2-Reducible paths containing a specified edge in $(2k+1)$-edge-connected graphs

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1 Introduction

We consider finite undirected multigraph without loops. Let G be a graph and let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. $\lambda(G)$ denotes the edge-connectivity of G. We allow repetition of vertices (but not edges) in a path or cycle. k is a natural number. When k is fixed and $\lambda(G) \geq k$, we call a path (or cycle) P 2-reducible if $\lambda(G-E(P)) \geq k-2$.

Let k (≥ 5) is odd, $\lambda(G) \geq k$, $T = \{u,v,s,t\} \subseteq V(G)$ and let $\{uv = f, vs\} \subseteq E(G)$. It is known that there is a 2-reducible s,t-path containing f ([5]). We here prove that there are distinct edges $g_i \neq f$ $(1 \leq i \leq (k-1)/2)$ incident to u such that there is a 2-reducible s,t-path P_i containing f and g_i. The result is useful to give a sufficient condition for existence of a 2-reducible s,t-path containing a given edge incident to u ([8]).

Let X, Y, $\{x\} \subseteq V(G)$, $X \cap Y = \emptyset$ and F, $\{f\} \subseteq E(G)$. We often denote $\{x\}$ by x, $X \cup \{x\}$ by $X + x$ and $F \cup \{f\}$ by $F + f$. We denote by $\partial(X,Y;G)$ the set of edges with one end in X and the other in Y, and define $\partial(X;G) := \partial(X, V(G)-X;G)$, $e(X,Y;G) := |\partial(X,Y;G)|$ and $e(X;G) := |\partial(X;G)|$. We denote $\lambda(X,Y;G)$ the maximal number of edge-disjoint paths between X and Y. We set $\lambda(X;G) := \min_{x \neq y \in X} \lambda(x,y;G)$ (note that $\lambda(G) = \lambda(V(G);G)$). In such expressions we often omit G.

Our result is the following.

Theorem 1 *If $k \geq 5$ is odd, $V(G) = W \cup S$, $W \cap S = \emptyset$, $\lambda(W) \geq k$, each vertex in S has even degree, $T = \{u,v,s,t\} \subseteq W$, $f_1 \in \partial(u,v)$, $f_2 \in \partial(v,s)$ and either $|T| = 4$ or $|T| = 3$ and $s = t$, then there are distinct g_i $(1 \leq i \leq (k-1)/2)$ in $\partial(u) - f_1$, such that for $1 \leq i \leq (k-1)/2$, G has a path P_i between s and t containing f_1 and g_i with $\lambda(W; G-E(P_i)) \geq k-2$.*

For $X \subseteq V(G)$, G/X denotes the graph obtained from G by identifying all the vertices in X and deleting any resulting loops. In G/X, X denotes the corresponding new vertex, each $x \in X$ is denoted by X and for $Y \subseteq V(G)$ with $Y \cap X \neq \emptyset$, Y denotes $(Y - X) \cup \{X\}$. For $x, y \in V(G)$, we write $P = P[x, y]$ to denote that P is a path between x and y, and we denote by $P(a, b)$ a subpath of P between a and b for $a, b \in V(P)$. $G - E(P)$ is often denoted by $G - P$. We often write $x \in P$ or $f \in P$ instead of $x \in V(P)$ or $f \in E(P)$, respectively. We sometimes give a path by the edge set. If $|X| \geq 2$, $|\overline{X}| \geq 2$ and $e(X) = k$, we call X and $\partial(X)$ a k-set and a k-cut respectively. For $a, b \in N(x)$ with $a \neq b$ ($a = b$, respectively) and for $f \in \partial(x, a)$ and $g \in \partial(x, b) - f$, $G_x^{a,b}$ and $G^{f,g}$ denotes the graph $(V(G), (E(G) + h) - \{f, g\})$, $((V(G), E(G) - \{f, g\})$, respectively), and is called a lifting of G at x, where h is a new edge between a and b. We call $G^{f,g}$ admissible if for each $y \neq z \in V(G) - x$, $\lambda(y, z; G^{f,g}) = \lambda(y, z; G)$. For $K \subseteq V(G) \cup E(G)$, we define $\mathcal{P}(G, s, t, K, X) := \{P \mid P = P[s, t]$ is a path in G containing K such that $\lambda(X; G - E(P)) \geq k - 2\}$ and we define $\mathcal{P}(G, s, t, K) := \mathcal{P}(G, s, t, K, W)$, where $W = \{x \in V(G) \mid e(x) \geq k\}$. We set $\overline{X} := V(G) - X$ and set $N(X; G) := \{a \in V(G) - X \mid e(a, X) > 0\}$. We say that $S \subseteq V(G)$ is dummy, if (1.1) below holds.

