On the Toughness of the Total Graph of a graph

Masakazu Nihei
Fujishiro High School
Fujishiro, Ibaraki, 300-1537
Japan

ABSTRACT. The toughness $t(G)$ of a noncomplete graph G is defined as

$$t(G) = \min\{|S|/\omega(G-S) \mid S \subset V(G), \omega(G-S) \geq 2\},$$

where $\omega(G-S)$ is the number of components of $G-S$. We also define $t(K_n) = +\infty$ for every n.

The total graph $T(G)$ of a graph G is the graph whose vertex set can be put in one-to-one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident.

In this article, we study the the toughness of the total graph $T(G)$ of a graph G on at least 3 vertices and give especially that $t(T(G)) = \kappa(G)$ if $\kappa(G) = \lambda(G)$ and $K(G) \leq 2$, where $\kappa(G)$ and $\lambda(G)$ are the vertex and the edge-connectivity of G respectively.

1 Introduction and Preliminaries

In this article, all graphs are finite, undirected, without loops or multiple edges. The toughness of a graph is an invariant first introduced by Chvátal [1]. He observed some relationships between this parameter and the existence of hamiltonian cycles or k-factors. The toughness is an interesting invariant in graph theory.

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges respectively.

We denote the order of G by $|G|$ and the number of connected components of G by $\omega(G)$. If S is a subset of G with $\omega(G-S) \geq 2$, we call it a cutset of G. If $S \subset V(G)$, $\langle S \rangle$ is the subgraph of G induced by S. We write $G-S$ for $\langle V(G)-S \rangle$.

A graph G is t-tough if the implication

$$\omega(G-S) > 1 \rightarrow |S| \geq t \cdot \omega(G-S)$$

holds for any $S \subset V(G)$.

A complete graph is t-tough for any real number t. If G is not complete, there exists the largest t such that G is t-tough. This number is denoted by $t(G)$ and is called the toughness of G. We define $t(K_n) = +\infty$ for every n. If G is not complete,

$$t(G) = \min\{|S|/\omega(G-S) \mid S \subset V(G), \omega(G-S) \geq 2\}.$$

The total graph $T(G)$ of a graph G is the graph whose vertex set can be put in one-to-one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident.

The main purpose of this article is to study the toughness of the total graph of a graph. In what follows, we assume that $|G| \geq 3$.

Here, in order to prove our results in the section 2, we describe some well-known results. We first give the definition of the subdivision graph of a graph.

The subdivision graph $S(G)$ of a graph G is obtained from G by inserting an additional vertex into each edge of G.

The square G^2 of a graph G has $V(G^2) = V(G)$ with u, v adjacent in G^2 whenever $d(u,v) \leq 2$ in G. Then the following result is well known.

Theorem A. ([3]) *Let G be a graph. Then $T(G) \cong S^2(G)$, where the symbol $\cong$ means isomorphism.*

Let us denote the vertex-connectivity of a graph G by $\kappa(G)$ and the edge-connectivity of G by $\lambda(G)$ respectively. Then we have $\kappa(G) \leq \lambda(G)$, which is proven in [2]. Moreover we have the following theorem, which is used in the proof of Theorem 1 in the next section.

Theorem B. ([1]) *Let G be a graph, then $t(G^2) \geq \kappa(G)$.*

2 Results

In order to prove our main theorem, we need the following two lemmas.

Lemma 1. *Let G be a graph with $K(G) \leq 2$, then $\kappa(S(G)) = \kappa(G)$.*

Proof: If G is disconnected, there is nothing to show. Hence we may assume that G is connected. Let κ (> 0) be the vertex-connectivity of graph G and M be a cutset of G such that $|M| = \kappa$.

From the definition of subdivision graph $S(G)$, we easily see that M is also a cut set of $S(G)$. Hence we have that $\kappa(S(G)) \le |M| = \kappa$. Therefore we may prove that $\kappa(S(G)) \ge \kappa$.

Let U be a cutset of $S(G)$. In $S(G)$, we call the vertices of G a α-vertices of $S(G)$ and call the other vertices β-vertices of $S(G)$. Then we can write that $U = X \cup Y$, where X is a set of α-vertices and Y is a set of β-vertices of $S(G)$ respectively.

Case 1. When $Y = \emptyset$.

Since $U = X$ and $K(G) \le 2$, X becomes a cutset of G. Hence we have that $|U| = |X| \ge \kappa$.

Case 2. When $X = \emptyset$.

$U = Y$ is a set of β-vertices of $S(G)$. Now, let $U = \{w_1, \ldots, w_r\}$ and let e_i be the edge of G which β-vertex w_i is inserted. Then since U is a cutset of $S(G)$, $F = \{e_1, \ldots, e_r\}$ becomes a edge-cutset of G. Hence, we obtain that $|U| = |F| \ge \lambda(G) \ge \kappa(G) = \kappa$.

Case 3. When $X \ne \emptyset$ and $Y \ne \emptyset$.

