Isomorphisms of Cyclic Abelian Covers of Symmetric Digraphs, III

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ABSTRACT. Let D be a connected symmetric digraph, Γ a group of automorphisms of D, and A a finite abelian group with some specified property. We discuss the number of isomorphism classes of g-cyclic A-covers of D with respect to a group Γ of automorphisms of D. Furthermore, we enumerate the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D for the cyclic group Z_{2^m} of order 2^m, where I is the trivial subgroup of $Aut\ D$.

1 Introduction

Graphs and digraphs treated here are finite and simple.

A graph H is called a *covering* of a graph G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. The projection $\pi : H \longrightarrow G$ is an *n-fold covering* of G if π is n-to-one. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $Aut\ H$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph and A a finite group. Let $D(G)$ be the arc set of the symmetric digraph corresponding to G. Then a mapping $\alpha : D(G) \longrightarrow A$ is called an *ordinary voltage assignment* if $\alpha(v, u) = \alpha(u, v)^{-1}$ for each $(u, v) \in D(G)$. The (*ordinary*) *derived graph* G^{α} derived from an ordinary voltage assignment α is defined as follows:

*Supported by Grant-Aid for Science Research (C)

$V(G^\alpha) = V(G) \times A$, and $((u,h),(v,k)) \in D(G^\alpha)$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$.

The graph G^α is called an A-*covering* of G . The A-covering G^α is an $| A |$-fold regular covering of G. Every regular covering of G is an A-covering of G for some group A (see [3]).

Let D be a symmetric digraph and A a finite group. A function $\alpha : A(D) \longrightarrow A$ is called *alternating* if $\alpha(y,x) = \alpha(x,y)^{-1}$ for each $(x,y) \in A(D)$. For $g \in A$, a *g-cyclic* A-*cover* $D_g(\alpha)$ of D is the digraph as follows:

$V(D_g(\alpha)) = V(D) \times A$, and $((u,h),(v,k)) \in A(D_g(\alpha))$ if and only if $(u,v) \in A(D)$ and $k^{-1}h\alpha(u,v) = g$.

The *natural projection* $\pi : D_g(\alpha) \longrightarrow D$ is a function from $V(D_g(\alpha))$ onto $V(D)$ which erases the second coordinates. A digraph D' is called a *cyclic* A-*cover* of D if D' is a g-cyclic A-cover of D for some $g \in A$. In the case that A is abelian, then $D_g(\alpha)$ is called simply a *cyclic abelian cover.* Furthermore the 1-cyclic A-cover $D_1(\alpha)$ of a symmetric digraph D can be considered as the A-covering G^α of the underlying graph G of D.

Let α and β be two alternating functions from $A(D)$ into A, and let Γ be a subgroup of the automorphism group $Aut\ D$ of D, denoted $\Gamma \leq Aut\ D$. Let $g,h \in A$. Then two cyclic A-covers $D_g(\alpha)$ and $D_h(\beta)$ are called Γ-*isomorphic*, denoted $D_g(\alpha) \cong_\Gamma D_h(\beta)$, if there exist an isomorphism $\Phi : D_g(\alpha) \longrightarrow D_h(\beta)$ and a $\gamma \in \Gamma$ such that $\pi\Phi = \gamma\pi$, i.e., the diagram

$$\begin{array}{ccc} D_g(\alpha) & \xrightarrow{\ \Phi\ } & D_h(\beta) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ D & \xrightarrow{\ \gamma\ } & D \end{array}$$

commutes. Let $I = \{1\}$ be the trivial group of automorphisms.

A general theory of graph coverings is developed in [4]. Z_2-coverings (double coverings) of graphs were dealed in [5] and [18]. Hofmeister [6] and, independently, Kwak and Lee [10] enumerated the I-isomorphism classes of n-fold coverings of a graph, for any $n \in N$. Dresbach [2] obtained a formula for the number of strong isomorphism classes of regular coverings of graphs with voltages in finite fields. The I-isomorphism classes of regular coverings of graphs with voltages in finite dimensional vector spaces over finite fields were enumerated by Hofmeister [7]. Hong, Kwak and Lee [8] gave the number of I-isomorphism classes of Z_n-coverings, $Z_p \bigoplus Z_p$-coverings and D_n-coverings, n:odd, of graphs, respectively. Sato [16] counted the Γ-isomorphism classes of Z_p-coverings of graphs for any prime $p(> 2)$.

