Minimizing $\beta + \Delta$ and well covered graphs

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Abstract

Let G be a graph. Let γ denote the minimum cardinality of a dominating set in G. Let β, respectively i, denote the maximum, respectively minimum, cardinality of a maximal independent set in G. We show $\gamma + \Delta \geq \lceil 2\sqrt{n} - 1 \rceil$, where n is the number of vertices of G. A straightforward construction shows that given any G' there exists a graph G such that $\gamma(G) + \Delta(G) = \lceil 2\sqrt{n} - 1 \rceil$ and G' is an induced subgraph of G, making classification of these $\gamma + \Delta$ minimum graphs difficult.

We then focus on the subclass of these graphs with the stronger condition that $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. For such graphs $i = \beta$ and thus the graphs are *well-covered*. If G is graph with $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$, we have $\beta = \lceil \frac{n}{\Delta+1} \rceil$. We give a catalogue of all well-covered graphs with $\Delta \leq 3$ and $\beta = \lceil \frac{n}{\Delta+1} \rceil$. Again we establish that given any G' we can construct G such that G' is an induced subgraph of G and G satisfies $\beta = \lceil \frac{n}{\Delta+1} \rceil$. In fact, the graph G can be constructed so that $\beta(G) + \Delta(G) = \lceil 2\sqrt{n} - 1 \rceil$. We remark that $\Delta(G)$ may be much larger than $\Delta(G')$.

We conclude the paper by analyzing integer solutions to $\lceil \frac{n}{\Delta+1} \rceil + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. In particular for each n, the values of Δ that satisfy the equation form an interval. When n is a perfect square, this interval contains only one value, namely $\sqrt{n}$. For each (n, Δ)

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solution to the equation, there exists a graph G with n vertices, maximum degree Δ, and $\beta = \lceil \frac{n}{\Delta+1} \rceil$.

1 Introduction

We consider finite, simple graphs $G = (V, E)$ with n vertices. For $v \in V$, the *open neighbourhood of* v is given by $N(v) = \{u \in V | uv \in E\}$, and the *closed neighbourhood of* v is given by $N[v] = \{v\} \cup N(v)$. For a set of vertices $X \subseteq V$, we define $N(X) = \cup_{v \in X} N(v)$ and $N[X] = \cup_{v \in X} N[v]$. The *private neighbourhood of* v *in* X is $PN(v, X) = N[v] - N[X - \{v\}]$. If $N[X] = V$, we say X is a *dominating set* of G. Let γ denoted the minimum cardinality of a dominating set in G. (For a general graph parameter p, we use $p(G)$ to explicitly state the graph when necessary for clarity.) Let β, respectively i, denote the maximum cardinality of an independent set, respectively the minimum cardinality of a maximal independent set, in G. Since every maximal independent set is a dominating set, it is clear

$$\gamma \leq i \leq \beta$$

[2]. For an in depth study of domination theory the reader is directed to [9, 10]. The maximum degree and minimum degree in G are denoted Δ and δ respectively.

The *Gallai-type* problem of characterizing graphs with $\gamma + \Delta = n$ and $i + \Delta = n$ is studied in [4] and [5]. It is the case that $i + \Delta \leq n$ for any graph. We study the complementary extremal problem of characterizing graphs with $\gamma + \Delta$ as small as possible. Specifically we show $\gamma + \Delta \geq \lceil 2\sqrt{n} - 1 \rceil$. We also demonstrate that a complete characterization of graphs which achieve this minimum value is unlikely. Thus we turn our focus to a restricted class of graphs; those for which $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. Clearly in such graphs $\beta = i = \gamma$.

If $i = \beta$, we say G is *well-covered;* that is, each maximal independent set has the same cardinality, for example, see [8]. The problem of determining whether or not a graph is *not* well-covered is NP-complete, [1]. However, characterizations of subclasses of the well-covered graphs do exist. For example, well-covered graphs with girth at least five have been characterized in [7], see also [6].

We establish that if $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$, then $\beta = \left\lceil \frac{n}{\Delta+1} \right\rceil$. This latter condition implies G is well-covered. We characterize all such graphs with $\Delta \leq 3$. We also present evidence that a complete characterization of these graphs is unlikely.

We complete the paper with a study of the integer solutions to

$$\left\lceil \frac{n}{\Delta + 1} \right\rceil + \Delta = \lceil 2\sqrt{n} - 1 \rceil$$

For any integer solution, there exists a graph with n vertices, and maximum degree Δ, such that $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$.

