TOUGH-MAXIMUM GRAPHS

S.A. Choudum* and N. Priya
Department of Mathematics
Indian Institute of Technology, Madras
Chennai - 600 036, INDIA
*e-mail: sac@pallava.iitm.ernet.in

Abstract. We characterize tough-maximum graphs, that is graphs having maximum number of edges among all graphs with given number of vertices and toughness.

1 Introduction

Chvàtal [3] introduced the toughness parameter of a graph and made a few interesting conjectures involving toughness, Hamilton cycles and regular factors. Since then there have been several papers on this interesting invariant.

For a graph G, let $p(G)$, $e(G)$, $\omega(G)$ respectively denote the number of vertices, edges and components. All our graphs are finite, undirected and simple. If $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are two vertex disjoint graphs, then (i) the **union** $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and (ii) the **join** $G_1 + G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. For any graph G, rG denotes the union of r *disjoint* copies of G. K_p is the complete graph on p vertices. A subset S of $V(G)$ is a vertex-cut of G, if either $G - S$ is disconnected or $G - S = K_1$. We follow [1] for general terminology. The **toughness** $t(G)$ of a graph G is defined as follows:

$$t(G) = \min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \text{ is a vertex-cut of } G\}.$$

We have slightly altered the original definition of Chvàtal, so that $t(K_p) = p-1$ instead of ∞. However, for incomplete graphs, both definitions coincide. A set $S \subseteq V(G)$ is called a t-set if $t(G) = \frac{|S|}{\omega(G-S)}$.

A graph G is called **t-maximal** if $t(G+x) > t(G)$, for every new edge x. The t-maximal graphs were characterized in [5] as follows.

Theorem A: A graph G is t-maximal if and only if (i) $G = K_p$ or (ii) $G = K_{p_1} \cup K_{p_2}$ or (iii) $G = K_s + \{K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}\}$.

A graph G with p vertices is called a **(p, t)-maximum** graph if it has maximum number of edges among all t-maximal graphs on p vertices with toughness t. In this paper, we characterize (p, t)-maximum graphs. A new vulnerability paremater, called the **tenacity**, was introduced in [4], on the lines of toughness. In [2], we have characterized tenacity maximum graphs.

2 Tough-Maximum Graphs

The characterization of (p, t)-**maximum** graphs is not straightforward even with the knowledge of Theorem A, since a graph G can achieve toughness t, either by the existence of a t-set S_1 such that $|S_1| = m$ and $\omega(G - S_1) = n$, or by the existence of a t-set S_2 such that $|S_2| = lm$ and $\omega(G - S_2) = ln$. In view of this, we have proceeded as follows to characterize t-maximum graphs:

1. We first characterize pairs (p, t) for which there exists a graph G with p vertices and toughness t.
2. We next compare the number of edges in the graphs G_1 and G_2 with t-sets S_1 and S_2 respectively such that $|S_1| = m$, $\omega(G_1 - S_1) = n$ and $|S_2| = lm$ and $\omega(G_2 - S_2) = ln$.

Clearly, $0 \leq t(G) \leq p-1$, and, moreover, $t(G) = 0$ iff G is disconnected, and $t(G) = p - 1$ iff $G = K_p$. So $K_{p-1} \cup K_1$ is the unique $(p, 0)$-maximum graph, and K_p is the unique $(p, p - 1)$-maximum graph. Therefore it is enough if we characterize incomplete, connected t-maximum graphs. Consequently, in the following, whenever $G = K_s + \{K_{p_1} \cup \cdots \cup K_{p_n}\}$ is t-maximal, we assume that $s \geq 1$ and $n \geq 2$.

We adopt the following terminology and notation.

- $\mathcal{T} = \{G : G \text{ is } t\text{-maximal }\}$.
- $\mathcal{H}(p, t) = \{G \in \mathcal{T} : p(G) = p \text{ and } t(G) = t\}$.
  If $\mathcal{H}(p, t) \neq \phi$, then (p, t) is called a t-**valid pair.**
- $\mathcal{H}^*(p, t) = \{G \in \mathcal{H}(p, t) : e(G) \geq e(H), \text{ for every } H \in \mathcal{H}(p, t)\}$.
- $\mathcal{H}(p, m, n) = \{G \in \mathcal{T} : p(G) = p$ and for some t-set S of G, $|S| = m$ and $\omega(G - S) = n\}$.
  If $\mathcal{H}(p, m, n) \neq \phi$, (p, m, n) is called a t-**valid triple.**
- $\mathcal{H}^*(p, m, n) = \{G \in \mathcal{H}(p, m, n) : e(G) \geq e(H), \text{ for every } H \in \mathcal{H}(p, m, n)\}$.