Cheng and Wells [1] discussed isomorphism classes of cyclic triple covers (1-cyclic Z_3-covers) of a complete symmetric digraph. Furthermore, Mizuno and Sato [13] gave a formula for the characteristic polynomial of a cyclic A-cover of a symmetric digraph, for any finite group A. Mizuno and Sato [12] discussed the number of Γ-isomorphism classes of cyclic V-covers of a connected symmetric digraph for any finite dimensional vector space V over the finite field $GF(p)(p > 2)$. For a connected symmetric digraph D, Mizuno and Sato [15] obtained a sufficient condition for two Γ-isomorphism classes of cyclic abelian covers of D to be of the same cardinality, and presented the number of I-isomorphism classes of g-cyclic Z_{p^m}-covers of D for any prime $p(> 2)$. For a connected cyclic A-covers, Mizuno, Lee and Sato [14] enumerated the number of I-isomorphism classes of connected g-cyclic A-covers of D, when A is the cyclic group Z_{p^m} and the direct sum of m copies of Z_p for any prime $p(> 2)$. Sato [17] gave a characterization for two cyclic abelian covers to be Γ-isomorphic, and counted the number of I-isomorphism classes of g-cyclic A-covers of a connected bipartite symmetric digraph for $A = Z_p^m, Z_{p^m}$.

In Section 2, we discuss the number of Γ-isomorphism classes of g-cyclic A-covers of a connected symmetric digraph D for any finite abelian group A. In Section 3, we treat the enumeration and the structure of Γ-isomorphism classes of orbit-cyclic A-covers of D. In Section 4, we count the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D.

2 Isomorphisms of cyclic abelian covers

Let D be a symmetric digraph and A a finite group. The group Γ of automorphisms of D acts on the set $C(D)$ of alternating functions from $A(D)$ into A as follows:

$$\alpha^{\gamma}(x, y) = \alpha(\gamma(x), \gamma(y)) \text{ for all } (x, y) \in A(D),$$

where $\alpha \in C(D)$ and $\gamma \in \Gamma$. Any voltage $g \in A$ determines a permutation $\rho(g)$ of the symmetric group S_A on A which is given by $\rho(g)(h) = hg,\ h \in A$.

From now on, assume that D is connected and A is abelian. Let G be the underlying graph, T a spanning tree of G and w a root of T.For any $\alpha \in C(D)$ and any walk W in G, the *net α-voltage* of W, denoted $\alpha(W)$, is the sum of the voltages of the edges of W. Then the *T-voltage* α_T of α is defined as follows:

$$\alpha_T(u, v) = \alpha(P_u) + \alpha(u, v) - \alpha(P_v) \text{ for each } (u, v) \in D(G) = A(D),$$

where P_u and P_v denote the unique walk from w to u and v in T, respectively. Note that $\alpha_T(u, v) = 0$ for each $(u, v) \in A(T)$. For a function $f : A(D) \longrightarrow A$, the *net f-value* $f(W)$ of any walk W is defined as the net α-voltage of W: $f(W) = f(a_1) + \cdots + f(a_n),\ W : a_1, \cdots, a_n$.

For a function $f : A(D) \longrightarrow A$, let $A_f(v)$ denote the subgroup of A generated by all net f-values of the closed walk based at $v \in V(D)$. Let $ord(g)$ be the order of $g \in A$. For a subset B of A, let $< B >$ denote the subgroup of A generated by B.

Sato [17] gave a characterization for two cyclic A-covers of D to be Γ-isomorphic. Let $d_T(u, v)$ be the distance between u and v in T.

Theorem 1. [17, Theorem 2] *Let D be a connected symmetric digraph, A a finite abelian group, $g, h \in A$ and $\alpha, \beta \in C(D)$. Furthermore, let G be the underlying graph of D, T a spanning tree of G and $\Gamma \leq Aut\ G$. Then the following are equivalent:*

1. $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$.
2. *There exist $\gamma \in \Gamma$ and an isomorphism $\sigma :< A_{\alpha_T - \epsilon g}(w) \cup \{2g\} > \longrightarrow < A_{\beta_{\gamma T} - \epsilon^\gamma h}(\gamma(w)) \cup \{2h\} >$ such that*

$$\beta^\gamma_{\gamma T}(u, v) - \epsilon^\gamma h = \sigma(\alpha_T(u, v) - \epsilon g) \text{ for each } (u, v) \in A(D)$$

and

$$\sigma(2g) = 2h,$$

where $(\alpha_T - g)(u, v) = \alpha_T(u, v) - g$, $(u, v) \in A(D)$, $w \in V(D)$ and

$$\epsilon^\gamma(u, v) = \begin{cases} 1 & \text{if the distance } d_{\gamma T}(\gamma u, \gamma v) \text{ is even,} \\ 0 & \text{otherwise} \end{cases}, \ \epsilon(u, v) = \epsilon^1(u, v).$$

Sketch of proof: Suppose that $D_g(\alpha) \cong {}_\Gamma D_h(\beta)$. By [15, Corollary 1] and [12, Theorem 3.1], there exist a family $(\pi_u)_{u \in V(D)} \in S_A^{V(D)}$ and $\gamma \in \Gamma$ such that

$$\rho(\beta^\gamma_{\gamma T}(u, v) - h) = \pi_v \rho(\alpha_T(u, v) - g)\pi_u^{-1} \text{ for each } (u, v) \in A(D).$$