(1.1) *$S = \emptyset, \{b\}$, or $\{b, b'\}, e(b') = k - 1, e(b, b') \leq e(b)/2$, and $2 \leq e(b) \leq k - 1$ is even.*

2 Preliminaries

We prepare some lemmas. Lemma 1 is obvious

Lemma 1 *If $\{x, y\} \subseteq X \subseteq V(G)$, $z \in \overline{X}$, $e(X) = k$, and $e(z, X; G) = k$, then $\lambda(x, y; G/\overline{X}) = \lambda(x, y; G)$.*

Lemma 2 (Mader [2] and Frank [1]) *If $x \in V(G)$, $3 \neq e(x) = k$ ($k = 2\alpha$ or $2\alpha + 1$) and there is no cut-edge incident to x, then there are distinct edges $\{f_1, \cdots, f_\alpha, g_1, \cdots, g_\alpha\} \subseteq \partial(x)$ such that G^{f_i, g_i} ($1 \leq i \leq \alpha$) are admissible.*

Lemma 3 (Mader [3])

(1) *If $\lambda(G) \geq 2$, $u \in V(G)$, and $\{f_1, f_2\} \subseteq \partial(u)$, then there is a cycle C containing f_1 and f_2 such that for each $x \neq y \in V(G)$, $\lambda(x, y; G - E(C)) \geq \lambda(x, y; G) - 2$.*

(2) *If $\lambda(G) \geq 2$, $\{s, t\} \subseteq W \subseteq V(G)$, $\lambda(W) \geq k \geq 4$ and $f \in \partial(s)$, then there is a path $P \in \mathcal{P}(G, s, t, f, W)$ such that $\lambda(s, t; G - E(P)) \geq k - 1$.*

Lemma 4 (Lemma 5 in [6] and [7]) *If $k \geq 3$ is odd, $V(G) = X \cup Y$, $X \cap Y = \emptyset$, $e(X) = k+1$, $W \subseteq V(G)$, $W \cap X \neq \emptyset \neq W \cap Y$, $\lambda(W; G/X) \geq k$, $\lambda(W; G/Y) \geq k$, each vertex in $\overline{W}$ has even degree and for some $x \in X$, $\lambda(x, Y) = k+1$, then $\lambda(W; G) \geq k$.*

Lemma 5 *If k is odd, S is dummy, $W = V(G) - S$, $\lambda(W) \geq k$, $V(G) = X \cup Y$, $X \cap Y = \emptyset$, $e(X) \leq k+1$, $X \cap W \neq \emptyset \neq Y \cap W$, $x \in W$, $y \in Y \cap W$ and $P_1[x, y]$ is a path in G/X such that $\lambda(W; G/X - E(P_1)) \geq k-2$, then one of the following holds.*

(1) *G has a path $P[x, y]$ such that $P/X = P_1$ and $\lambda(W; G - E(P)) \geq k-2$.*

(2) *$e(X) = k+1$ and $x \in X$.*

(3) *$e(X) = k+1$, $x \in Y$ and there are X_1 and X_2 so that $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, $e(X_1) = e(X_2) = k$ and either $E(P_1) \cap \partial(X) = \emptyset$ or $|E(P_1) \cap \partial(X_i)| = 1$ $(i = 1, 2)$.*

Proof. Assume that (1) and (2) do not hold.

Case 1. $x \in X$.

$e(X) = k$ (otherwise (2) occurs). Let $g \in E(P_1) \cap \partial(X)$. By Lemma 3 (2), there is a path $P_2 \in \mathcal{P}(G/Y, Y, x, g)$ such that $\lambda(Y, x; G/Y - P_2) = k-1$. Let $P := P_1 \cup P_2$ in G. By Lemma 4 with $k-2$ instead of k, $\lambda(W; G - P) \geq k-2$ and (1) holds, a contradiction.

Case 2. $x \in Y$.

Let $G_1 := G + h$, where h is a new edge between x and y. Let C be a cycle in G_1/X with $C_1 = P_1 + h$. By Corollary 6 in [7], (3) follows.

Lemma 6 (Theorem 1.1 in [5]) *If $k \geq 5$ is odd, $\lambda(G) \geq k$, $T = \{u, v, s, t\} \subseteq V(G)$, $|T| = 4$, $f \in \partial(v, s)$, $g \in \partial(v, u)$ and $e(X) \geq k+1$ for each $X \subseteq V(G)$ with $X \cap T = \{s, u\}$, then $\mathcal{P}(G, s, t, \{f, g\}) \neq \emptyset$.*

Lemma 7 *Suppose that $k \geq 3$ is odd, S is dummy, $W = V(G) - S$, $Z = \{x_1, x_2\} \subseteq W$, $e(x_1) = e(x_2) = k$, $e(x_1, x_2) = (k-1)/2$ and $\lambda(W/Z) \geq k$. Then*

