# The intersection problem for twin bowtie and near bowtie systems*


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**Abstract**

In this article the intersection problem for twin bowtie and near bowtie systems is completely solved.


## 1 Introduction

A *Steiner triple system of order v* (briefly $STS(v)$) is a pair $(S, \mathcal{T})$ where $\mathcal{T}$ is a collection of edge disjoint triangles (called *triples*) which partitions the edge set of the complete undirected graph $K_v$, with vertex set $S$.

It is well known that an $STS(v)$ has $t_v = \frac{1}{6}v(v-1)$ triples and a necessary and sufficient condition for existence is $v \equiv 1, 3 \pmod 6$.

A *bowtie* in the complete graph $K_v$ is a pair of triangles having exactly one vertex in common. A *bowtie system of order v* (briefly $BS(v)$) is a pair $(S, \mathcal{B})$ where $\mathcal{B}$ is a collection of edge disjoint bowties which partitions the edge set of $K_v$, with vertex set $S$. A *near bowtie system of order v* (briefly $NBS(v)$) is a pair $(S, \mathcal{B})$, where $\mathcal{B}$ is a collection of edge disjoint bowties and exactly one triangle which partitions the edge set of $K_v$, with vertex set $S$.

In what follows we will denote the triangle with vertices $a, b$, and $c$ by $\{a, b, c\}$ or $abc$, the bowtie consisting of the triangles $\{a, b, c\}$ and $\{a, d, e\}$ by $(\{a, b, c\},$


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$\{a,d,e\}$) or $abc$ juxtaposed with $ade$, and the set of bowties of a near bowtie system $(S,\mathcal{B})$ by $\tilde{\mathcal{B}}$.

It is obvious that if we separate the bowties in a bowtie system or in a near bowtie system into triangles we get a Steiner triple system. In [3] P. Horák and A. Rosa proved the following theorem.

**Theorem 1.** *Every Steiner triple system can be partitioned into a bowtie system or a near bowtie system (depending on whether or not the number of triples is even or odd).*

It can be shown easily that there is a $BS(v)$ $[NBS(v)]$ $(S,\mathcal{B})$ if and only if $v \equiv 1,9 \pmod{12}$ $[v \equiv 3,7 \pmod{12}]$ and $|\mathcal{B}| = \frac{t_v}{2}$ $[|\mathcal{B} - \{abc\}| = \frac{t_v-1}{2}]$. Therefore in saying that a certain property concerning $BS(v)$ $[NBS(v)]$ is true it is understood that $v \equiv 1,9 \pmod{12}$ $[v \equiv 3,7 \pmod{12}]$.

The bowtie systems $(S,\mathcal{B}_1)$ and $(S,\mathcal{B}_2)$ [near bowtie systems *with the same triangle*] are said to be *twin* provided the separation of the bowties in $\mathcal{B}_1$ and $\mathcal{B}_2$ gives the same Steiner triple system.

Various papers have dealt with the investigation of possible numbers of blocks that two designs, with the same parameters and based on the same $v$-set, may have in common. C.C. Lindner and A. Rosa [4] considered this problem for Steiner triple systems; M. Gionfriddo and C.C. Lindner [2], G. Lo Faro [5] and others, for Steiner quadruple systems; E.J. Billington and D.G. Hoffman [1] for certain balanced ternary designs and G. Lo Faro [6] for extended triple systems.

The intersection problem for Steiner triple systems would easily yield appropriate results for bowtie and near bowtie systems . For this reason , in this paper we consider the intersection problem for twin bowtie systems and twin near bowtie systems.

Let $J_B(v)$ $[J_N(v)]$ denote the set of non-negative integers $k$ such that there exists a pair of twin bowtie systems [twin near bowtie systems] intersecting in $k$ bowties. Let $I_B(v) = \{0,1,\ldots,\frac{t_v}{2}\} - \{\frac{t_v}{2} - 1\}$ $[I_N(v) = \{0,1,\ldots,\frac{t_v-1}{2}\} - \{\frac{t_v-1}{2} - 1\}]$.

The aim of this paper is to prove the following result:

**Main Theorem.** *$J_B(v) = I_B(v)$ and $J_N(v) = I_N(v)$.*

## 2 Auxiliary constructions

In this section we give some constructions which are the main tools in what follows.