Let $(u, v) \in D(T)$. Then we have $\pi_v = \rho(h)\pi_u\rho(-g) = \rho(-h)\pi_u\rho(g)$, and so

$$\rho(2h) = \pi_u \rho(2g)\pi_u^{-1}.$$

Let $P : (u, v), (v, z)$ be any path of length two in D. Then we have

$$\rho(\beta^\gamma_{\gamma T}(P) - 2h) = \pi_z \rho(\alpha_T(P) - 2g)\pi_u^{-1}.$$

If $(u, v), (v, z) \in D(T)$, then we have $\pi_u = \pi_z$. Since D is connected, for any $z \in V(D)$, we have

$$\pi_z = \begin{cases} \pi_w & \text{if } d_T(w, z) \text{ is even,} \\ \rho(-h)\pi_w\rho(g) & \text{otherwise,} \end{cases}$$

where $w \in V(D)$.

Let $(v,z) \in A(D) \setminus D(T)$. If $d_T(v,z)$ is even, then we have

$$\rho(\beta^{\gamma}_{\gamma T}(v,z) - h) = \pi_w \rho(\alpha_T(v,z) - g)\pi_w^{-1}.$$

In the case that $d_T(v,z)$ is odd, we have

$$\rho(\beta^{\gamma}_{\gamma T}(v,z)) = \pi_w \rho(\alpha_T(v,z))\pi_w^{-1}.$$

Therefore it follows that $\rho(\beta^{\gamma}_{\gamma T}(v,z) - \epsilon^{\gamma} h) = \pi_w \rho(\alpha_T(v,z) - \epsilon g)\pi_w^{-1}$ for each $(v,z) \in A(D)$, where

$$\epsilon = \begin{cases} 1 & \text{if } d_T(v,z) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Hence there exists an isomorphism $\sigma :< A_{\alpha_T - \epsilon g}(w) \cup \{2g\} > \longrightarrow < A_{\beta_{\gamma T} - \epsilon^{\gamma} h}(\gamma(w)) \cup \{2h\} >$ such that

$$\beta^{\gamma}_{\gamma T}(u,v) - \epsilon^{\gamma} h = \sigma(\alpha_T(u,v) - \epsilon g) \text{ for each } (u,v) \in A(D).$$

and $\sigma(2g) = 2h$.

The converse is omitted. □

For $(v,z) \in A(T)$, we have $d_T(v,z) = 1$, and so $\epsilon = 0$. Note that, if G is bipartite and $g \notin A_{\alpha_T}(w)$, then $2g \notin A_{\alpha_T - \epsilon g}(w)$ may arise.

An finite group $\mathcal{B}$ is said to have the *isomorphism extension property (IEP)*, if every isomorphism between any two isomorphic subgroups $\mathcal{E}_1$ and $\mathcal{E}_2$ of $\mathcal{B}$ can be extented to an automorphism of $\mathcal{B}$ (see [8]). For example, the cyclic group Z_n for any $n \in N$, the dihedral group D_n for odd $n \geq 3$, and the direct sum of m copies of Z_p have the IEP.

Corollary 1. *Let D, A, T and Γ be as in Theorem 1, $g \in A$ and $\alpha, \beta \in C(D)$. Assume that A has the IEP. Then the following are equivalent:*

1. $D_g(\alpha) \cong {}_{\Gamma} D_g(\beta)$.
2. *There exists $\gamma \in \Gamma$ and $\sigma \in Aut\ A$ such that*

   $$\beta^{\gamma}_{\gamma T}(u,v) - \epsilon^{\gamma}(u,v)g = \sigma(\alpha_T(u,v) - \epsilon(u,v)g) \text{ for each } (u,v) \in A(D)$$

   and

   $$\sigma(2g) = 2g.$$

Corollary 2. *Let D, A, T and Γ be as in Theorem 1, $g \in A$ and $\alpha, \beta \in C(D)$. Assume that A has the IEP. Then the following are equivalent:*

1. $D_g(\alpha) \cong {}_{I} D_g(\beta)$.

2. *There exists* $\sigma \in Aut\ A$ *such that*

$$\beta_T(u,v) - \epsilon(u,v)g = \sigma(\alpha_T(u,v) - \epsilon(u,v)g) \text{ for each } (u,v) \in A(D)$$

and

$$\sigma(2g) = 2g.$$

Let D be a connected symmetric digraph, G its underlying graph and A a finite abelian group. The set of ordinary voltage assignments of G with voltages in A is denoted by $C^1(G;A)$. Note that $C(D) = C^1(G;A)$. Furthermore, let $C^0(G;A)$ be the set of functions from $V(G)$ into A. We consider $C^0(G;A)$ and $C^1(G;A)$ as additive groups. The homomorphism $\delta : C^0(G;A) \longrightarrow C^1(G;A)$ is defined by $(\delta s)(x,y) = s(x) - s(y)$ for $s \in C^0(G;A)$ and $(x,y) \in A(D)$. For each $\alpha \in C^1(G;A)$, let $[\alpha]$ be the element of $C^1(G;A)/Im\delta$ which contains α.