(1) *If for some $X \subseteq V(G) - x_2$ with $X \cap W \neq \emptyset$, $e(X) \leq k-1$, then $x_1 \in X$, $N(x_2) \cap X = \{x_1\}$ and $N(x_1) \cap \overline{X} = \{x_2\}$.*

(2) *If $N(x_1) \cap N(x_2) \neq \emptyset$, then $\lambda(W) \geq k$.*

Proof. (1) $x_1 \in X$ since $\lambda(W/Z) \geq k$. If there is $y \in N(x_2) \cap X - x_1$, then $e(x_2, X) \geq (k+1)/2$ and $e(X) \geq e(X + x_2) + 1 \geq k+1$, a contradiction. Thus $N(x_2) \cap X = x_1$. Similerly $N(x_1) \cap \overline{X} = x_2$.
(2) If there is $y \in N(x_1) \cap N(x_2)$, then for each $Y \subseteq V(G) - x_2$ with $x_1 \in Y$, either $y \in N(x_2) \cap Y$ or $y \in N(x_1) \cap \overline{Y}$, and so by (1), $e(Y) \geq k$. Thus $\lambda(W) \geq k$.

3 Proof of Theorem 1

In this section, let $\alpha := (k-1)/2$, $F(G) = F(G,s,t) = F(G,s,t,f_1) := \{g \in \partial(u) - f_1 \mid \mathcal{P}(G,s,t,\{f_1,g\}) \neq \emptyset\}$, for $g \in F(G,s,t,f_1)$, let $I(G,g) = I(G,s,t,g) := \{P \in \mathcal{P}(G,s,t,\{f_1,g\}) \mid P \text{ has no repeated vertices}\}$ and let $\mathcal{P}(G,K) := \mathcal{P}(G,s,t,K)$. If $x \in W$ and $e(x) \geq k+2$, then by Lemma 2 for some $g,h \in \partial(x) - \{f_1,f_2\}$, $G^{g,h}$ is admissible. If $|F(G^{g,h})| \geq \alpha$, then $|F(G)| \geq \alpha$ and therefore we may assume

(3.1) *$e(x) = k$ or $k+1$ for each $x \in W$.*

(3.2) *$e(x) = k$ for each $x \in W - \{u,v\}$.*

Proof. If $x \in W - \{u,v\}$, $e(x) = k+1$ and $\partial(x) = \{g_1, \cdots, g_{k+1}\}$, then we replace x and $\partial(x)$ by the graph in Figure 1, in which heavy edges represent $\alpha = (k-1)/2$ parallal edges, producing a new graph G'. If the result holds in G', then it also holds in G.

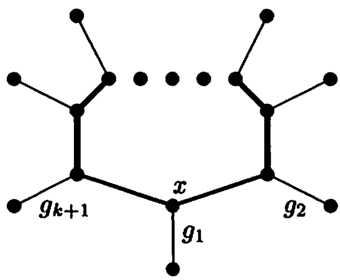

Figure 1

A minimal counterexample G of Theorem 1 with the additional conditions (3.1) and (3.2) is considered.

(3.3) *If $g \in F(G)$, then $I(G,g) \neq \emptyset$.*

Proof. We choose $P \in \mathcal{P}(G,\{f_1,g\})$ with minimal $|R(P)|$, where $R(P)=$ {repeated vertices in P}. Assume $I(G,g) = \emptyset$, then there is $x \in R(P)$. By (3.1) and (3.2), $x \in S$. There are edge-disjoint paths $P_1[u,s]$ and $P_2[u,t]$ in P. For $i = 1,2$, if P_i has a repeated vertex, then we can take simple subpath of P_i instead of P_i. Thus P_1 and P_2 are simple. If $g \in P_2$, then $P_2 \cup \{f_1,f_2\} \in I(G,g)$, and so $g \in P_1$ and $f_1 \in P_2$. Since $x \in V(P_1) \cap V(P_2)$, let $P_3 := \{f_2,f_1\} \cup P_1(u,x) \cup P_2(x,t)$. Then $P_3 \in \mathcal{P}(G,\{f_1,g\})$ and $|R(P_3)| \leq |R(P)| - 1$, contrary to the minimality of $|R(P)|$.

(3.4) $|T| = 4$.

Proof. Otherwise $s = t$ and by Theorem 9 in [7], $|F(G)| \geq \alpha$.

(3.5) $S = \emptyset$.

Proof. If there is $x \in S$, then $|F(G_x)| \geq \alpha$ for an admissible lifting G_x of G at x (see Lemma 2) and so $|F(G)| \geq \alpha$.