Let $K_{v+7}$ be the complete graph on $Z_{v+7}$ vertices. The edges of $K_{v+7}$ fall into $v+7$ disjoint classes $P_1, P_2, \ldots, P_{v+7}$, where the edge $\{i,k\}$ is in $P_j$ if and only if $i-k \equiv j \pmod{v+7}$. In [7] R.G. Stanton and I.P. Goulden proved the following result:

(*) The graph $K_{v+7}$ may be factored into a set of $v+7$ triangles covering $P_1, P_2, P_3$ and a set of $v$ one-factors covering the other $P_j$.

Put $S = \{a_i : i = 1,2,\dots,v\}$ and factor the complete graph $K_{v+7}$ with vertex set $X = \{i : i = 1,2,\dots,v+7\}$, $S \cap X = \emptyset$, by (*). Let $T = \{\{i, i+1, i+3\} : i = 1,2,\dots,v+7\}$ (mod $v+7$) be the set of triangles and $\mathcal{F} = \{F_i : i = 1,2,\dots,v\}$ be the set of 1-factors where $F_i = \{\{x_{i,j}, y_{i,j}\}, j = 1,2,\dots,\frac{v+7}{2}\}$ with $x_{i,1} = 1$, for every $i = 1,2,\dots,v$.

**Construction A:** $v$ *to* $2v+7$, $v \equiv 1,9 \pmod{12}$.

Let $(S,\mathcal{B})$ be a $BS(v)$.
Put $S^* = S \cup X$ and $\mathcal{B}^* = \mathcal{B} \cup \mathcal{D} \cup \mathcal{L}$, where $\mathcal{D} = \bigcup_{i=1}^{v} \mathcal{D}_i$,

$$\mathcal{D}_i = \left\{(\{a_i, x_{i,2j+1}, y_{i,2j+1}\}, \{a_i, x_{i,2j+2}, y_{i,2j+2}\}) : j = 0,1,\dots,\tfrac{v+3}{4}\right\},$$

and

$$\mathcal{L} = \left\{(\{2i, 2i-1, 2i+2\}, \{2i, 2i+1, 2i+3\}) : i = 1,2,\dots,\tfrac{v+7}{2}\right\}$$

(mod $v+7$).
Then $(S^*,\mathcal{B}^*)$ is a $BS(2v+7)$.

**Construction A*:** $v$ *to* $2v+7$, $v \equiv 3,7 \pmod{12}$.

Let $(S,\mathcal{B})$ be a $NBS(v)$ with triangle $a_{v-2}a_{v-1}a_v$.
Put $S^* = S \cup X$, $\mathcal{L}$ as in Construction $A$ and

$$\mathcal{B}^* = \tilde{\mathcal{B}} \cup \mathcal{L} \cup \mathcal{D} \cup \mathcal{H} \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\},$$

where $\mathcal{D} = \bigcup_{i=1}^{v} \mathcal{D}_i$,

$$\mathcal{D}_i = \left\{(\{a_i, x_{i,2j}, y_{i,2j}\}, \{a_i, x_{i,2j+1}, y_{i,2j+1}\}) : j = 1,2,\dots,\tfrac{v+5}{4}\right\},$$

and

$$\mathcal{H} = \left\{(\{1, y_{2i+1,1}, a_{2i+1}\}, \{1, y_{2i+2,1}, a_{2i+2}\}) : i = 0,1,\dots,\tfrac{v-3}{2}\right\}.$$

Then $(S^*,\mathcal{B}^*)$ is a $BS(2v+7)$.

Put $S = \{a_i : i = 1,2,\dots,v\}$ and let $\mathcal{F} = \{F_i : i = 1,2,\dots,v\}$ be a 1-factorization of $K_{v+1}$ on $X = \{i : i = 1,2,\dots,v+1\}$, where $S \cap X = \emptyset$.
Put $F_i = \{\{x_{i,j}, y_{i,j}\}, j = 1,2,\dots,\frac{v+1}{2}\}$ with $x_{i,1} = 1$, for every $i = 1,2,\dots,v$.