The automorphism group $Aut\ A$ acts on $C^0(G;A)$ and $C^1(G;A)$ as follows:

$$(\sigma s)(x) = \sigma(s(x)) \text{ for } x \in V(D),$$

$$(\sigma\alpha)(x,y) = \sigma(\alpha(x,y)) \text{ for } (x,y) \in A(D),$$

where $s \in C^0(G;A)$, $\alpha \in C^1(G;A)$ and $\sigma \in Aut\ A$.

We state a characterization for two cyclic A-covers to be Γ-isomorphic, for any finite abelian group A with the IEP.

Theorem 2. *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group with the IEP, α, $\beta \in C(D)$, $g, h \in A$ and $\Gamma \leq Aut\ D$. Then the following are equivalent:*

1. $D_g(\alpha) \cong_\Gamma D_h(\beta)$.

2. *There exist* $\sigma \in AutA$, $\gamma \in \Gamma$ *and* $s \in C^0(G;A)$ *such that*

$$\beta - \epsilon h = \sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \delta s \text{ and } \sigma(2g) = 2h.$$

Proof: By Theorem 1 and Corollary 1 of [15]. □

Now we consider the number of Γ-isomorphism classes of cyclic A-covers of a connected symmetric digraph D. Let G be the underlying graph of D, A a finite abelian group with the IEP and $\Pi = Aut\ A$. For any $g \in A$, set

$$\Pi_g = \{\sigma \in \Pi \mid \sigma(g) = g\}.$$

Then Π_g is a subgroup of Π.

Let $\Gamma \leq Aut\ D$ and $g \in A$. Set $H^1(G;A) = C^1(G;A)/Im\delta$. An actions of $\Pi_{2g} \times \Gamma$ on $H^1(G;A)$ are defined as follows:

$$(\sigma,\gamma)[\alpha] = [\sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g] = \{\sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g + \delta s \mid s \in C^0(G;A)\},$$

where $\sigma \in \Pi_{2g}$, $\gamma \in \Gamma$ and $\alpha \in C^1(G;A)$. Then this action is well-defined by the following result.

Proposition 1. *$\sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g$ is alternating.*

Proof: Since $\sigma \in \Pi_{2g}$, we have $\sigma(2g) = 2g$, i.e., $\sigma(g) - g = g - \sigma(g)$. Let $(u,v) \in A(D)$. If $d_T(u,v)$ is odd, then we have $\epsilon(u,v) = \epsilon(v,u) = \epsilon^\gamma(u,v) = \epsilon^\gamma(v,u) = 0$, and so

$$\begin{aligned}\sigma\alpha^\gamma(v,u) - \epsilon^\gamma(v,u)\sigma(g) + \epsilon(v,u)g &= \sigma\alpha^\gamma(v,u) = -\sigma\alpha^\gamma(u,v)\\ &= -\{\sigma\alpha^\gamma(u,v) - \epsilon^\gamma(u,v)\sigma(g) + \epsilon(u,v)g\}.\end{aligned}$$

In the case that $d_T(u,v)$ is even, we have $\epsilon(u,v) = \epsilon(v,u) = \epsilon^\gamma(u,v) = \epsilon^\gamma(v,u) = 1$, and so

$$\begin{aligned}\sigma\alpha^\gamma(v,u) - \epsilon^\gamma(v,u)\sigma(g) + \epsilon(v,u)g &= -\sigma\alpha^\gamma(u,v) - \sigma(g) + g\\ &= -\sigma\alpha^\gamma(u,v) + \sigma(g) - g\\ &= -\{\sigma\alpha^\gamma(u,v) - \epsilon^\gamma(u,v)\sigma(g) + \epsilon(u,v)g\}.\end{aligned}$$

Therefore, $\sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g$ is alternating. □

Theorem 2 implies that the number of Γ-isomorphism classes of g-cyclic A-covers of D is equal to that of $\Pi_{2g}\times\Gamma$-orbits on $H^1(G;A)$. Let $Iso(D, A, g, \Gamma)$ be the number of Γ-isomorphism classes of g-cyclic A-covers of D.

We estimate the number $Iso(D, A, g, \Gamma)$ by the number of some orbits on the 1-cohomology group.

