expression of LOTTs

To show the main result in this paper, we must provide some more preliminaries. For a subtree $T(u) = (V(u), E(u))$ of $T = (V, E)$ defined for an edge $(u, \cdot) \in E$, let

$$t^T(u) = [t_1^T(u), \ldots, t_{|E(u)|}^T(u)]$$

be a subsequence of $t^T$ in which the traffics of edges in $E(u)$ are arranged in descending order. We must distinguish $t^T(u)$ from $t^{T(u)} = [t_1^{T(u)}, \ldots, t_{|E(u)|}^{T(u)}]$. If $E(u) = \emptyset$, then we set $t^T(u) = [\,]$.

For two disjoint sets $V_1$ and $V_2$ with $|V_1| \geq |V_2| > 0$, let $V = V_1 \cup V_2$, $n = |V|$ and $n' = |V_2|$. Then $0 < n' \leq \frac{n}{2}$ is obvious. Given a tree $T_1 = (V_1, E_1) \in \mathcal{T}_{V_1,L}$ with $v \in V_1$ and $\deg(v) < L$, we define

$$\begin{aligned} \mathcal{T}_{V,L}^{T_1} &= \{(V, E) \in \mathcal{T}_{V,L} | (V_2, E_2) \in \mathcal{T}_{V_2,L}, \\ &\quad u \in V_2, \deg(u) < L, E = E_1 \cup E_2 \cup \{(v, u)\}\}. \end{aligned}$$

For a tree $T \in \mathcal{T}_{V,L}^{T_1}$, let $T(v) = (V(v), E(v))$ and $T(u) = (V(u), E(u))$ be subtrees of $T$ defined for $(v, u) \in E$ (then $T(v) = T_1$ is obvious). Then we obtain the following three lemmas.

**Lemma 2** *If $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}^{T_1}$, then it also minimizes the subsequence $t^T(u)$ lexicographically in $\mathcal{T}_{V,L}^{T_1}$.*

**Proof** We can easily find that if $t^T(u)$ is not lexicographically minimum, then $t^T$ cannot be lexicographically minimum. □

**Lemma 3** *Suppose that $L \geq 3$ holds. If $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}^{T_1}$, then $u \in V^{T(u)}$ holds, that is, $u$ is one of the maximum traffic vertices of $T(u)$.*

**Proof** Since the lemma holds obviously when $n' \leq 2$ is satisfied, we consider the case of $n' \geq 3$. Let $T \in \mathcal{T}_{V,L}^{T_1}$ be a tree satisfying $u \in V^{T_2}$ where $T_2 = (V_2, E_2) = T(u)$, and $T' = (V, E_1 \cup E_2 \cup \{(v, u')\})$ where $u' \in V_2 \setminus V^{T_2}$ and $\deg(u') < L$ hold in $T_2$. Then there exists an edge $(u', w) \in E_2$ such that $|V(w)| > |\bar{V}(w)|$ holds for the subtree $T(w) = (V(w), E(w))$ of $T_2$ defined for $(u', w)$. Suppose that $t(T_2) = t_1^{T_2} = n''(n' - n'')$ holds for a certain $n''$ $(\leq \frac{n'}{2})$. Then we easily find that $T$ satisfies $t_1^T(u) = n''(n - n'')$. On the other hand, since $n'' \leq \frac{n'}{2} < |V(w)| < \frac{n}{2}$ holds, $T'$ satisfies $t((u', w), T') = |V(w)|(n - |V(w)|) > n''(n - n'')$. Hence, $t^{T'}$ cannot be lexicographically minimum. □

**Remark** It is easy to find that Lemma 3 is not always true in the case of $L = 2$.

**Lemma 4** *Supposing that $L \geq 3$ holds, we find that $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}^{T_1}$ if and only if $T(u)$ minimizes $t^{T(u)}$ lexicographically in $\mathcal{T}_{V_2,L}(L-1, u)$.*

**Proof** Since the case of $n' \leq 3$ is trivial, suppose that $n' \geq 4$ holds. Note that, for any subtree $T(u) = (V(u), E(u))$ satisfying $u \in V^{T(u)}$, both $t_i^{T(u)}$ and $t_i^T(u)$ correspond to the same edge in $T(u)$ for all $i = 1, \ldots, n'-1$. In fact, for any edge $e \in E(u)$, if $t(e, T(u))$ is expressed by $n''(n'-n'')$ for some $n''$ $(0 < n'' \leq \frac{n'}{2})$, then $t(e, T) = n''(n-n'')$ holds. Consider another tree $T' \in \mathcal{T}_{V,L}^{T_1}$ such that $T'(u)$ is a subtree of $T'$ defined for $(v, u)$ and $u \in V^{T'(u)}$ holds. By Lemma 2, we have only to show that if $t^{T'(u)} \succ t^{T(u)}$ then $t^{T'}(u) \succ t^T(u)$ holds. Let $d$ be the smallest index $i$ satisfying $t_i^{T'(u)} > t_i^{T(u)}$, that is,

$$t_1^{T'(u)} = t_1^{T(u)}, \ldots, t_{d-1}^{T'(u)} = t_{d-1}^{T(u)}, t_d^{T'(u)} > t_d^{T(u)}.$$