2 $\gamma + \Delta$ is minimum

Proposition 2.1 *Let G be a graph with n vertices. Then $\gamma(G) + \Delta(G) \geq \lceil 2\sqrt{n} - 1 \rceil$.*

Proof: Let $D \subseteq V(G)$ be a dominating set of size γ. Each vertex in D can dominate at most $\Delta + 1$ vertices. Hence $\gamma \cdot (\Delta + 1) \geq n$. This implies that $\gamma + \Delta \geq \frac{n}{\Delta+1} + \Delta$. Fix n and let $f(\Delta) = \frac{n}{\Delta+1} + \Delta$. Consider Δ as a real variable and then we have the following:

$$f(\Delta) = \frac{n + \Delta^2 + \Delta}{\Delta + 1}$$
$$f'(\Delta) = \frac{\Delta^2 + 2\Delta + (1 - n)}{(\Delta + 1)^2}$$

Solving for the feasible root of f', we find f has a global minimum on the domain $0 \leq \Delta \leq n - 1$. This minimum is achieved when $\Delta = \sqrt{n} - 1$. Hence $\gamma + \Delta \geq f(\Delta) \geq 2\sqrt{n} - 1$. Restricting Δ and γ to integer values gives the result. ▮

Corollary 2.2 *Let G be a graph with the property $\gamma + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. Then $\gamma = \lceil \frac{n}{\Delta+1} \rceil$.*

We examine the problem of finding integer solutions to $\lceil \frac{n}{\Delta+1} \rceil + \Delta = \lceil 2\sqrt{n} - 1 \rceil$ in the final section of the paper. At this point we show that given an integer solution to this equation, there is a graph G with n vertices, maximum degree Δ, and domination number $\lceil \frac{n}{\Delta+1} \rceil$. Moreover, the following theorem shows that a forbidden subgraph classification of these minimum $\gamma + \Delta$ graphs is impossible.

Theorem 2.3 *Let n and D be integers such that $\lceil \frac{n}{D+1} \rceil + D = \lceil 2\sqrt{n} - 1 \rceil$. Then there exists a graph G such that $|V(G)| = n$, $\Delta(G) = D$, and $\gamma(G) = \lceil \frac{n}{D+1} \rceil$. Moreover, given any graph H on $0 \leq p \leq n - \lceil \frac{n}{D+1} \rceil$ vertices, with $\Delta(H) < D$, the graph G can be selected so that H is an induced subgraph of G.*

Proof: Let n and D be integers as above. Let $g = \lceil \frac{n}{D+1} \rceil$. Consider a graph H satisfying the above hypothesis. To $V(H)$ we add two disjoint sets of vertices X and Y such that $|X| = g$ and $|Y| = n - g - p$. Let

$V(G) = V(H) \cup X \cup Y$. Partition the vertices in $V(H) \cup Y$ into g parts, say $V_1, V_2, \ldots, V_g$ such that $|V_i| = D$ for $i = 1, 2, \ldots, g-1$ and $|V_g| = n - g - (g-1) \cdot D$. Note that $|V_g| \leq D$. Label the vertices of X as $x_1, x_2, \ldots, x_g$. Join each x_i to all the vertices of V_i. We now observe that $\gamma(G) \leq g$, $\Delta(G) = D$. Thus $\gamma(G) + \Delta(G) \leq \lceil 2\sqrt{n} - 1 \rceil$. ∎

We note that a strengthening of this result to $\Delta(H) \leq D$ is not possible. Consider $n = 9$, $\Delta = 2$, and $H = C_5$. Each vertex in a dominating set of G would need to dominate itself and two other private neighbours. Since $D = \Delta(H) = 2$, no edges can be added to the vertices in the C_5. However, C_5 does not have the property that any dominating set has each vertex dominating exactly three private neighbours.