Theorem 3. *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group with the IEP, $g, h \in A$ and $\Gamma \leq Aut\ D$. Assume that $\kappa(g) = h$ for some $\kappa \in Aut\ A$. Then*

$$Iso(D, A, g, \Gamma) = Iso(D, A, h, \Gamma).$$

Proof: Set $\Pi = Aut\ A$. By Theorem 2, we have

$$Iso(D, A, g, \Gamma) = \frac{1}{|\Pi_{2g}| \cdot |\Gamma|} \sum_{(\sigma,\gamma)\in\Pi_{2g}\times\Gamma} |H^1(G;A)^{(\sigma,\gamma)}|.$$

Let κ be an automorphism of A such that $\kappa(g) = h$. Furthermore, let $(\sigma, \gamma) \in \Pi_{2g}\times\Gamma$. Then $[\alpha] \in H^1(G;A)^{(\sigma,\gamma)}$ if and only if $\sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g = \alpha + \delta s$ for some $s \in C^0(G;A)$, i.e.,

$$\kappa\sigma\kappa^{-1}(\kappa\alpha)^\gamma - \epsilon^\gamma\kappa\sigma\kappa^{-1}(h) + \epsilon h = \kappa\alpha + \delta(\kappa s).$$

Since $\kappa\sigma\kappa^{-1} \in \Pi_{2h}$, $[\alpha] \in H^1(G;A)^{(\sigma,\gamma)}$ if and only if $[\kappa\alpha] \in H^1(G;A)^{(\kappa\sigma\kappa^{-1},\gamma)}$. By the fact that a mapping $[\alpha] \longmapsto [\kappa\alpha]$ is bijective, we have

$$|H^1(G;A)^{(\sigma,\gamma)}| = |H^1(G;A)^{(\kappa\sigma\kappa^{-1},\gamma)}| \quad \text{for each } (\sigma,\gamma) \in \Pi_{2g}\times\Gamma.$$

Since $\Pi_{2h} = \kappa \Pi_{2g} \kappa^{-1}$, it follows that

$$Iso(D, A, g, \Gamma) = Iso(D, A, h, \Gamma).$$

□

Now, we consider the number of Γ-isomorphism classes of g-cyclic A-covers of D in the case of $ord(2g) = 1$.

Proposition 2. *Let D, G, A and Γ be as in Theorem 2. If $ord(2g) = 1$, then the number of Γ-isomorphism classes of g-cyclic A-covers of D is equal to that of Γ-isomorphism classes of A-coverings of G.*

Proof: Since $ord(2g) = 1$, we have $\Pi_{2g} = \Pi = Aut\ A$ and $g = -g$. For $(u, v) \in A(D)$, $((u, h), (v, k)) \in A(D_g(\alpha))$ if and only if $k = \alpha(u, v) + h - g$, i.e., $h = \alpha(v, u) + k - g$. Then $((u, h), (v, k)) \in A(D_g(\alpha))$ if and only if $((v, k), (u, h)) \in A(D_g(\alpha))$. Thus, $D_g(\alpha)$ is an A-covering $G^{\alpha - g}$ of G. Therefore the result follows. □

Let D be a connected symmetric digraph, p prime and $F_p = GF(p)$ the finite field with p elements. Let F_p^r be the r-dimensional vector space over F_p. Then the additive group F_p^r has the IEP and the general linear group $GL_r(F_p)$ is the automorphism group of F_p^r. Furthermore, $GL_r(F_p)$ acts transitively on $F_p^r \backslash \{0\}$.

Corollary 3. *Let D be a connected symmetric digraph, G its underlying graph and $\Gamma \leq Aut\ D$. Let g, h be any two elements of $F_2^r \backslash \{0\}$. Then*

$$Iso(D, F_2^r, g, \Gamma) = Iso(D, F_2^r, h, \Gamma).$$

Furthermore, $Iso(D, F_2^r, g, \Gamma)$ is equal to that of Γ-isomorphism classes of F_2^r-coverings of G.

Proof: Note that $2g = 2h = 0$. By Theorem 3 and Proposition 2. □

In the case of $p > 2$, the similar result to Corollary 3 is obtained by [12,15].

For a connected symmetric digraph D, let $B(D) = m - n + 1$ be the *Betti-number* of D, where $m = \mid A(D) \mid /2$ and $n = \mid V(D) \mid$. We give the number of I-isomorphism classes of g-cyclic F_2^r-covers of D for any $g \in F_2^r$.

Corollary 4. *Let D be a connected symmetric digraph and $g \in F_2^r$. Then the number of I-isomorphism classes of g-cyclic F_2^r-covers of D is*

$$Iso(D, F_2^r, g, I) = 1 + \sum_{h=1}^{r} \frac{(2^B - 1)(2^{B-1} - 1) \cdots (2^{B-h+1} - 1)}{(2^h - 1)(2^{h-1} - 1) \cdots (2 - 1)},$$

where $B = B(D)$.

Proof: By Corollary 2 of [11]. □

3 Isomorphisms of orbit-cyclic abelian covers

Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $\Gamma \leq Aut\ D$ and $\Pi = Aut\ A$. For a nonidentity element g of A, the Π-orbit on A containing g is denoted by $\Pi(g)$. A cyclic A-cover $D_h(\alpha)$ of D is called *$\Pi(g)$-cyclic* if $h \in \Pi(g)$. Let $\mathcal{D}_k$ be the set of all k-cyclic A-covers of D for any $k \in A$, and let $\mathcal{D} = \bigcup_{h\in\Pi(g)} \mathcal{D}_h$. Then $\mathcal{D}$ is the set of all $\Pi(g)$-cyclic A-covers of D. Let $\mathcal{D}/\cong_\Gamma$ and $\mathcal{D}_h/\cong_\Gamma$ be the set of all Γ-isomorphism classes over $\mathcal{D}$ and $\mathcal{D}_h$, respectively. The Γ-isomorphism class of $\mathcal{D}_h$ containing $D_h(\alpha)$ is denoted by $[D_h(\alpha)]$.