Remark: If $(p, t = \frac{m}{n})$ is a t-valid pair, then (p, m, n) need not be a t-valid triple, but (p, r, s) is a t-valid triple for some r, s with $\frac{m}{n} = \frac{r}{s}$. This and similar statements are implicitly made in [6].

Theorem 1: *If $G = K_s + (D_1 \cup D_2 \cup \cdots \cup D_n)$ is a t-maximal graph, then $V(K_s)$ is the unique t-set, and so $t(G) = \frac{s}{n}$.*

Proof: By the definition of $t(G)$, we have $t(G) \le \dfrac{|V(K_s)|}{\omega(G - V(K_s))} = \dfrac{s}{n}$. Next, let S be any t-set of G. Since S is a vertex-cut and every vertex-cut of G contains $V(K_s)$, we have $V(K_s) \subseteq S$. Moreover, if $x \in S - V(K_s)$, and $S' = S - \{x\}$, we have

$$\frac{|S'|}{\omega(G-S')} \le \frac{|S|-1}{\omega(G-S)} < \frac{|S|}{\omega(G-S)},$$

a contradiction to the fact that S is a t-set. Hence, $S = V(K_s)$ and $t(G) = \frac{s}{n}$. □

Theorem 2: *An integral triple (p, m, n) is t-valid if and only if $p \ge m+n$. Hence, if (p, km, kn) is t-valid for some $k \ge 2$, then $(p, (k-1)m, (k-1)n)$ is also t-valid.*

Proof: Suppose (p, m, n) is t-valid. Then there exists a graph G on p vertices with a t-set S such that $|S| = m$ and $\omega(G - S) = n$. If $D_1, D_2, \ldots, D_n$ are the components of $G - S$, then $p = |S| + \sum_{i=1}^{n} |D_i| \ge m + n$. On the other hand, for any triple (p, m, n) with $p \ge m+n$, $G := K_m + (K_{p-m-n+1} \cup (n-1)K_1)$ is a graph on p vertices with toughness $\frac{m}{n}$ (by Theorem 1). Hence, the integral triple (p, m, n) is t-valid. □

Theorem 3: *If $p \ge m + n$, then $\mathcal{H}^*(p, m, n)$ contains only one graph, namely*

$$G^*(p, m, n) := K_m + (K_{p-m-n+1} \cup (n-1)K_1).$$

Proof: Let $p \ge m + n$ and $G^* \in \mathcal{H}^*(p, m, n)$. Since G^* is t-maximal, $G^* \simeq K_s + (K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_k})$. By Theorem 1, $s = m$, $k = n$. We next claim that at most one $p_i \ne 1$. If, on the contrary, for some i and j, $1 < p_i \le p_j$, let $x \in V(K_{p_i})$, and define a graph G' as follows: $V(G') = V(G^*)$, and $E(G') = E(G^*) - \{(x, y) : y \ne x,\ y \in V(K_{p_i})\} \cup \{(x, z) : z \in V(K_{p_j})\}$. By Theorem A, G' is t-maximal and by Theorem 1, $t(G') = \frac{m}{n}$ and so $G' \in \mathcal{H}(p, m, n)$. But, $e(G') = e(G^*) - (p_i - 1) + (p_j) > e(G^*)$, a contradiction, since $G^* \in \mathcal{H}^*(p, m, n)$. Hence the claim is true; that is $G^* \simeq G^*(p, m, n)$. □

Let

- $G^*(p, km, kn) = K_{km} + (K_{p-km-kn+1} \cup (kn-1)K_1)$.