Corollary 2.4 *Given any graph H, there exists a graph G such that $\gamma(G) + \Delta(G) = \lceil 2\sqrt{|V(G)|} - 1 \rceil$ and H is an induced subgraph of G.*

Proof: Let $D = \max\{\Delta(H) + 1, \left\lceil \sqrt{|V(H)|} \right\rceil\}$ and $n = (D+1)^2$. ∎

3 $\beta + \Delta$ is minimum

In light of the *non-structure* results above, we restrict our attention to graphs that satisfy the stronger condition that $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. Since $\beta(G) \geq i(G) \geq \gamma(G)$ for any graph G, we have $\beta + \Delta \geq \gamma + \Delta$, and $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$ implies $\gamma + \Delta = \lceil 2\sqrt{n} - 1 \rceil$

Defintion 3.1 *A graph G is* well-covered *if each maximal independent set has the same cardinality. That is, $\beta(G) = i(G)$.*

Proposition 3.2 *Let G be a graph on n vertices such that $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. Then*

1. *G is well-covered;*
2. *each component of G is well-covered;*
3. *$\beta = \gamma = \lceil \frac{n}{\Delta+1} \rceil$.*

Since any maximal independent set is a dominating set, and each vertex is such a set can dominate at most $\Delta + 1$ vertices, we have $|S| \cdot (\Delta + 1) \geq n$, for any maximal independent set S in a graph with n vertices. This gives the following result.

Proposition 3.3 *Let G be a graph with n vertices such that $\beta = \lceil \frac{n}{\Delta+1} \rceil$, then G is well-covered.*

In the special case that $(\Delta + 1)|n$, we can classify the graphs with $\beta = \frac{n}{\Delta+1}$.

Theorem 3.4 *Let G be a graph on n vertices where $\beta = \frac{n}{\Delta+1}$. Then $G = \beta K_{\Delta+1}$.*

Proof: Let $D \subseteq V(G)$ be a maximum independent set. Thus D is also a dominating set where each vertex can dominate at most $\Delta + 1$ vertices. Since $\beta \cdot (\Delta + 1) = n$, each vertex in D must have exactly $\Delta + 1$ private neighbours.

Let $v \in D$ and let x, y be private neighbours of v. Since G is well covered, x and y must be adjacent. Otherwise, $D \cup \{x, y\} - \{v\}$ is an independent set of size $\beta + 1$.

Thus, for any vertex $v \in D$: v has degree Δ, $N(v)$ is complete, and v has no common neighbours with any other vertex in D. That is, $G = \beta K_{\Delta+1}$. ∎

Corollary 3.5 *Let G be a graph on n vertices where $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$ and $(\Delta + 1)|n$. Then $G = \beta K_{\Delta+1}$.*

Corollary 3.6 *Let G be a graph on n vertices where $n = t^2$ for some integer t. Then $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$ if and only if $G = tK_t$.*

Proof: From Proposition 2.1 we know that $\frac{n}{\Delta+1} + \Delta$ has a minimum value of $2\sqrt{n} - 1 = 2t - 1$ when $\Delta = \sqrt{n} - 1 = t - 1$. Thus $(\Delta + 1)|n$. ∎

In the proof of Theorem 3.4 we observed that the private neighbours of any vertex in D must form a clique; otherwise, an independent set of size $|D| + 1$ exists. As a consequence we have the following result.

Corollary 3.7 *Let G be a well-covered graph. Suppose D is an maximal independent set in G. Then D is a dominating set and the private neighbours of each $v \in D$ form a clique.*

In Theorem 3.4 we observe that a disjoint union of cliques is the only graph on n vertices with $\beta = \frac{n}{\Delta+1}$. If $(\Delta + 1) \not| n$, it is easy to construct a graph with $\beta = \lceil \frac{n}{\Delta+1} \rceil$ by using a disjoint union of cliques. However, the following result shows it is impossible to classify all graphs with $\beta = \lceil \frac{n}{\Delta+1} \rceil$ in terms of forbidden subgraphs.

Theorem 3.8 *Let H be a graph. Then there exists a well-covered graph G such that H is an induced subgraph of G. Moreover, G can be constructed so that $\beta(G) + \Delta(G) = \lceil 2\sqrt{n} - 1 \rceil$.*

Proof: Let H be a graph on p vertices, say $\{1, 2, \ldots, p\}$. Consider p copies of K_t, each with a vertex labeled 0. Identify vertex 0 in copy i with vertex i in H, for $i = 1, 2, \ldots p$. We now have a graph on tp vertices with maximum degree $\Delta(H) + t - 1$. We add to this graph, m disjoint copies of $K_{t+\Delta(H)}$. Call this graph G. We now have $|V(G)| = tp + m(t + \Delta(H))$, $\Delta(G) = \Delta(H)+t-1$, and $\beta(G) = m+p$. We know that $\beta+\Delta \geq \lceil 2\sqrt{n}-1 \rceil$. If we find values of t and m such that $\beta + \Delta < 2\sqrt{n}$, then we can conclude that $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. Fix m and consider t as a real variable. Solving

$$m + p + \Delta(H) + t - 1 < 2\sqrt{tp + m(t + \Delta(H))}$$

for t gives

$$\begin{aligned} &m + 1 + p - \Delta(H) - 2\sqrt{m + p - p\Delta(H)} \\ &< t < \quad m + 1 + p - \Delta(H) + 2\sqrt{m + p - p\Delta(H)} \end{aligned}$$