Theorem 4. *Let D be a connected symmetric digraph,* A *a finite abelian group with the IEP, $\Gamma \leq Aut\ D$ and $\Pi = Aut\ A$. Furthermore, let $g \in A \setminus \{0\}$. Then*

$$|\mathcal{D}/\cong_\Gamma| = Iso(D, A, h, \Gamma) \text{ for each } h \in \Pi(g).$$

Proof: For any $h \neq g \in \Pi(g)$ and any $\tau \in C(D)$, let

$$\beta = \sigma^{-1}(\tau^\gamma + \epsilon\sigma(g) - \epsilon^\gamma h - \delta s),\ \gamma \in \Gamma,\ s \in C^0(G; A),$$

where σ is an automorphism of A such that $\sigma(g) = h$, and G is the underlying graph of D. By Theorem 2, we have $D_g(\beta) \cong_\Gamma D_h(\tau)$.

For each $h \neq g \in \Pi(g)$, we define a map $\Phi_h : \mathcal{D}_g/\cong_\Gamma \longrightarrow \mathcal{D}_h/\cong_\Gamma$ by

$$\Phi_h([D_g(\beta)]) = [D_h(\tau)],$$

where $D_g(\beta) \cong_\Gamma D_h(\tau)$. Since $\cong_\Gamma$ is an equivalence relation over $\mathcal{D}$, Φ_h is injective. By Theorem 3, we have

$$|\mathcal{D}_g/\cong_\Gamma| = |\mathcal{D}_h/\cong_\Gamma| < \infty.$$

Thus Φ_h is a bijection. Therefore it follows that

$$|\mathcal{D}/\cong_\Gamma| = Iso(D, A, h, \Gamma).$$

□

Let $A = F_2^r$. Then a cyclic F_2^r-covers $D_g(\alpha)$ is called *nonzero-cyclic* if g is not equal to the nuit 0 of F_2^r.The set $\mathcal{D}$ is the set of all nonzero-cyclic F_2^r-covers.

Corollary 5. *Let D be a connected symmetric digraph. Then*

$$|\mathcal{D}/\cong_I| = 1 + \sum_{h=1}^{r} \frac{(2^B-1)(2^{B-1}-1)\cdots(2^{B-h+1}-1)}{(2^h-1)(2^{h-1}-1)\cdots(2-1)},$$

This is a generalization of Corollary 9 in [17].

Now, we state the structure of Γ-isomorphism classes of $\Pi(g)$-cyclic A-covers of D.

Theorem 5. *Let D be a connected symmetric digraph, A a finite abelian group with the IEP, $\Gamma \leq Aut\ D$ and $\Pi = Aut\ A$. Let σ_h be a fixed automorphism of A such that $\sigma_h(g) = h$ for $h \in \Pi(g)$. Then any Γ-isomorphism class of $\Pi(g)$-cyclic A-covers of D is of the form*

$$\bigcup_{h \in \Pi(g)} \{D_h(\sigma_h \beta) \mid \beta = \sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g + \delta s,\ \sigma \in \Pi_{2g},\ \gamma \in \Gamma,\ s \in C^0(G; A)\},$$

where $\alpha \in C(D)$ and G is the underlying graph of D.

Proof: Let $\tau \in C(D)$, $h \neq g \in \Pi(g)$ and $[[D_h(\tau)]]$ the Γ-isomorphism class of $\mathcal{D}$ containing $D_h(\tau)$. By the first half of the proof of Theorem 4, there exists a g-cyclic A-cover $D_g(\alpha)$ such that $[[D_h(\tau)]] = [[D_g(\alpha)]]$.

In the proof of Theorem 4, the map Φ_h is a bijection from $\mathcal{D}_g/\cong_\Gamma$ into $\mathcal{D}_h/\cong_\Gamma$ for any $h \neq g \in \Pi(g)$. Thus there exists an h-cyclic A-cover $D_h(\beta)$ such that $D_g(\alpha) \cong_\Gamma D_h(\beta)$ for any $h \neq g \in \Pi(g)$. We define a map $\Psi_h : [D_g(\alpha)] \longrightarrow [D_h(\beta)]$ by

$$\Psi_h(D_g(\alpha')) = D_h(\sigma_h\alpha'),\ \alpha' = \sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g + \delta s,$$

where $\sigma \in \Pi_{2g}$, $\gamma \in \Gamma$, $s \in C^0(G; A)$ and $\sigma_h(g) = h$. By Theorem 2, Ψ_h is well-defined. It is clear that Ψ_h is injective.