**Construction B:** *v to* $2v+1$, $v \equiv 1,9 \pmod{12}$.

Let $(S, \mathcal{B})$ be a $BS(v)$.
Put $S^* = S \cup X$ and $\mathcal{B}^* = \mathcal{B} \cup \mathcal{D} \cup \mathcal{H} \cup \{\{a_v, 1, y_{v,1}\}\}$, where $\mathcal{D} = \bigcup_{i=1}^{v} \mathcal{D}_i$,

$$\mathcal{D}_i = \left\{(\{a_i, x_{i,2j}, y_{i,2j}\}, \{a_i, x_{i,2j+1}, y_{i,2j+1}\}) : j = 1, 2, \ldots, \tfrac{v-1}{4}\right\},$$

and

$$\mathcal{H} = \left\{(\{1, y_{2i+1,1}, a_{2i+1}\}, \{1, y_{2i+2,1}, a_{2i+2}\}) : i = 0, 1, \ldots, \tfrac{v-3}{2}\right\}.$$

Then $(S^*, \mathcal{B}^*)$ is a $NBS(2v+1)$.

**Construction B*:** *v to* $2v+1$, $v \equiv 3,7 \pmod{12}$.

Let $(S, \mathcal{B})$ be a $NBS(v)$.
Put $S^* = S \cup X$ and $\mathcal{B}^* = \mathcal{B} \cup \mathcal{D}$, where $\mathcal{D} = \bigcup_{i=1}^{v} \mathcal{D}_i$,

$$\mathcal{D}_i = \left\{(\{a_i, x_{i,2j+1}, y_{i,2j+1}\}, \{a_i, x_{i,2j+2}, y_{i,2j+2}\}) : j = 0, 1, \ldots, \tfrac{v-3}{4}\right\}.$$

Then $(S^*, \mathcal{B}^*)$ is a $NBS(2v+1)$.

# 3 Basic lemmas

**Lemma 1.** *$J_B(v) \subsetneq I_B(v)$ and $J_N(v) \subsetneq I_N(v)$.*

*Proof.* It is seen instantly that $J_B(v) \subsetneq I_B(v)$ and $J_N(v) \subsetneq I_N(v)$; in other words it is impossible to have two twin bowtie systems and two twin near boetie systems which have all but one bowtie the same. □

**Lemma 2.** *For $v \equiv 1,9 \pmod{12}$, $\{\frac{t_v}{2}, \frac{t_v}{2}+1, \ldots, \frac{t_{2v+7}}{2}\} - \{\frac{t_{2v+7}}{2} - 1\} \subseteq I_B(2v+7)$.*

*Proof.* Let $(S, \mathcal{B})$ be a $BS(v)$ and $X$, $\mathcal{D}$ and $\mathcal{L}$ as in Construction $A$. Consider the following permutations on $\{1, 2, \ldots, \frac{v+7}{2}\}$:

$$\alpha_{p_i} = \left(2p_i + 1, 2p_i + 2, \ldots, \tfrac{v+7}{2}\right);$$

$p_i \in \{0, 1, \ldots, \frac{v-1}{4}, \frac{v+7}{4}\}, i = 1, 2, \ldots, v$. It is straightforward to see that $|\mathcal{D}_i \cap \alpha_{p_i}(\mathcal{D}_i)| = p_i$, where

$$\alpha_{p_i}(\mathcal{D}_i) = \{(\{a_i, x_{i,\alpha_{p_i}(2j+1)}, y_{i,\alpha_{p_i}(2j+1)}\}, \{a_i, x_{i,\alpha_{p_i}(2j+2)}, y_{i,\alpha_{p_i}(2j+2)}\}) : j = 0, 1, \ldots, \tfrac{v+3}{4}\}.$$

Put

$$\alpha_{p_1 p_2 \ldots p_v}(\mathcal{D}) = \bigcup_{i=1}^{v} \alpha_{p_i}(\mathcal{D}_i)$$

and

$$\mathcal{L}' = \left\{ (\{2i+1, 2i, 2i+3\}, \{2i+1, 2i+2, 2i+4\}) : i = 0, 1, \ldots, \tfrac{v+5}{2} \right\}$$

(mod $v+7$).