(3.6) *If X is a k-set, then $X \cap T = \{u, v\}$ or $\{s, t\}$.*

Proof. Case 1. $|X \cap T| \leq 1$.
$|F(G/X, s, t)| \geq \alpha$. Let $g \in F(G/X)$ and let $P_1 \in I(G/X, g)$. By Lemma 5, there is $P \in \mathcal{P}(G, \{f_1, g\})$ such that $P/X = P_1$. If $X \cap T \neq u$ then $g \in F(G)$. Thus let $X \cap T = u$. If $F(G/X) \subseteq \partial(u)$, then $F(G/X) \subseteq F(G)$ and if otherwise there is $g \in F(G/X) \cap \partial(x)$ for some $x \in X - u$. By (3.4), $|F(G/\overline{X}, x, x, f_1)| \geq \alpha$, and thus $|F(G)| \geq \alpha$.

Case 2. $|X \cap T| = 2$.
Let $X \cap T = \{u, s\}$ or $\{u, t\}$. We choose minimal X. Let $K := \partial(u, X - u)$. $|K| \geq e(u)/2 \geq \alpha$, since $e(X - u) \geq k = e(X)$. For each $g \in K$, by (3.5) and Lemma 6, there is $P_1 \in \mathcal{P}(G/\overline{X}, \overline{X}, s, \{f_1, g\})$ since X is minimal if $X \cap T = \{u, s\}$ and there is $P_1 \in \mathcal{P}(G/\overline{X}, \overline{X}, t, \{f_1, g\})$ if $X \cap T = \{u, t\}$. By Lemma 5, there is $P \in \mathcal{P}(G, s, t, \{f_1, g\})$ such that $P/\overline{X} = P_1$.

(3.7) *If X is a k-set and $X \cap T = \{s, t\}$, then $X = \{s, t, x\}$ for some $x \in W - T$.*

Proof. Otherwise $|X| \geq 4$ by (3.2). Let $X_1 := \overline{X}$, $X_2 = X$, $\partial(X_1) = \{h_1, h_2, \ldots, h_k\}$, $h_1 = f_2$ and let $V(h_i) \cap X_2 = s$ for $1 \leq i \leq r$ and $\neq s$ for $r + 1 \leq i \leq k$. We construct new graph G_1 as follows.

$V(G_1) = X_1 \cup \{s, t, x\}$, $G_1/\{s, t, x\} = G/X_2$,

$V(h_i; G_1) \cap \{s, t, x\} = s \ (1 \leq i \leq r), \ = t \, (r + 1 \leq i \leq r + \alpha), = x \, (r + \alpha + 1 \leq i \leq k)$,

$e(s, t; G_1) = k - (r + \alpha)$, $e(s, x; G_1) = \alpha$ and $e(t, x; G_1) = r$.

Then $\lambda(G_1) \geq k$ and by the minimality of G, $|F(G_1)| \geq \alpha$. Let $g \in F(G_1)$, $P_1 \in I(G_1, g)$ and let $E(P_1) \cap \partial(X_1) = \{h_i, h_j\}$ for $i < j$.

(3.7.1) *We can choose P_1 such that $i \leq r$.*

Proof. Otherwise $h_i \in \partial(t; G_1)$ and $h_j \in \partial(x; G_1)$. If $P_1(u, x)$ contains f_1, then we take $\{f_1, f_2\} \cup P_1(u, t)$ instead of P_1, and if $P_1(u, x)$ contains g, then we take $\{f_2, f_1, h\} \cup P_1(u, x)$ instead of P_1, where $h \in \partial(x, t; G_1)$.

By (3.7.1), (3.6) and Lemma 6, there is $P_2 \in \mathcal{P}(G/X_1, s, t, \{h_i, h_j\})$. Let P be a path in G with $P/X_2 = P_1$ and with $P/X_1 = P_2$. Then $P \in \mathcal{P}(G, \{f_1, g\})$ and we have $|F(G)| \geq \alpha$, a contradiction. Thus $|X_2| = 3$.

(3.8) *If X is a $(k+1)$-set with $X \cap T = v$, then for some $x \in W$, $X = \{v, x\}$, $e(v) = k$ and $e(v, x) = \alpha$.*

Proof. Let $g \in F(G/X) - F(G)$ and let $P_1 \in I(G/X, g)$. Then P_1 can not be extended to a path in $I(G, g)$. By Lemma 5 and (3.6), (3.8) follows.

(3.9) $\partial(u, \{s, t\}) \subseteq F(G)$.

Proof. If $g \in \partial(u, t)$, then $\{f_2, f_1, g\} \in I(G, g)$ by (3.6). If $g \in \partial(u, s)$, then by Lemma 6, $\mathcal{P}(G, \{f_1, g\}) \neq \emptyset$, and so $g \in F(G)$.