Now, let $D_h(\eta)$ be any element of $[D_h(\beta)]$. Then we have $D_h(\eta) \cong_\Gamma D_g(\alpha)$. By Theorem 2, there exist $\sigma' \in \Pi$, $\nu \in \Gamma$ and $t \in C^0(G; A)$ such that $\eta = \sigma'\alpha^\nu - \epsilon^\nu\sigma'(g) + \epsilon h + \delta t$ and $\sigma'(2g) = 2h$. Let $\mu = \sigma_h^{-1}\sigma'\alpha^\nu - \epsilon^\gamma\sigma_h^{-1}\sigma'(g) + \epsilon g + \delta(\sigma_h^{-1}t)$. Then we have

$$\sigma_h^{-1}\sigma' \in \Pi_{2g} \text{ and } \Psi_h(D_g(\mu)) = D_h(\eta).$$

Therefore Ψ_h is surjective, i.e., bijective. Hence it follows that

$$\begin{aligned}&[D_h(\beta)]\\ =&\{D_h(\sigma_h\alpha') \mid \alpha' = \sigma\alpha^\gamma - \epsilon^\gamma\sigma(g) + \epsilon g + \delta s,\ \sigma \in \Pi_{2g},\ \gamma \in \Gamma,\ s \in C^0(G; A)\},\end{aligned}$$

and so the result follows. □

Corollary 6. [15,Theorem 5] *Let D be a connected symmetric digraph, G its underlying graph, A a finite abelian group with the IEP and $\Gamma \leq Aut\ D$. Suppose that $g \in A$ has odd order. Let σ_h be a fixed automorphism of A such that $\sigma_h(g) = h$ for $h \in \Pi(g)$. Then any Γ-isomorphism class of $\Pi(g)$-cyclic A-covers of D is of the form*

$$\bigcup_{h \in \Pi(g)} \{D_h(\sigma_h\beta) \mid \beta = \sigma\alpha^\gamma + \delta s,\ \sigma \in \Pi_g,\ \gamma \in \Gamma,\ s \in C^0(G; A)\},$$

where $\alpha \in C(D)$.

Corollary 7. *Let D be a connected symmetric digraph, $\Pi = GL_r(F_2)$ and $\Gamma \leq Aut\ D$. Set $e = (10\cdots 0)^t \in F_2^r$. Furthermore, A_g be a fixed element of $GL_r(F_2)$ such that $A_g e = g$ for each $g \neq 0 \in F_2^r$. Then any Γ-isomorphism class of nonzero-cyclic F_2^r-covers of D is of the form*

$$\bigcup_{g \neq 0} \{D_g(A_g\beta) \mid \beta = B\alpha^{\gamma} - \epsilon^{\gamma} Be + \epsilon e + \delta s,\ , B \in \Pi,\ \gamma \in \Gamma,\ s \in C^0(G; F_2^r)\},$$

where $\alpha \in C(D)$.

4 Isomorphisms of cyclic Z_{2^m}-covers

Let Z_n be the cyclic group of order n. Then Z_n has the IEP.

We shall consider the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of a connected symmetric digraph D, for any $g \in Z_{2^m}$. Set $\Pi_{2g} = \{\sigma \in Aut\ Z_{2^m} \mid \sigma(2g) = 2g\}$.

Theorem 6. *Let D be a connected symmetric digraph, $g \in Z_{2^m}$ and $ord(2g) = 2^{m-\mu}$. Set $B = B(D)$. Then the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D is*

$$Iso(D, Z_{2^m}, g, I) = \begin{cases} 2^{mB-\mu} + 2^{(m-\mu)B-1}(2^{\mu(B-1)} - 1)/(2^{B-1} - 1) & \text{if } \mu \neq m \text{ and } B > 1, \\ 2^{m-\mu-1}(\mu + 2) & \text{if } \mu \neq m \text{ and } B = 1, \\ 2^{m(B-1)+1} - 1 + (2^{m(B-1)} - 1)/(2^{B-1} - 1) & \text{if } \mu = m \text{ and } B > 1, \\ m + 1 & \text{if } \mu = m \text{ and } B = 1, \end{cases}$$

Proof: Let G be the underlying graph of D and T a spanning tree of G. Set $E(G) \backslash E(T) = \{u_1v_1, \cdots, u_Bv_B\}$, where $B = B(D)$ is the Betti number of D. Assume that $d_T(u_i, v_i)$ is odd if $1 \leq i \leq l$, and even otherwise. Then, let

$$\mathcal{F}_T(Z_{2^m}; B) = \{(a_1 - g, \cdots, a_l - g, a_{l+1}, \cdots, a_B) \mid (a_1, \cdots, \cdots, a_B) \in (Z_{2^m})^B\}.$$

By Corollary 2, the number of I-isomorphism classes of g-cyclic Z_{2^m}-covers of D is equal to that of Π_{2g}-orbits on $\mathcal{F}_T(Z_{2^m}; B)$. By Burnside's Lemma, we have

$$Iso(D, Z_{2^m}, g, I) = \frac{1}{\mid \Pi_{2g} \mid} \sum_{\sigma \in \Pi_{2g}} \mid \mathcal{F}_T(Z_{2^m}; B)^{\sigma} \mid .$$