Clearly, if $m > p(\Delta(H) - 1)$, the solution interval, for t, contains integer values. ∎

We remark that attaching a clique to each vertex of H in the above proof, ensures the resulting graph is well-covered. Graphs where the *pendant edges* form a matching play an important role in the study of well-covered graphs, [3, 6, 7]. (An edge is a pendant edge if it is incident with a vertex of degree one. Such graphs may be viewed as having a K_2 attatched to each vertex.)

This section concludes with a catalogue of graphs with $\beta = \lceil \frac{n}{\Delta+1} \rceil$ and Δ is small.

Proposition 3.9 *Suppose G is a graph on n vertices with $\beta = \lceil \frac{n}{\Delta+1} \rceil$.*

1. *If $\Delta = 0$, then $G = \beta K_1$.*
2. *If $\Delta = 1$, then $G = \beta K_2$ if n is even and $G = (\beta - 1)K_2 \cup K_1$ if n is odd.*

We now classify graph with $\Delta = 2$. Let $A = \{2K_2, C_4, P_4\}$ and $B = \{C_7, C_5 \cup K_2\}$. We delay the proofs our our catalogue theorems until the end of the paper.

Theorem 3.10 *Let G be a graph on $n = 3t + r$ vertices, where t and r are integers with $0 \leq r < 3$. Suppose $\Delta = 2$ and $\beta = \lceil \frac{n}{\Delta+1} \rceil$. Then G is one of the following graphs:*

1. *If $r = 0$, then $G = tK_3$.*

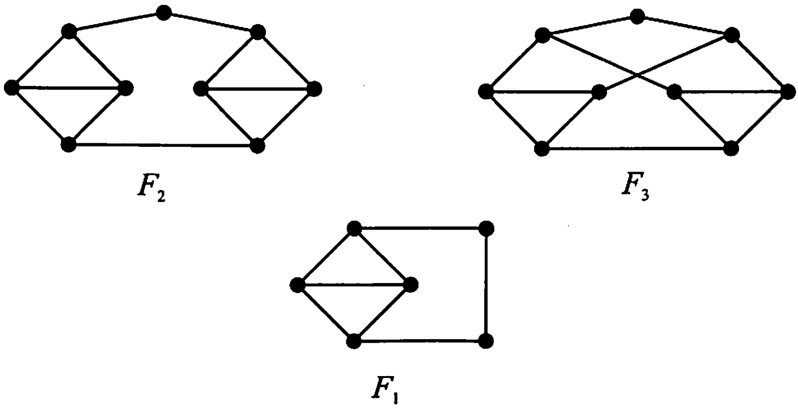

Figure 1: The graphs F_1, F_2, and F_3

2. If $r = 1$, *then* $G =$

  (a) $tK_3 \cup K_1$,
  (b) $(t-1)K_3 \cup H$, *where* $H \in A$,
  (c) $(t-2)K_3 \cup H$, *where* $H \in B$, *or*
  (d) $(t-3)K_3 \cup 2C_5$.

3. If $r = 2$, *then* $G = tK_3 \cup K_2$ *or* $(t-1)K_3 \cup C_5$.

The catalogue for $\Delta = 3$ uses the graphs F_1, F_2, and F_3 that appear in Figure 1. The following generalization of the disjoint union of two graphs is also used. Observe that $K_3 \cup K_3$ is a well-covered graph with $\beta = 2$. Also, if we add any edge xy with x in the first K_3 and y in the second, we still have a well-covered graph with $\beta = 2$. (Of course we now have $\Delta = 3$.) In fact, we can add any subset of a perfect matching between the two K_3's and retain a well-covered graph with $\beta = 2$ and $\Delta = 3$. This idea is generalized in the following definition.

Defintion 3.11 *Let* G_1, G_2, *and* G_3 *be graphs. The notation*

$$G_1 \cup G_2 \tilde{\cup} G_3$$

refers to any graph obtained from the graph

$$G_1 \cup G_2 \cup G_3$$

by adding edges between G_2 *and* G_3, *under the restriction that* $\Delta(G_1 \cup G_2 \tilde{\cup} G_3) = \Delta(G_1 \cup G_2 \cup G_3)$.