(3.10) *If X is a $(k+1)$-set and $X \cap T = \{u, v\}$, then either $\partial(u, \overline{X}) \subseteq F(G)$ or $X = \{u, v\}$ and $e(u) = e(v) = k$.*

Proof. Assume that there is $g \in \partial(u, \overline{X}) - F(G)$. Let $V(g) \cap \overline{X} = x$. By (3.9), $x \neq s, t$. By Lemma 6 and (3.6), there is $P_1 \in \mathcal{P}(G/X, \{f_2, g\})$. P_1 can not be extended to a path in $\mathcal{P}(G, \{f_2, g\})$. Thus by Lemma 5, $X = \{u, v\}$ and $e(u) = e(v) = k$.

(3.11) *If $x \in V(G) - \{s, v, u\}$ and $h \in \partial(x, v)$ then h is contained in no k-cut.*

Proof. Assume that there is a k-set X with $h \in \partial(X)$. By (3.6) and (3.7), say $X = \{s, t, z\}$ where $x = t$ or z. $|F(G/X, X, X, f_1)| \geq \alpha$. Let $g \in F(G/X, X, X, f_1)$. We can choose $P_1 \in I(G/X, X, X, g)$ such that $f_2 \in P_1$. Let $\{f_2, g_1\} = \partial(X) \cap E(P_1)$ and let $y \in V(g_1) \cap X$. If $y \neq s$, then there is $P \in I(G, s, t, g)$ such that $P/X = P_1$. When $y = s$, $(P_1 - f_2) \cup \{h\} \in I(G, g)$ if $x = t$ and $(P_1 - f_2) \cup \{h, h_1\} \in I(G, g)$ for $h_1 \in \partial(z, t)$ if $x = z$. Hence $g \in F(G)$ and $| F(G) | \geq \alpha$.

(3.12) *If $e(v) = k + 1$, then $N(v) = T - v$.*

Proof. Otherwise for some $x \in \overline{T}$, there is $h \in \partial(v, x)$. By (3.11), $\lambda(W - x, G - h) \geq k$ and $|F(G - h)| \geq \alpha$, and thus $|F(G)| \geq \alpha$.

(3.13) $e(u) = k$.

Proof. Otherwise $e(u) = k + 1$ by (3.1). If $N(u) \subseteq T$, then $e(u, \{s, t\}) \geq \alpha$, contrary to (3.9). Thus there is $g \in \partial(u, x)$ for some $x \in W - T$. If $\lambda(W - x, G - g) \geq k$, then $|F(G - g)| \geq \alpha$, and so $|F(G)| \geq \alpha$. Thus and by (3.7), $\{s, t, x\}$ is a k-set. Let $h \in \partial(x, t)$. $\{f_2, f_1, g, h\} \in I(G, g)$, and so $g \in F(G)$. Hence $\partial(u) - \partial(u, v) \subseteq F(G)$, contrary to $|F(G)| < \alpha$.

(3.14) *If* $\{x_1, x_2\} \subseteq W - T$, $h \in \partial(x_1, x_2)$ *and* h *is contained in no* k*-cut, then*

(i) $e(x_1, x_2) = \alpha$,

(ii) $F(G-h) - F(G) \neq \emptyset$,

(iii) *for each* $g \in F(G-h) - F(G)$, $G - h$ *has paths* $P_1[v,s]$ *and* $P_2[v,t]$ *so that* $\{f_1, g\} \subseteq E(P_1)$, $V(P_1) \cap V(P_2) = \{v\}$, *for* $(r,s) = (1,2)$ *or* $(2,1)$, $x_r \in P_1$, $x_s \in P_2$ *and* $\lambda(W - \{x_1, x_2\}; G - h - E(P_1 \cup P_2)) = k-2$,

(iv) *for path* $P^* := \{f_2, h\} \cup P_1(v, x_r) \cup P_2(x_s, t)$ *in* G *and for some* $Z \subseteq V(G) - V(P_1(s, x_r))$ *with* $V(P_2(v, x_s)) \subseteq Z$, $e(Z; G - E(P^*)) = k-3$.

Proof. In $G-h$, let $S' := \{x_1, x_2\}$ and let $W' = W - \{x_1, x_2\}$. $|F(G-h)| \geq \alpha$ and $|F(G)| < \alpha$ and thus (ii) follows. Let $g \in F(G-h) - F(G)$. By (3.3), we choose $P \in I(G-h, g)$ with minimal length. Let $P_1 := P(v,s)$ and $P_2 := P(v,t)$. $e(\{x_1, x_2\}; G - P) \leq k-3$, otherwise $P \in I(G, g)$, contrary to $g \notin F(G)$. Thus $e(x_1, x_2) = \alpha$ and $\{x_1, x_2\} \subseteq V(P)$. If $\{x_1, x_2\} \subseteq P_i$ (i=1 or 2), then we can replace $P_i(x_1, x_2)$ by $h_1 \in \partial(x_1, x_2; G-h)$, contrary to the minimality of P. Thus $x_r \in P_1$, $x_s \in P_2$ for $(r,s) = (1,2)$ or $(2,1)$. Then $f_2 \notin P$, $\{f_1, g\} \subseteq P_1$ and thus (iii) follows. For P^* given in (iv), $\lambda(G - P^*) = k-3$, thus (iv) easily follows.