Certainly, if $R(v)$ is the set of all non-negative intergers $h$ such that $h = \sum_{i=1}^{v} p_i$, $p_i \in \{0, 1, \ldots, \frac{v-1}{4}, \frac{v+7}{4}\}$, then $R(v) = \{0, 1, \ldots, \frac{v(v+7)}{4} - 2, \frac{v(v+7)}{4}\}$. It is easy to see that for each $h \in R(v)$ there exists:

(i) a pair of twin $BS(2v+7)$'s, $(S \cup X, \mathcal{B} \cup \mathcal{D} \cup \mathcal{L})$ and $(S \cup X, \mathcal{B} \cup \alpha_{p_1 p_2 \ldots p_v}(\mathcal{D}) \cup \mathcal{L})$ with exactly $\frac{t_v}{2} + \frac{v+7}{2} + h$ bowties in common;

(ii) a pair of twin $BS(2v+7)$'s, $(S \cup X, \mathcal{B} \cup \mathcal{D} \cup \mathcal{L})$ and $(S \cup X, \mathcal{B} \cup \alpha_{p_1 p_2 \ldots p_v}(\mathcal{D}) \cup \mathcal{L}')$ with exactly $\frac{t_v}{2} + h$ bowties in common.

Thus we have $\{\frac{t_v}{2}, \frac{t_v}{2} + 1, \ldots, \frac{t_{2v+7}}{2}\} - \{\frac{t_{2v+7}}{2} - 1\} \subseteq I_B(2v+7)$. □

**Lemma 3.** *For $v \equiv 1, 9 \pmod{12}$, $J_B(v) \subseteq J_B(2v+7)$.*

*Proof.* Let $k \in J_B(v)$. If $(S, \mathcal{B}_1)$, $(S, \mathcal{B}_2)$ is a pair of twin $BS(2v+7)$'s with $|\mathcal{B}_1 \cap \mathcal{B}_2| = k$, $X$, $\mathcal{D}$ and $\mathcal{L}$ as in Construction $A$, then $(S \cup X, \mathcal{B}_1 \cup \mathcal{D} \cup \mathcal{L})$, $(S \cup X, \mathcal{B}_2 \cup \alpha_0(\mathcal{D}) \cup \mathcal{L}')$ (where $\alpha_0(\mathcal{D}) = \alpha_{00\ldots0}(\mathcal{D})$) is a pair of twin $BS(2v+7)$'s with exactly $k$ bowties in common. □

**Corollary 4.** *For $v \equiv 1, 9 \pmod{12}$, $J_B(v) = I_B(v)$ implies $J_B(2v+7) = I_B(2v+7)$.*

*Proof.* By Lemmas 2 and 3 we have only to prove that $(\frac{t_v}{2} - 1) \in J_B(2v+7)$. Obviously $(\frac{t_v}{2} - 1) \in R(v)$ and so if $(S, \mathcal{B}_1)$, $(S, \mathcal{B}_2)$ is a pair of twin $BS(v)$'s with $|\mathcal{B}_1 \cap \mathcal{B}_2| = 0$, then we can obtain a pair of twin $BS(2v+7)$'s, $(S \cup X, \mathcal{B}_1 \cup \mathcal{D} \cup \mathcal{L})$ and $(S \cup X, \mathcal{B}_2 \cup \alpha_{p_1 p_2 \ldots p_v}(\mathcal{D}) \cup \mathcal{L}')$, with exactly $(\frac{t_v}{2} - 1)$ bowties in common. □

**Lemma 5.** *For $v \equiv 3, 7 \pmod{12}$, $v \geq 7$, $\{\frac{t_v+1}{2}, \frac{t_v+3}{2}, \ldots, \frac{t_{2v+7}}{2}\} - \{\frac{t_{2v+7}}{2} - 1\} \subseteq I_B(2v+7)$.*

*Proof.* Let $(S, \mathcal{B})$ be a $NBS(v)$ with triangle $a_{v-2}a_{v-1}a_v$ and $X$, $\mathcal{L}, \mathcal{D}$ and $\mathcal{H}$ as in Construction $A^*$.
Consider the following permutations:

$$\alpha_{p_i} = \left(2p_i + 2, 2p_i + 3, \ldots, \tfrac{v+7}{2}\right),$$

$p_i \in \{0, 1, \ldots, \frac{v-3}{4}, \frac{v+5}{4}\}, i = 1, 2, \ldots, v$, on the set $\{1, 2, \ldots, \frac{v+7}{2}\}$;