Theorem 3.12 *Suppose G is graph with $\beta = \left\lceil \frac{n}{\Delta+1} \right\rceil$ and $\Delta = 3$. Further suppose $|V(G)| = 4t + r$ where $0 \le r < 4$. Then G is one of the following graphs:*

1. *If $r = 0$, then $G = tK_4$.*
2. *If $r = 1$, then $G =$*
   (a) $tK_4 \cup K_1$,
   (b) $(t-1)K_4 \cup K_3 \tilde{\cup} K_2$,
   (c) $(t-1)K_4 \cup C_5$,
   (d) $(t-2)K_4 \cup K_3 \tilde{\cup} K_3 \tilde{\cup} K_3$,
   (e) $(t-2)K_4 \cup K_3 \tilde{\cup} F_1$,
   (f) $(t-2)K_4 \cup F_2$, *or*
   (g) $(t-2)K_4 \cup F_3$,
3. *If $r = 2$, then $G =$*
   (a) $tK_4 \cup K_2$
   (b) $(t-1)K_4 \cup K_3 \tilde{\cup} K_3$, *or*
   (c) $(t-1)K_4 \cup F_1$.
4. *If $r = 3$, then $G = tK_4 \cup K_3$.*

4 Feasible Δ, β pairs

We conclude the paper with a discussion of solutions to $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$.

Proposition 4.1 *Let n be a positive integer. Define t and k such that $n = t^2 - k$ where $0 \le k < 2t - 1$. Then (n, Δ) is a pair of integers satisfying*

$$\left\lceil \frac{n}{\Delta + 1} \right\rceil + \Delta = \lceil 2\sqrt{n} - 1 \rceil$$

if and only if Δ is an integer with

$$t - \sqrt{k} - 1 \le \Delta \le t + \sqrt{k} - 1, \text{ when } 0 \le k < t$$

or

$$t - \sqrt{4(k-t)+1} - \frac{3}{2} \le \Delta \le t + \sqrt{4(k-t)+1} - \frac{3}{2}, \text{ when } t \le k < 2t - 1$$

Proof: Suppose $n = t^2 - k$ where $0 \le k < t$. Then $(t-0.5)^2 < n \le t^2$ and we have $\lceil 2\sqrt{n} - 1 \rceil = 2t - 1$. Consequently, we need to solve

$$2t - 1 = \left\lceil \frac{n}{\Delta + 1} \right\rceil + \Delta = \left\lceil \frac{n + \Delta^2 + \Delta}{\Delta + 1} \right\rceil$$

Since the right hand side of this equation has a minimum value (over all Δ) of $2\sqrt{n} - 1 > 2t - 2$. We reduce our problem to solving

$$2t - 1 \ge \left(\frac{n + \Delta^2 + \Delta}{\Delta + 1} \right) = \left(\frac{t^2 - k + \Delta^2 + \Delta}{\Delta + 1} \right)$$

The solution is

$$t - \sqrt{k} - 1 \le \Delta \le t + \sqrt{k} - 1$$

The other case is similar. ∎

Corollary 4.2 *Let n be a fixed integer. The feasible values of Δ for any graph on n vertices with $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$ form an interval.*

When $n = t^2$ for some integer t, we know from the above proposition that any $\beta + \Delta$ minimum graph, say G, must have $\Delta = t - 1$. Thus, $(\Delta + 1) | n$ and $G = tK_t$. This is precisely a restatement of Corollary 3.6.

Corollary 4.3 *Let $n = t^2 - 1$ for some integer t, and let G be a graph on n vertices with $\beta + \Delta = \lceil 2\sqrt{n} - 1 \rceil$. Then G is one of the following graphs:*

- $(t+1)K_{t-1}$
- $(t-1)K_{t+1}$
- $(t-1)K_t \cup K_{t-1}$
- $K_3 \cup C_5$

Proof: Since $n = t^2 - 1$, we have $t - 2 \le \Delta \le t$. (To avoid trivialities, we assume that $t > 1$.) If $\Delta = t-2$ or $\Delta = t$, then $(\Delta+1)|n$, and by Corollary 3.5 we have $G = (t+1)K_{t-1}$ or $G = (t-1)K_{t+1}$, respectively.