We denote by $\Omega(x_1, x_2)$ the set of (g, P_1, P_2, P^*, Z) given in(3.14).

(3.15) *If* $\{x_1, x_2 . x_3, x_4\} \subseteq W - T$, $h \in \partial(x_2, x_3)$, h *is contained in no* k*-cut and* $e(x_i, x_{i+1}) = \alpha$ $(1 \leq i \leq 3)$, *then for* $(i,j) = (2,3)$ *or* $(3,2)$, $N(x_i) \cap T = \emptyset$, $\{u\}$, *or* $\{s\}$ *and* $N(x_j) \cap T = \emptyset$, $\{v\}$, *or* $\{t\}$.

Proof. By (3.14), there is $(g, P_1, P_2, P^*, Z) \in \Omega(x_2, x_3)$. For $(i,j) = (2,3)$ or $(3,2)$, $x_i \in P_1$ and $x_j \in P_2$. Then $N(x_i) \cap T = \emptyset$, $\{u\}$, or $\{s\}$ and $N(x_j) \cap T = \emptyset$, $\{v\}$, or $\{t\}$.

(3.16) (A) *If* X *is a minimal* k*-set with* $X \cap T = \{v, u\}$, *then for* $\{z_1, z_2\} \subseteq X - T$, $e(z_1, z_2) = 0$.

(B) *If* G *has no* k*-set, then for* $\{z_1, z_2\} \subseteq W - T$, $e(z_1, z_2) = 0$.

Proof. We shall prove (A) and (B) simultaneously. Assume that $e(z_1, z_2) > 0$. In (A), let $X_1 := X$ and $X_2 := \overline{X}$, then $X_2 = \{s, t, y\}$ for some $y \in W$ by (3.7). In (B), let $X_1 := V(G) - \{s, t\}$ and $X_2 := \{s, t\}$. Let D be the component of $G - (X_2 \cup T)$ containing z_1 and let $V(D) = \{x_1, x_2, \ldots, x_n\}$. $n \geq 2$ by $\{z_1, z_2\} \subseteq V(D)$. By (3.14), we may let $e(x_i, x_{i+1}) = \alpha$ $(1 \leq i \leq n-1)$. For some $1 \leq r \leq n-1$, we choose $(g, P_1, P_2, P^*, Z) \in$

$\Omega(x_r, x_{r+1})$ such that $P_1 \cup P_2$ has the minimal length in G/X_2. In (A), we can choose Z in X_1, otherwise $\lambda(G - P^*) \geq k - 2$. Let $T_0 := X_2 - s$ and let $h_0 \in P^* \cap \partial(V(D), T_0)$. $E(P^*/X_2) \subseteq \{f_2, f_1, g\} \cup E(D) \cup \{h_0\}$. Since $e(Z; G - P^*) \leq k - 3 = 2\alpha - 2$ and $e(x_i, x_{i+1}) = \alpha$, we have

(3.16.1) $|P^* \cap E(D) \cap \partial(Z)| \leq 2$ *and if the equlity holds* $\partial(Z; G - P^*) \subseteq E(D)$.

(3.16.2) $V(P_1) \cap X_2 = \{s\}$.

Proof. Otherwise $V(P_1) \cap X_2 = \{y\}$ and (A) occurs. For $h \in \partial(y, t)$, $\{f_2, h\} \cup P_1(v, y) \in I(G, g)$, a contradiction.

For some $\{y_1, \ldots, y_p\} \subseteq V(D)$, we may let $V(P^*) \cap X_1 = \{v, u, y_1, \ldots, y_p\}$ (in this order in P^*). $e(y_i, T_0) = 0$ for $1 \leq i \leq p - 1$ by the minimality of $P_1 \cup P_2$. Let $h_1 \in P_1 \cap \partial(s, V(D))$ (see (3.16.2)) and let $h_2 \in P_2 \cap \partial(v, V(D))$. By (iv) in (3.14) we have $V(h_1) \subseteq \overline{Z}$ and $V(h_2) \subseteq Z$.

Case 1. $|P^* \cap E(D) \cap \partial(Z)| = 2$.