Clearly

(1) $$e_k := e(G^*(p, km, kn)) = \binom{km}{2} + km(p - km) + \binom{p - km - kn + 1}{2}.$$

Let $(p, t = \frac{m}{n})$ be a pair where $\frac{m}{n} = \frac{m'}{n'}$ and $gcd(m', n') = 1$. It follows by Theorem 2 that if $(p, \frac{m}{n})$ is t-valid, then $(p, \frac{m'}{n'})$ is t-valid. Hence, throughout the following, we assume that m, n are integers with $gcd(m, n) = 1$, and if $(p, \frac{m}{n})$ is t-valid, then let L denote the largest integer such that $p \geq Lm+Ln$. By Theorem 1, $t(G^*(p, km, kn)) = \frac{m}{n}$, for every $k, 1 \leq k \leq L$. Our next aim is to identify the maximum e_i in the sequence $(e_1, e_2 \ldots, e_L)$ *generated by the t-valid pair* $(p, \frac{m}{n})$.

Theorem 4: *If* $e(G^*(p, m, n)) \leq e(G^*(p, 2m, 2n))$, *then* $L \leq 2$.

Proof: By the hypothesis,

$$\binom{m}{2} + m(p-m) + \binom{p-m-n+1}{2} \leq \binom{2m}{2} + 2m(p-2m) + \binom{p-2m-2n+1}{2}.$$

On simplification, we have $p \leq 3m + \frac{3n}{2} - \frac{m}{n} - \frac{1}{2}$. If $L \geq 3$, then $3m + 3n \leq p \leq 3m + \frac{3n}{2} - \frac{m}{n} - \frac{1}{2}$, which is a contradiction. Therefore $L \leq 2$. □

Theorem 5: *Suppose* $L \geq 3$. *If* $e_{L-1} \geq e_L$, *then* $e_{L-2} > e_{L-1}$. *So, if* $(p, t = \frac{m}{n})$ *is t-valid and* $e_{L-1} \geq e_L$, *then* $e_1 > e_2 > \cdots > e_{L-1} \geq e_L$.

Proof: Since, $e_k := \binom{km}{2} + km(p-km) + \binom{p-km-kn+1}{2}$, and $e_{L-1} \geq e_L$, using the identity $\binom{a+b}{2} = \binom{a}{2} + \binom{b}{2} + ab$, we deduce that

$$(2) \qquad Lm^2 - m(p - Lm + m) + \binom{m+n}{2} + (p - Lm - Ln + 1)(m+n) \geq \binom{m}{2} + (Lm - m)m.$$

To prove $e_{L-2} > e_{L-1}$, we have to show

$$\binom{Lm-2m}{2} + (Lm-2m)(p-Lm+2m) + \binom{p-Lm+2m-Ln+2n+1}{2} > \binom{Lm-m}{2} + (Lm-m)(p-Lm+m) + \binom{p-Lm+m-Ln+n+1}{2},$$

which on simplification reduces to (2). □

By similar arithmetic, we have

Theorem 6: *If for some* k, $3 \leq k \leq L-2$, $e_k \leq e_{k+1}$, *then* $e_{k+1} < e_{k+2}$. □

Theorems 4, 5 and 6 imply the following monotonic property of the sequence $(e_1, e_2, \ldots, e_L)$.

Corollary 1: Let $(p, \frac{m}{n})$ be a t-valid pair. The sequence $(e_1, e_2, \ldots e_L)$ generated by it is

(1) of length 2 with $e_1 \leq e_2$, or

(2) monotonically decreasing: $e_1 > e_2 > \ldots > e_{L-1} \geq e_L$, or

(3) parabolic: $e_1 > e_2 > \ldots e_{k-1} \geq e_k < e_{k+1} < \ldots e_L$ with $e_1 = e_L$ or $e_1 < e_L$ or $e_1 > e_L$. □

As examples, we display four t-valid pairs $(p, t = \frac{m}{n})$ and the corresponding four discrete curves $(e_1, e_2, \ldots, e_L)$ described in (2) and (3) in Figure 1.