Now suppose $\Delta = t - 1$ and let D be a maximum independent set of G. Consequently, $|D| = t$. Now each vertex in D has t private neighbours with the exception of one vertex which has $t-1$ private neighbours, or $t-2$ vertices of D have t private neighbours and 2 vertices, say u and v, each have $t - 1$ private neighbours and have one common neighbour say x. In the former case, it follows that $G = (t-1)K_t \cup K_{t-1}$. We complete the proof by showing in the latter case, $t = 3$ and $G = K_3 \cup C_5$.

First, if $t = 2$, then $n = 3$ and $\Delta = 1$. Consequently, $G = K_2 \cup K_1 = (t-1)K_t \cup K_{t-1}$. Hence $t \geq 3$. Now consider $S = N[\{u, v\}]$. The subgraph induced by S must have independence number 2; otherwise, we can find an independent set of size $\beta + 1$. The private neighbours of u and v must form cliques of size $t - 1$. Thus each member of $PN(u, S)$ can be adjacent to at most one vertex in $S \backslash PN(u, S)$. Similarly for the members $PN(v, S)$, they have at most one neighbour in $S \backslash PN(v, S)$. The vertex x can have at most $t - 3$ neighbours in $N(\{u, v\})$. Hence, there is at least one vertex in $PN(u, S)$, say y, which is not adjacent to x, and there is at least one vertex in $PN(u, S)$, say z, which is not adjacent to x. The vertices y and z must be adjacent to avoid an independent set of size three, namely $\{x, y, z\}$. We claim there can be no other member $y' \in N(u)$ such that $y' \notin N(x)$. Such a vertex y' gives the independent set $\{x, y', z\}$. (Note that z has degree Δ.) Similarly, there is no member, other that z, of $N(v)$ which is not adjacent to x. Since $|N(u) \cup N(v) - \{x\}| = 2t - 4$, and x is adjacent to all neighbours of u and v except y and z, we have $2t - 4 - 2 \leq t - 3$. Thus, $t \leq 3$ and we conclude $G = K_3 \cup C_5$. ∎

5 Proofs of catalogue theorems

Proof of Theorem 3.10: Suppose G is a graph satisfying the above hypothesis. Since $\Delta = 2$, G is a union of paths and cycles. If $r = 0$, then by Corollary 3.5 $G = tK_3$.

Suppose $r = 2$. This implies $\beta = t + 1$. Let D be a maximum independent set, (and thus D is a dominating set). Observe that $n = 3t + 2$ and $\beta \cdot (\Delta + 1) = 3t + 3 = n + 1$. Two situations can occur. First, there is some vertex $v \in D$ such that v dominates exactly two vertices (itself and one other) and every other vertex in D dominates exactly three vertices. As observed above, the private neighbours of each vertex in D form a clique, i.e. $G = tK_3 \cup K_2$.

The second possibility is that each vertex in D dominates exactly three vertices. Thus some vertex $x \in V - D$ is dominated by two distinct vertices u and v. Let $N(u) = \{x, y\}$ and $N(v) = \{x, z\}$. The vertices y and z must be adjacent, otherwise $D \cup \{x, y, z\} - \{u, v\}$ is an independent set of size $\beta + 1$. Hence $G = (t-1)K_3 \cup C_5$.

Finally suppose $r = 1$. This implies $\beta = t + 1$ and $\beta \cdot (\Delta + 1) = n + 2$. Let D be an independent set of size β. Partition D into D_1 and D_2. The vertices in D_1 each have three private neighbours and D_2 contains the remainder of the vertices. Thus, D_1 consists of $|D_1|$ copies of K_3 and $N[D_2]$ is K_3 free. Since $n \equiv 1 \bmod 3$, it must be the case that $|D_2| \equiv 1 \bmod 3$. Thus, $1 \leq |D_2|$. Also $|D_2| \leq 4$. To see this, suppose that $|D_2| = k$. Thus

$|D_1| = t+1-k$. Each vertex in D_1 dominates 3 vertices and each vertex in D_2 dominates itself, at most one other private neighbour, and at most one other vertex which is a common neighbour of two vertices in D_2. Thus $|N[D_2]| \leq 2.5k$. This gives $3(t+1-k)+2.5k \geq 3t+1$ and $k \leq 4$.