First we consider the case that $y_1 \notin Z$. $|\partial(Z) \cap \{f_1, g\}| = 1$ and $Z \cap V(D) \subseteq P^*$. Let $Z \cap V(D) = \{y_l, y_{l+1}, \ldots, y_m\}$. Then $\{x_r, x_{r+1}\} = \{y_{l-1}, y_l\}$ and $h_2 \in \partial(v, y_l)$. $N(Z \cap V(D); G - P^*) - \{y_{l-1}, y_{m+1}\} \subseteq Z$ by (3.16.1). Since $e(y_i; Z - V(D)) = 1$ $(l \leq i \leq m)$, $|Z \cap V(D)| = e(Z - V(D); G - P^*) \geq k - 2 \geq 3$. By (3.15) and $e(\{y_1, \ldots y_{p-1}\}, T_0) = 0$, $e(y_{l+1}, u) = e(y_{l+2}, v) = 1$. Then for $h_3 \in \partial(v, y_{l+2})$, $P_1 \cup \{h_3\} \cup P_2(y_{l+2}, t) \in I(G, g)$, a contradiction. Next we consider the case that $y_1 \in Z$. $\overline{Z} \cap V(D) \subseteq P^*$. Let $\overline{Z} \cap V(D) = \{y_l, y_{l+1}, \ldots, y_m\}$. Then $h_1 \in \partial(s, y_m)$. Since $e(y_i, \overline{Z} \cap \overline{V(D)}) = 1\,(l \leq i \leq m)$, $|\overline{Z} \cap V(D)| = e(Z \cup V(D); G - P^*) \geq k - 2 \geq 3$. By (3.15) and $e(\{y_1, \ldots, y_{p-1}\}, T_0) = 0$, $e(y_{m-1}, v) = 1$, contrary to $v \in Z$.

Case 2. $|P^* \cap E(D) \cap \partial(Z)| = 1$.

For some $1 \leq l \leq p - 1$, there is $h \in P^* \cap \partial(y_l, y_{l+1}) \cap \partial(Z)$. We may let $y_l = x_r$ and $y_{l+1} = x_{r+1}$. By $P_1(s, x_r) \subseteq \overline{Z}$, $\{x_1, \cdots, x_r\} \subseteq \overline{Z}$ and $\{x_{r+1}, \cdots, x_n\} \subseteq Z$. Since $|\partial(Z) \cap \{f_1, g\}| = 1$, we have $t \notin Z$, $|\partial(Z) \cap P^*| = 4$ and $e(Z) = k + 1$. By (3.10) and $g \notin F(G)$, we have $u \in \overline{Z}$. Then by (3.8), $Z = \{v, x_n\}$, $e(v) = k$, $e(v, x_n) = \alpha$, $r + 1 = n$ and $e(x_n, T_0) = 1$. If $n \geq 3$, then by the same argument for $\{x_{n-1}, x_{n-2}\}$ instead of $\{x_r, x_{r+1}\}$, we have $n - 2 = 1$ and $e(v, x_1) = \alpha$, a contradiction. Thus $n = 2$ and $h_1 \in \partial(s, x_1)$.

It is easy to see (3.16.3) and we have (3.16.4) by (3.9).

(3.16.3) $e(v, T_0) = e(x_1, T_0) = 0$.

(3.16.4) $e(u, X_2) < \alpha$.

If $e(u,\{v,x_1\}) \geq \alpha+2$, then $e(\{x_1,x_2,v,u\}) \leq e(\{x_1,x_2,v\})-3 = k-1$, a contradiction. Thus we have

(3.16.5) $e(u,\{v,x_1\}) \leq \alpha+1$.

By (3.16.4) and (3.16.5), $e(u,x_3) > 0$ for some $x_3 \in X_1 - \{u,v,x_1,x_2\}$. If $|X_1| = 5$, then $e(X_2) = k$ by the parity. $N(x_3;G/X_2) = \{u,v,X_2\}$. By $e(T_0,x_2) = 1$ and (3.16.3), $e(T_0,\{u,x_3\}) \geq \alpha$. By (3.16.4), $e(T_0,x_3) > 0$. Then $\partial(u,x_3) \subseteq F(G)$. $e(u,X_2+x_3) = k - e(u,\{v,x_1\}) \geq \alpha$ by (3.16.5), contrary to $|F(G)| < \alpha$. If $|X_1| \geq 6$, say $x_4 \in X_1 - \{v,u,x_1,x_2,x_3\}$. If $e(x_3,x_4) > 0$, then by the same argument, $e(v,x_i) = \alpha \quad (i = 3 \text{ or } 4)$, a contradiction. Thus $e(x_3,x_4) = 0$. $e(T \cup X_2) \leq 3k-4$ if $e(X_2) = k$, and so G has no k-set. Since $e(T) \leq 4k-4$, we have $|X_1| = 6$. $\partial(u,x_i) - F(G) \neq \emptyset$ (i=3 or 4), otherwise $|F(G)| \geq \alpha$ by (3.16.5), say for $i = 3$. Then $e(x_3,t) = 0$ and $N(x_3) = T - t$. Let $g_1 \in \partial(u,x_3)$, $g_2 \in \partial(x_3,s)$, $h_2 \in \partial(v,x_2)$, $h_0 \in \partial(x_2,t)$ and let $P := \{g_2,g_1,f_1,h_2,h_0\}$. Since $g_1 \notin F(G)$, $e(Y;G-P) < k-2$ for some $Y \subseteq V(G) - s$. If $\{g_1,g_2\} \subseteq \partial(Y)$, then $e(Y) \geq e(Y-x_3)+3 \geq k+3$ since $e(x_3,v) < \alpha$ and $N(x_3) = T-t$. Thus $\partial(Y) \cap E(P) = E(P) - g_i \quad (i = 1 \text{ or } 2)$ and $e(Y) = k+1$. $x_1 \in Y$, otherwise $N(x_2) \subseteq \overline{Y}$. $e(Y) = e(Y-x_2)+1$, and so $e(Y-x_2) = k$, contray to $\{u,x_1\} \subseteq Y - x_2$. Now (3.16) is proved.