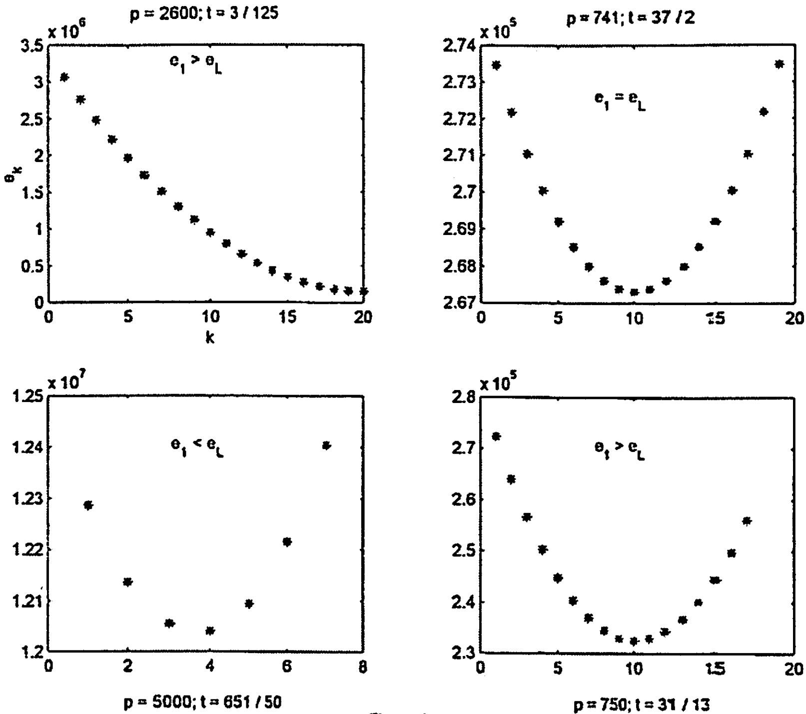

Figure 1

It is now easy to characterize (p,t)-maximum graphs as follows.

Theorem 7: *Let G be a connected incomplete graph with p vertices and $t(G) = t = \frac{m}{n}$ where $gcd(m,n) = 1$. Let L be the largest integer such that $p \geq Lm + Ln$, and e_k be the combinatorial number defined in (1). Then G is (p,t)-maximum iff it is isomorphic to*

$$G^* = \begin{cases} G^*(p,m,n) \text{ or } G^*(p,2m,2n), & \text{if } e_1 = e_2, \\ G^*(p,2m,2n), & \text{if } e_1 < e_2, \\ G^*(p,m,n), & \text{if } e_1 > e_2 \text{ and } L = 2, \\ G^*(p,m,n), & \text{if } L \geq 3 \text{ and } e_{L-1} \geq e_L, \\ G^*(p,m,n), & \text{if } L \geq 3,\ e_{L-1} < e_L \text{ and } e_1 \geq e_L, \\ G^*(p,Lm,Ln), & \text{if } L \geq 3,\ e_{L-1} < e_L \text{ and } e_1 \leq e_L. \end{cases}$$

Proof: The first two alternatives follow by Theorem 4, the third is obvious, the fourth follows by Theorem 5, and the fifth and sixth follow by Theorems 4, 6 and Corollary 1. □

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory and Applications, Macmillan, London, 1976.

[2] S.A. Choudum and N. Priya, *Tenacity Maximum Graphs*, to appear in The Journal of Combinatorial Mathematics and Combinatorial Computing.

[3] V. Chvàtal, *Tough Graphs and Hamiltonian Circuits*, Discrete Mathematics **5** (1973), 215-228.

[4] M. Cozzens, D. Moazzami and S. Stueckle, *The Tenacity of a Graph*, in "Proceedings of the Seventh Quadrennial International Conference on the Theory and Applications of Graphs", Y. Alavi and A. Schwenk (eds.), John Wiley & Sons, Inc., 1995, pp. 1111-1122.

[5] W.D. Goddard and H.C. Swart, *On Some Extremal Problems in Connectivity*, in "Graph Theory, Combinatorics and Applications, Proceedings of the Sixth Quadrennial International Conference on the Theory and Applications of Graphs", Y. Alavi, G. Chartrand, O.R. Oellermann and A.J. Schwenk (eds.), John Wiley & Sons, New York, 1991, pp. 535-551.

[6] M.D. Plummer, *A Note on Toughness and Tough Components*, Congressus Numerantum **125** (1997), 179-192.