$$\beta_q = (2q_i + 1, 2q_i + 2, \ldots, v - 1),$$

$q \in \{0, 1, \ldots, \frac{v-5}{2}, \frac{v-1}{2}\}$, on the set $\{1, 2, \ldots, v-1\}$. If

$$\beta_q(\mathcal{H}) = \{(\{1, y_{\beta_q(2i+1),1}, a_{\beta_q(2i+1)}\}, \{1, y_{\beta_q(2i+2),1}, a_{\beta_q(2i+2)}\}) : i = 0, 1, \ldots, \frac{v-3}{2}\},$$

then it is straightforward to see that $|\mathcal{D}_i \cap \alpha_{p_i}(\mathcal{D}_i)| = p_i$ and $|\mathcal{H} \cap \beta_q(\mathcal{H})| = q$. The same argument used in Lemma 2 gives:

(i) $(S \cup X, \tilde{\mathcal{B}} \cup \mathcal{L} \cup \mathcal{D} \cup \mathcal{H} \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\})$, $(S \cup X, \tilde{\mathcal{B}} \cup \mathcal{L} \cup \alpha_{p_1 p_2 \ldots p_v}(\mathcal{D}) \cup \beta_q(\mathcal{H}) \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\})$ is a pair of twin $BS(2v+7)$'s with exactly $\frac{t_v - 1}{2} + \frac{v+7}{2} + q + 1 + \sum_{i=1}^{v} p_i$ bowties in common;

(ii) $(S \cup X, \tilde{\mathcal{B}} \cup \mathcal{L} \cup \mathcal{D} \cup \mathcal{H} \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\})$, $(S \cup X, \tilde{\mathcal{B}} \cup \mathcal{L}' \cup \alpha_{p_1 p_2 \ldots p_v}(\mathcal{D}) \cup \beta_q(\mathcal{H}) \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\})$ is a pair of twin $BS(2v+7)$'s with exactly $\frac{t_v - 1}{2} + q + 1 + \sum_{i=1}^{v} p_i$ bowties in common.

Thus we have $\{\frac{t_v+1}{2}, \frac{t_v+3}{2}, \ldots, \frac{t_{2v+7}}{2}\} - \{\frac{t_{2v+7}}{2} - 1\} \subseteq I_B(2v+7)$. □

**Lemma 6.** *For $v \equiv 3, 7 \pmod{12}$, $k \in J_N(v)$ implies $(k+1) \in J_B(2v+7)$.*

*Proof.* Let $k \in J_B(v)$. If $(S, \mathcal{B}_1)$, $(S, \mathcal{B}_2)$ is a pair of twin $NBS(2v+7)$'s (with the same triangle $a_{v-2}a_{v-1}a_v$) such that $|\mathcal{B}_1 \cap \mathcal{B}_2 - \{a_{v-2}a_{v-1}a_v\}| = k$, $X$, $\mathcal{D}$ and $\mathcal{L}$ as in Construction $A$, then

$$(S \cup X, \tilde{\mathcal{B}}_1 \cup \mathcal{L} \cup \mathcal{D} \cup \mathcal{H} \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\}),$$
$$(S \cup X, \tilde{\mathcal{B}}_2 \cup \mathcal{L}' \cup \alpha_0(\mathcal{D}) \cup \beta_0(\mathcal{H}) \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\})$$

is a pair of twin $BS(2v+7)$'s with exactly $k+1$ bowties in common, where $X$, $\mathcal{L}, \mathcal{D}$ and $\mathcal{H}$ are as in Construction $A^*$. □

**Corollary 7.** *For $v \equiv 3, 7 \pmod{12}$, $v \geq 7$, $J_N(v) = I_N(v)$ implies $J_B(2v+7) = I_B(2v+7)$.*

*Proof.* By Lemmas 5 and 6 we have only to prove that $\{0, \frac{t_v-1}{2} - 1, \frac{t_v-1}{2}\} \subseteq J_B(2v+7)$. It is easy to see that $\{\frac{t_v-1}{2} - 1, \frac{t_v-1}{2}\} \subseteq J_B(2v+7)$ and so we will prove that $0 \in J_B(2v+7)$. Let $(S, \mathcal{B}_1)$, $(S, \mathcal{B}_2)$ be a pair of twin $NBS(v)$'s with the same triangle $a_{v-2}a_{v-1}a_v$. If