By (3.9), $W - T \neq \emptyset$. Let $W - T = \{x_1,\cdots,x_n\}$. By (3.7), (3.16) and $e(T) \leq 4k-4$, $n \leq 3$. First let $n = 1$ or 3, then $e(v) = k+1$ by the parity. By (3.12), $N(v) = T - v$. Then $N(x_i) = T - v \ (1 \leq i \leq n)$ and thus $n = 1$. Then $\partial(u,x_1) \subseteq F(G)$, contrary to $|F(G)| < \alpha$. Next let $n = 2$, then $e(v) = k$. If G has a k-set, let $X_1 = \{u,v,x_1\}$, $X_2 = \{s,t,x_2\}$ and $e(X_1) = e(X_2) = k$. $e(u,X_2) < \alpha$ since $\partial(u,X_2) \subseteq F(G)$ by (3.9), and so $e(v,x_1) = e(u,X_2) < \alpha$. If $e(x_1,X_2-s) > 0$, then $\partial(u,x_1) \subseteq F(G)$, a contradiction. Thus $e(x_1,X_2) = e(x_1,s)$. Then $e(v,\{x_2,t\}) > 0$ by $\alpha+1 \leq e(\{x_2,t\},X_1) = e(\{x_2,t\},\{v,u\})$ and it easily follows that $\partial(u,x_1) \subseteq F(G)$, contrary to $|F(G)| < \alpha$. Thus G has no k-set.

Case 1. $e(v,x_1) = \alpha$.

Since $|F(G/\{v,x_1\})| \geq \alpha$, there is $g \in F(G/\{v,x_1\}) - F(G)$. Let $P \in I(G/\{v,x_1\},g)$. Then there is $h \in \partial(v,x_1) \cap P$ and $\lambda(G-P) < k-2$. By (3.9), $g \in \partial(u,x_2)$. $V(P)$ is $\{s,x_2,u,v,x_1,t\}$ in this order and $e(v,t) = 0$. By $e(v,s;G-P) = e(v,s) > 0$ and Lemma 7(2), $e(x_1,s) = 0$. Then $N(x_1) = T - s$. For some $X \subseteq V(G) - v$, $e(X;G-P) = k-3$. Since $e(u,x_1) < \alpha$ (otherwise $e(\{u,v,x_1\}) \leq k$)), $e(x_1,t) \geq 2$ and $e(x_1,t;G-P) > 0$. By Lemma 7(1), $\{x_1,u,t\} \subseteq X$ and $s \in \overline{X}$. Then $|\partial(X) \cap P| = 3$, contrary to $e(X;G-P) = k-3$.

Case 2. $e(v,x_i) < \alpha \quad (i = 1,2)$.

By (3.9), for $i = 1$ or 2, $e(u,x_i) > 0$ and $\partial(u,x_i) - F(G) \neq \emptyset$, say for $i = 1$. Then $e(x_1, t) = 0$ and $N(x_1) = T - t$. Then $e(v, t) = 0$ and $N(t) = \{u, s, x_2\}$. Since $\partial(u, x_2) \subseteq F(G)$ and by (3.9), we have

(3.17) $e(u, \{v, x_1\}) \geq \alpha + 2$.

$e(v, x_2) > 0$, otherwise $\{u, s\}$ is a separating set, a contradiction. Let $h_1 \in \partial(s, x_1)$, $g \in \partial(x_1, u)$, $h_2 \in \partial(v, x_2)$, $h_3 \in \partial(x_2, t)$ and let $P := \{h_1, g, f_1, h_2, h_3\}$. Since $g \notin F(G)$, for some $X \subseteq V(G) - x_1$, $e(X; G - P) < k - 2$. If $\{g, f_1\} \subseteq \partial(X)$, then by (3.17), $e(X) \geq e(X - u) + 3 \geq k + 3$, a contradiction. Thus $E(P) \cap \partial(X) = E(P) - g$ or $E(P) - f_1$ and $e(X) = k+1$. Since $|X|$ is even, $\overline{X} = \{x_1, x_2\}$, a contradiction. This completes the proof of Theorem 1.

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