Observe that since D_1 dominates $3|D_1|$ vertices, D_2 dominates a total of $3|D_2|-2$ vertices. If $|D_2| = 1$, then $N[D_2] = K_1$. If $|D_2| = 2$, then we must construct a well covered graph on 4 vertices such that each independent set has size 2. The possibilities are $2K_2, P_4$ and C_4. If $|D_2| = 3$, then $|N[D_2]| = 7$. The possibilities for $N[D_2]$ are C_7 and $C_5 \cup K_2$. (Recall, $N[D_2]$ is a K_3-free union paths and cycles with $\beta = 3$.) If $|D_2| = 4$, then $|N[D_2]| = 10$. The only possibility is $2C_5$. ▮

Proof of Theorem 3.12: Let G be a graph satisfying the hypothesis. If $r = 0$, then by Corollary 3.5 $G = tK_4$.

Suppose that $r = 3$. Then n is odd and $\beta \cdot 4 = n+1$. Thus in any dominating set we have one of two situations: either all vertices have 4 private neighbours with the exception of one vertex who has 3 private neighbours; or all but two vertices have 4 private neighbours and the remaining two vertices have 3 private neighbours and one neighbour in common. We show that latter cannot occur and thus $G = tK_4 \cup K_3$.

Since n is odd, there is a vertex of even degree. The only possibility is there exists a vertex u of degree 2. Consider the graph $H = G \backslash N[u]$. We can add u to any maximal independent set in H, to obtain a maximal independent set in G of cardinality one larger. Thus H is well covered. Since H has $4t$ vertices, $\beta(H) = t$ and H is well covered, we know that $H = tK_4$. Hence, $N[u]$ sends no edges to H. Consequently u has three private neighbours which must form a clique. That is, $G = tK_4 \cup K_3$. This complete $r = 3$.

Now suppose that $r = 2$. Observe that $\beta = t+1$ and $\beta \cdot 4 = n+2$. We cannot have $\delta(G) = 0$, for such a vertex would dominate only itself (in any dominating set) resulting with a dominating set that dominates at most $4t+1 = n-1$ vertices. Consider the case $\delta(G) = 1$. Let u be a vertex of degree 1 and let v be its neighbour. Consider any β set of G containing u. Each vertex other that u must have 4 private neighbours to ensure $\beta \cdot 4 = n+2$. Thus each vertex in $G \backslash N[u]$ has degree three. Hence $G = tK_4 \cup K_2$.

We now examine the case $\delta(G) = 2$. Consider a vertex u with $\deg(u) = 2$ and let $H = G \backslash N[u]$. Again H is well covered with $\beta(H) = \beta(G) - 1$. Since $|V(H)| = 4(t-1)+3$, we know that $H = (t-1)K_4 \cup K_3$. Let w, v be the two neighbours of u. If $wv \in E(G)$, then the only possible edges between H and $N[u]$ are between the two K_3's. Thus $G = (t-1)K_4 \cup K_3 \tilde{\cup} K_3$. On the other hand, if $wv \notin E(G)$, we consider a β-set, say S, containing

u. We know that the private neighbours of u form a clique, thus w must dominated by some x in $V(H) \cap S$. Clearly the only vertices in H that can send edges to $N[u]$ are in the K_3. Let the x, y, z be the vertices of the K_3 in H. Observe the subgraph induced by $\{y, z, w, v\}$ must have independence number 2. Otherwise we can remove $\{x, u\}$ from S and add the three independent vertices from $\{y, z, w, v\}$ to from an independent set of size $\beta + 1$, contrary to the definition of β. To avoid an independent set of size three in $\{y, w, v\}$, we need one of the edges yw or yv to be present in G. Similarly, we need one of the zv or zw to be present. In both cases, $G = (t-1)K_4 \cup F_1$.

Suppose $\delta(G) = 3$. We recall $\beta = t + 1$ and $4\beta = n + 2$. Since G is 3-regular, any maximum independent set, say S, must dominate the graph in such a way that there are two vertices x, y (not in S) such that x and y are each adjacent to two vertices in S. First consider the case that x and y are both adjacent to the same two vertices in S, say u, v. Both u and v have one other neighbour each, say z and z' respectively. If $xy \in E(G)$, then z and z' cannot have degree 3. It is easy to verify that without loss of generality, xz and yz' are both edges. Finally, zz' must be an edge for G to be 3-regular. In this case $G = (t-1)K_4 \cup K_3 \tilde{\cup} K_3$. Next consider the case that $N(x) \cap S = \{t, u\}$ and $N(y) \cap S = \{w, v\}$. Both t and u must have two other neighbours, say t', t'' and u', u'', respectively. Since t' and t'' are private neighbours of t, they are adjacent. Similarly, u' and u'' are adjacent. The subgraph induced by $\{t', t'', x, u', u''\}$ cannot contain an independent set of size three. However, the vertex x has degree at most one in this induced subgraph. Without loss of say x is not adjacent to t', u', u''. Now this set must have an independent pair of vertices since t' can only be adjacent to one of u' or u'', resulting in an independent set of size three $N[\{t, u\}]$. This contradicts the fact that S is a maximum independent set. This completes the case that x and y do not have a common neighbour in S.