Let us set $Y = \{z_1, \ldots, z_p\}$, and let $e_i = u_i v_i$ be the edge of G which β-vertex z_i is inserted. Then, if the set $M = \{e_1, \ldots, e_p\}$ is a edge-cutset of G, it is clear that $|U| \ge |M| \ge \lambda(G) \ge \kappa(G) = \kappa$. Therefore we may assume that M is not a edge-cutset of G. Then we identify z_i with either u_i or v_i. From now, we use v_i. Note that vertex v_i does not always differ from vertex v_j though $z_i \ne z_j$.

From the definition of $S(G)$, we easily see that $F = X \cup \{v_1, \ldots, v_k\}$ $(1 \le k \le p)$ is a cutset of G. This implies that $|U| \ge |F| \ge \kappa$, which completes the proof. □

Lemma 2. *Let G be a graph and $\lambda(G)$ be the edge-connectivity of G. Then there exists a cutset M of $S^2(G)$ such that $|M| \le 2\lambda(G)$.*

Proof: For brevity, we shall denote $\lambda(G)$ by λ. Since it is clear when λ is zero, we may assume that λ is a positive integer.

Since the edge-connectivity of G is λ, there exists an edge set $F = \{e_1, e_2, \ldots, e_\lambda\}$ such that $G - F$ is disconnected. Here we distinguish two cases.

Case 1. When $G - F$ has isolated vertices.

Let us denote such a vertex by u and let the degree of u in G be p $(1 \le p \le \lambda)$. So let $N_G(u) = \{v_1, v_2, \ldots, v_p\}$ and let w_i $(i = 1, 2, \ldots, p)$ be a vertex inserted into the edge uv_i $(i = 1, 2, \ldots, p)$ of G when we construct $S^2(G)$. Then we easily see that $M = \{w_1, w_2, \ldots, w_p, v_1, v_2, \ldots, v_p\}$ is a cutset of $S^2(G)$ and $|M| \le 2\lambda$.

Case 2. When not Case 1.

Let w_i $(i = 1, 2, \ldots, \lambda)$ be a vertex inserted into the edge e_i of G when we construct $S^2(G)$. Here let us choose either of the two endvertices of e_i, and denote it z_i. Then we can also easily check that the set $M = \{w_1, \ldots, w_\lambda, z_1, \ldots, z_k\}$ $(1 \leq k \leq \lambda)$ is a cutset of $S^2(G)$ and $|M| \leq 2\lambda$. This completes the proof. □

Theorem 1. *Let G be a graph with $K(G) \leq 2$, then*

$$\kappa(G) \leq t(T(G)) \leq \lambda(G).$$

Proof: Let λ be the edge-connectivity of G. Since $T(G) = S^2(G)$, from Lemma 2, there exists a cutset M of $T(G)$ such that $|M| \leq 2\lambda$. Hence, we have

$$t(T(G)) \leq |M|/k(T(G) - M) \leq 2\lambda/2 = \lambda \tag{1}$$

On the other hand, $t(T(G)) = t(S^2(G)) \geq \kappa(S(G))$ from Theorem B. By the way, from Lemma 1, $\kappa(S(G)) = \kappa(G)$. Hence we have

$$t(T(G)) \geq \kappa(G). \tag{2}$$

Combining (1) with (2), we obtain the desired result. □

Corollary 1. *Let G be a graph with $K(G) \leq 2$. If $\kappa(G) = \lambda(G)$, then $t(T(G)) = \kappa(G)$.*

Corollary 2. *If a connected graph G has a bridge, $t(T(G)) = 1$.*

The middle graph $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. Then we can easily see that the middle graph $M(G)$ of a graph G is a spanning subgraph of $T(G)$. Hence we have that $t(M(G) \leq t(T(G))$ (see [1], Propositionl.1) . Moreover we have in [4] that $t(M(G)) = \lambda(G)/2$. Therefore, from Theorem 1, we obtain the following result.

Theorem 2. *Let G be a graph with $K(G) \leq 2$, then*

$$\max\{2, \lambda(G)/2\} \leq t(T(G)) \leq \lambda(G).$$

3 Conclusion

The following conjecture is an interesting one, which is stated in [5].

Conjecture. *A 2-tough graph on at least 3 vertices is hamiltonian.*

If this conjecture is true, we can have that the total graph of any 2-connected graph is hamiltonian.

Conversely, if we can show that there exists a 2-connected graph G such that $T(G)$ is not hamiltonian, we can give a counter-example of the above conjecture. But we have not found such a graph.

References

[1] V. Chvátal, Tough graphs and Hamiltonian circuits, *Discrete Math.* **5** (1973), 215–228.

[2] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Second edition, Wadsworth & Brooks / Cole, Advanced Books & Software, Monterey, CA (1986).

[3] M. Behzat, A criterion for the planarity of a total graph, *Proc. Cambridge Philos. Soc.* **63** (1967), 679–681.

[4] M. Nihei, On the toughness of the middle graph of a graph, *Ars Combin.* **60** (2001), 55–58.

[5] D. Bauer, H.J. Broersma, J. van den Heuvel and H.J. Veldman, On Hamiltonian properties of 2-tough graphs, *J. Graph Theory* **6** (1994), 539–543.