$$\mathcal{H}' = \{(\{1, y_{2i,1}, a_{2i}\}, \{1, y_{2i+1,1}, a_{2i+1}\}) : i = 1, 2, \ldots, \frac{v-3}{2}\} \cup \{(\{1, y_{1,1}, a_1\}, \{1, y_{v,1}, a_v\})\},$$

then

$$(S \cup X, \tilde{\mathcal{B}}_1 \cup \mathcal{L} \cup \mathcal{D} \cup \mathcal{H} \cup \{(\{a_v, 1, y_{v,1}\}, \{a_v, a_{v-2}, a_{v-1}\})\}),$$

$$(S \cup X, \tilde{\mathcal{B}}_2 \cup \mathcal{L}' \cup \alpha_0(\mathcal{D}) \cup \mathcal{H}' \cup \{(\{a_{v-1}, 1, y_{v-1,1}\}, \{a_{v-1}, a_{v-2}, a_v\})\})$$

is a pair of twin $BS(2v + 7)$'s with zero bowties in common. This prove the corollary. □

The same argument used in previous lemmas works to prove the following corollary.

**Corollary 8.**

*(i)* *For* $v \equiv 1, 9 \pmod{12}$, $J_B(v) = I_B(v)$ *implies* $J_N(2v+1) = I_N(2v+1)$.

*(ii)* *For* $v \equiv 3, 7 \pmod{12}$, $v \geq 7$, $J_N(v) = I_N(v)$ *implies* $J_N(2v + 1) = I_N(2v + 1)$.

# 4 Small cases

In this section we will deal with the cases $v = 3, 7, 9$ and 13.

**v=3.**
There is precisely one $NBS(3)$: $S = \{1,2,3\}, \mathcal{B} = \{123\}$. So $J_N(3) = 0 = I_N(3)$.

**v=7.**
Let $(S, \mathcal{T})$ be the following $STS(7)$ on the set $S = \{1,2,3,4,5,6,7\}$:

$$\mathcal{T} = \{123, 145, 167, 246, 257, 347, 356\}.$$

We can obtain the following pairwise twin $NBS(7)$'s $(S, \mathcal{B}_i)$, $i = 1, 2, 3$:

$$\mathcal{B}_1 = \{123145, 617624, 725734, 356\};$$

$$\mathcal{B}_2 = \{145167, 246257, 347312, 356\};$$

$$\mathcal{B}_3 = \{123145, 246257, 734716, 356\}.$$

Then it is easy to check that

$$|\tilde{\mathcal{B}}_1 \cap \tilde{\mathcal{B}}_2| = 0, \ |\tilde{\mathcal{B}}_1 \cap \tilde{\mathcal{B}}_3| = 1, \ |\tilde{\mathcal{B}}_1 \cap \tilde{\mathcal{B}}_1| = 3.$$

So $J_N(7) = I_N(7)$.

**v=9.**
Let $(S, \mathcal{T})$ be the following $STS(9)$ on the set $S = \{1,2,3,4,5,6,7,8,9\}$:

$$\mathcal{T} = \{123, 456, 789, 147, 258, 369, 159, 267, 348, 357, 168, 249\}.$$

We can obtain the following pairwise twin $BS(9)$'s $(S, \mathcal{B}_i)$, $i = 1, 2, 3, 4, 5$:

$$\mathcal{B}_1 = \{123147, 546528, 978936, 519537, 627618, 438429\};$$

$$\mathcal{B}_2 = \{231258, 645639, 789714, 159168, 267249, 348357\};$$

$$\mathcal{B}_3 = \{123147, 645639, 879825, 159168, 267249, 348357\};$$

$$\mathcal{B}_4 = \{123147, 546528, 978915, 639618, 267249, 348357\};$$

$$\mathcal{B}_5 = \{123147, 546528, 978936, 159168, 267249, 348357\};$$

$$\mathcal{B}_6 = \{123147, 546528, 978936, 519537, 267249, 834816\}.$$

Then it is easy to check that

$$|\mathcal{B}_1 \cap \mathcal{B}_2| = 0,\ |\mathcal{B}_1 \cap \mathcal{B}_3| = 1,\ |\mathcal{B}_1 \cap \mathcal{B}_4| = 2,$$

$$|\mathcal{B}_1 \cap \mathcal{B}_5| = 3,\ |\mathcal{B}_1 \cap \mathcal{B}_6| = 4,\ |\mathcal{B}_1 \cap \mathcal{B}_1| = 6.$$

So $J_B(9) = I_B(9)$.