Then it is easy to see that

$$t_1^{T'}(u) = t_1^T(u), \ldots, t_{d-1}^{T'}(u) = t_{d-1}^T(u), t_d^{T'}(u) > t_d^T(u)$$

hold, which implies that $t^{T'}(u) \succ t^T(u)$ is obtained. □

For a fixed vertex $v \in V$ and a fixed integer $M$ $(2 \leq M \leq L)$, Theorems 2 and 3 say that if $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}(M, v)$ then $t^T = [t_1^T, \ldots, t_{n-1}^T]$ satisfies

$$t_i^T = \begin{cases} (n'+1)(n-n'-1) & \text{for } i = 1, \ldots, r \\ n'(n-n') & \text{for } i = r+1, \ldots, m^T \end{cases}$$

for

$$m^T = \begin{cases} n-1 & \text{if } n-1 \leq M \\ M & \text{if } n-1 > M \end{cases}$$

and nonnegative integers $n'$ and $r$ with $n-1 = n'm^T + r$ $(0 \leq r < m^T)$. Also, it is obvious that $[t_{m^T+1}^T, \ldots, t_{n-1}^T]$ is the subsequence of $t^T$ in which every elements in $t^T(u_i)$ $(i = 1, \ldots, m^T)$ are rearranged in descending order. Further, we obtain the following theorem:

**Theorem 4** *Supposing that $L \geq 3$ holds, we find that $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}(M, v)$ if and only if $T$ satisfies the condition in Theorem 3 and each $T(u_i)$ minimizes $t^{T(u_i)}$ lexicographically in $\mathcal{T}_{V(u_i),L}(L-1, u_i)$ $(i = 1, \ldots, m^T)$.*

**Proof** It comes directly from Lemma 4. □

Further, for a tree $T = (V, E) \in \mathcal{T}_{V,L}$ and a vertex $v \in V$ $(m = \deg(v))$, we define a property called $(n, L, v)$-*balancedness* recursively as follows:

(i) If $n = 1$, then $T$ is $(1, L, v)$-balanced.

(ii) If $n > 1$, then let $T(u_i)$ $(i = 1, \ldots, m)$ be subtrees of $T$ defined for $(v, u_i)$ $(i = 1, \ldots, m)$. If

$$m = \begin{cases} n-1 & \text{if } n-1 \leq L-1 \\ L-1 & \text{if } n-1 > L-1 \end{cases}$$

and

$$T(u_i) \text{ is } \begin{cases} (n'+1, L, u_i)\text{-balanced} & \text{for } i = 1, \ldots, r \\ (n', L, u_i)\text{-balanced} & \text{for } i = r+1, \ldots, m \end{cases}$$

for nonnegative integer $n'$ and $r$ satisfying $n - 1 = n'm + r$ $(0 \leq r < m)$, then $T$ is $(n, L, v)$-balanced.

From this definition, any tree $T \in \mathcal{T}_{V,2}$ with $v \in V$ and $\deg(v) = 1$ is $(n, 2, v)$-balanced. Also, if $T$ is an $(n, L, v)$-balanced tree for $L \geq 3$, then $T \in \mathcal{T}_{V,L}(L-1, v)$ holds.

**Theorem 5** *Suppose that $L \geq 3$ holds, and consider a tree $T \in \mathcal{T}_{V,L}(L-1, v)$. Then $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}(L-1, v)$ if and only if $T$ is $(n, L, v)$-balanced.*

**Proof** Since the theorem holds obviously for $n = 1$, we consider the case of $n > 1$. For a tree $T \in \mathcal{T}_{V,L}(L-1, v)$, let $m = \deg(v)$, and let $T(u_i)$ $(i = 1, \ldots, m)$ be subtrees of $T$ defined for $(v, u_i)$ $(i = 1, \ldots, m)$. Assume that the theorem holds for each subtree $T(u_i)$ $(i = 1, \ldots, m)$. If $T$ is $(n, L, v)$-balanced, then it follows from the inductive assumption and Theorem 4 that $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}(L-1, v)$. If $T$ is not $(n, L, v)$-balanced, then consider the following two cases: (i) $m$, $n'$ and $r$ do not satisfy the condition of $(n, L, v)$-balancedness, (ii) $T(u_i)$ is not $(n'+1, L, u_i)$-balanced (nor $(n', L, u_i)$-balanced) for a certain $i$. In case (i), we find from Theorem 3 that $T$ does not minimize $t^T$ lexicographically. In case (ii), it follows from the inductive assumption and Theorem 4 that $T$ does not minimize $t^T$ lexicographically. □

From the above discussion, we obtain the following main result:

**Theorem 6** *For a given set $V$ with $|V| = n$ and a given integer $L$ $(\geq 2)$, let $T$ be a tree belonging to $\mathcal{T}_{V,L}$ and $v$ a vertex with $v \in V^T$. Also, let $m^T = \deg(v)$, and let $T(u_i)$ $(i = 1, \ldots, m^T)$ be subtrees of $T$ defined for edges $(v, u_i)$ $(i = 1, \ldots, m^T)$. Then $T$ minimizes $t^T$ lexicographically in $\mathcal{T}_{V,L}$ if and only if*

$$m^T = \begin{cases} n-1 & \text{if } n-1 \leq L \\ L & \text{if } n-1 > L \end{cases}$$

*holds and*

$$T(u_i) \text{ is } \begin{cases} (n'+1, L, u_i)\text{-balanced} & \text{for } i = 1, \ldots, r \\ (n', L, u_i)\text{-balanced} & \text{for } i = r+1, \ldots, m^T \end{cases}$$

*for nonnegative integers* $n'$ *and* $r$ *satisfying* $n-1 = n'm^T + r$ $(0 \leq r < m^T)$.

**Proof** Obviously, the theorem holds for $n \leq 3$ or $L = 2$. In other cases, we can prove the theorem by applying Theorem 4 for $M = L$ and Theorem 5. □

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