We conclude with the case that $N(x) \cap S = \{u, v\}$ and $N(y) \cap S = \{v, w\}$. Let $T = N(\{u, v, w\})$. We have $|T| = 7$ and the subgraph induced by T in G, say H, has maximum degree 2 and independence number 3. This implies that H is not bipartite. Both x and y have degree one in H. Thus H consists of $K_3 \cup P_4$ or $C_5 \cup P_2$. In both cases we get the subgraph of G induced by $N[\{u, v, w\}]$ contains an independent set of size 4, a contradiction.

Our final case to consider is $r = 1$. Thus, $n = 4t + 1$ and $\beta = t + 1$. Let u be a vertex of minimum degree. Since n is odd, we have $\deg(u) \leq 2$. If $\deg(u) = 0$, then $G = tK_4 \cup K_1$. If $\deg(u) = 1$, then let $H = G \backslash N[u]$. Now $|V(H)| = 4(t-1) + 3$. Hence $H = (t-1)K_4 \cup K_3$. We have $G = (t-1)K_4 \cup K_3 \tilde{\cup} K_2$.

Suppose $\deg(u) = 2$. Again let $H = G \backslash N[u]$. We have $|V(H)| = 4(t-1) + 2$. Let $N(u) = \{x, y\}$. First assume $xy \in E(G)$. We consider all possibilities for H. If $H = (t-1)K_4 \cup K_2$, then $G = (t-1)K_4 \cup K_3 \tilde{\cup} K_2$. If $H = (t-2)K_4 \cup K_3 \tilde{\cup} K_3$, then $G = (t-2)K_4 \cup K_3 \tilde{\cup} K_3 \tilde{\cup} K_3$. Finally, if $H = (t-2)K_4 \cup F_1$, then $G = (t-2)K_4 \cup K_3 \tilde{\cup} F_1$.

We now consider the case $xy \notin E(G)$. Begin by assuming that $H = (t-1)K_4 \cup K_2$. Let w be a vertex in the K_2 component in H, and let $S = N[\{u, w\}]$. We have $|S| = 5$ and the subgraph in G induced by S is well covered with independence number 2. If this subgraph is triangle free, then it must be C_5. If it contains a triangle then it must be $K_3 \tilde{\cup} K_2$.

Next consider $H = (t-2)K_4 \cup K_3 \tilde{\cup} K_3$. Let the two K_3s have vertex sets $\{w_1, w_2, w_3\}$ and $\{v_1, v_2, v_3\}$. Let $S = N[\{u, w_1, v_1\}]$. Observe that the subgraph induced by S is well covered and has independence number three. If any w_i or v_i $(i = 1, 2, 3)$ has degree two, then we can apply the case above ($xy \in E(G)$) with this degree 2 vertex equal to u. Thus assume all vertices in the two triangles have degree three. Since the private neighbours of u form a clique, we must have x adjacent to a vertex of the triangles, say w_1. If y is not adjacent to any vertex in $\{v_1, v_2, v_3\}$, then each of x, w_2, and w_3 must be adjacent to some v_i to ensure each v_i has degree 3. At this point y is only adjacent to u and every other vertex in G has degree 3. This contradicts our assumption that u has minimum degree. Hence suppose without loss of generality that $yv_1 \in E(G)$. We may also assume that $v_2w_2 \in E(G)$; otherwise, only x and y can send edges to $\{v_2, w_2, v_3, w_3\}$. The latter implies we have a vertex in this set with degree less than three. Consider v_3. If $v_3w_3 \in E(G)$, then $\{v_3, w_2, x, y\}$ forms an independent set of size four, a contradiction. If $v_3y \in E(G)$, then we must have $w_3x \in E(G)$. Thus $G = (k-2)K_4 \cup F_2$. If $v_3x \in E(G)$, then $w_3y \in E(G)$ and $G = (k-2)K_4 \cup F_3$.

We conclude with the case that $H = (t-2)K_4 \cup F_1$. The graph F_1 contains two nonadjacent vertices of degree three, say r and s. The set x, y, r, s forms an independent set of size four, implying that G contains an independent set of size $t+2$. This is impossible. This completes the proof.
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