For $\rho \in \Pi$, let $\mathcal{F}(\rho) = \{h \in Z_{2^m} \mid \rho(h) = h\}$. Then we have

$$\mid \mathcal{F}_T(Z_{2^m}; B)^{\sigma} \mid = \mid (Z_{2^m})^{\sigma} \mid ^B = \mid \mathcal{F}(\sigma) \mid ^B.$$

But we have

$$\Pi_{2g} = \{\lambda \in Z_{2^m} \mid (\lambda, 2^m) = 1 \text{ and } \lambda 2g = 2g\}.$$

Then

$$\lambda \in \Pi_{2g} \Leftrightarrow 2\lambda g \equiv 2g \ (mod\ 2^m) \quad \Leftrightarrow \quad 2g(\lambda - 1) \equiv 0 \ (mod\ 2^m)$$
$$\Leftrightarrow \ \lambda - 1 \in < ord(2g) > .$$

Thus we have $\mid \Pi_{2g} \mid = 2^m/ord(2g)$. That is, $\mid \Pi_{2g} \mid = 2^\mu$ if $2g \in K_\mu(m)$, where $K_\mu(m) = \{k \in Z_{2^m} \mid k \in < 2^\mu >, \ k \notin < 2^{\mu+1} >\}$. Let $ord(2g) = 2^{m-\mu}$. If $ord(2g) = 1$, then $2g = 2^m$. Otherwise $\Pi_{2g} = \{2^{m-\mu}\nu + 1 \mid \nu = 0, 1, \cdots, 2^\mu - 1\}$.

By Lemma 3 of [8], $\mid \mathcal{F}(\rho) \mid = 2^k$ if $\rho - 1 \in K_k(m)$. Thus we have

$$\mid \{\lambda \in \Pi_{2g} \mid \mid \mathcal{F}(\lambda) \mid = 2^{m-\mu+t}\} \mid = 2^{\mu-t-1} \ (0 \le t \le \mu - 1),$$
$$\mid \{\lambda \in \Pi_{2g} \mid \mid \mathcal{F}(\lambda) \mid = 2^m\} \mid = 1.$$

Therefore the result follows. Specially, the third and fourth parts of the formula are given by Theorem 8 of [8]. □

Finally, we give an example for the computation of the number $Iso(D, A, h, \Gamma)$ when Γ is non-trivial. Let $D = KD_3$ be the complete symmetric digraph with three vertices 1,2,3, and $A = Z_4 = \{0, 1, 2, -1\}$ (the additive group). Furthermore, let $\Gamma = S_3$ be the symmetric group of degree 3 and $g = 1$. Then we have $G = K_3$(the complete graph with three vertices), $\Pi = Aut\ Z_4 = \{1, -1\}$ and $\Pi_2 = \Pi$. Thus, the number $Iso(KD_3, Z_4, 1, S_3)$ of S_3-isomorphism classes of 1-cyclic Z_4-covers of KD_3 is equal to that of $\Pi \times S_3$-orbits on $H^1(K_3; Z_4)$.

Let T be the spanning tree of K_3 such that $E(T) = \{23, 13\}$. By [15, Corollary 1], we have

$$D_1(\alpha) \cong {}_I D_1(\alpha_T)$$

for each $\alpha \in C(KD_3)$. Let $C_T(KD_3) = \{\alpha_T \mid \alpha \in C(KD_3)\}$. Then $Iso(KD_3, Z_4, 1, S_3)$ is equal to that of $\Pi \times S_3$-orbits on $C_T(KD_3)$.

All T-voltages are given as follows:

$$\alpha_i(1,2) = i, \alpha_i(2,3) = \alpha_i(1,3) = 0, \ i = 0, 1, 2, 3.$$

But, we have

$$\alpha_3 = \alpha_1^\gamma, \ \gamma = (12) \text{ and } \alpha_2 = \sigma\alpha_0 - \epsilon\sigma(1) + \epsilon \cdot 1, \ \sigma = -1.$$

Furthermore, if $\alpha_1 = \sigma\alpha_0^\gamma - \epsilon^\gamma\sigma(1) + \epsilon \cdot 1 + \delta s$, then we have $-1 = -\sigma(1) + 1 = 1$, a contradiction. Thus, the $\Pi \times S_3$-orbits on $C_T(KD_3)$ are given as follows: $\{\alpha_0, \alpha_2\}, \{\alpha_1, \alpha_3\}$. Therefore it follows that

$$Iso(KD_3, Z_4, 1, S_3) = 2,$$

and the representatives of S_3-isomorphism classes of 1-cyclic Z_4-covers of KD_3 are $D_1(\alpha_0), D_1(\alpha_1)$.

Acknowledgement. The author would like to thank the referee for helpful comments and useful suggestions.

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