**v=13.**

By an argument similar to Lemma 5 it is easy to see that $I_B(13) - \{0, 1, 3, 5, 10\} \subseteq J_B(13)$. Let $(S, \mathcal{T})$ be the following $STS(13)$ on the set $S = \{1, 2, \ldots, 13\}$:

$$\begin{aligned}\mathcal{T} = \{&\{1,2,5\},\{2,3,6\},\{3,4,7\},\{4,5,8\},\{5,6,9\},\\ &\{6,7,10\},\{7,8,11\},\{8,9,12\},\{9,10,13\},\{10,11,1\},\\ &\{11,12,2\},\{12,13,3\},\{13,1,4\},\{1,3,8\},\{2,4,9\},\\ &\{3,5,10\},\{4,6,11\},\{5,7,12\},\{6,8,13\},\{7,9,1\},\\ &\{8,10,2\},\{9,11,3\},\{10,12,4\},\{11,13,5\},\{12,1,6\},\\ &\{13,2,7\}\}.\end{aligned}$$

We can obtain the following pairwise twin $BS(13)$'s $(S, \mathcal{B}_i)$, $i = 1, 2, 3, 4, 5, 6$:

$$\begin{aligned}\mathcal{B}_1 = \{&(\{2,1,5\},\{2,3,6\}),(\{4,3,7\},\{4,5,8\}),(\{6,5,9\},\{6,7,10\}),\\ &(\{8,7,11\},\{8,9,12\}),\ (\{10,9,13\},\{10,11,1\}),\ (\{12,11,2\},\{12,13,3\}),\\ &(\{1,13,4\},\{1,3,8\}),\ (\{4,2,9\},\{4,6,11\}),\ (\{5,3,10\},\{5,7,12\}),\\ &(\{8,6,13\},\{8,10,2\}),\ (\{9,7,1\},\{9,11,3\}),\ (\{12,10,4\},\{12,1,6\}),\\ &(\{13,11,5\},\{13,2,7\})\}.\end{aligned}$$

$$\mathcal{B}_2 = \{(\{1,2,5\},\{1,12,6\}), (\{3,2,6\},\{3,4,7\}), (\{5,4,8\},\{5,6,9\}), (\{7,6,10\},\{7,8,11\}), (\{9,8,12\},\{9,10,13\}), (\{11,10,1\},\{11,12,2\}), (\{13,12,3\},\{13,1,4\}), (\{3,1,8\},\{3,5,10\}), (\{6,4,11\},\{6,8,13\}), (\{7,5,12\},\{7,9,1\}), (\{10,8,2\},\{10,12,4\}), (\{11,9,3\},\{11,13,5\}), (\{2,4,9\},\{2,13,7\})\}.$$

$$\mathcal{B}_3 = \{(\{2,1,5\},\{2,3,6\}), (\{7,3,4\},\{7,8,11\}), (\{8,4,5\},\{8,9,12\}), (\{9,5,6\},\{9,10,13\}), (\{10,6,7\},\{10,11,1\}), (\{2,11,12\},\{2,4,9\}), (\{3,12,13\},\{3,5,10\}), (\{4,13,1\},\{4,6,11\}), (\{8,1,3\},\{8,6,13\}), (\{7,5,12\},\{7,13,2\}), (\{1,7,9\},\{1,12,6\}), (\{10,8,2\},\{10,12,4\}), (\{11,9,3\},\{11,13,5\})\}.$$

$$\mathcal{B}_4 = \{(\{2,1,5\},\{2,3,6\}), (\{4,3,7\},\{4,5,8\}), (\{6,5,9\},\{6,7,10\}), (\{11,7,8\},\{11,10,1\}), (\{8,9,12\},\{8,1,3\}), (\{13,9,10\},\{13,6,8\}), (\{2,11,12\},\{2,4,9\}), (\{3,12,13\},\{3,5,10\}), (\{4,13,1\},\{4,6,11\}), (\{7,5,12\},\{7,13,2\}), (\{1,7,9\},\{1,12,6\}), (\{10,8,2\},\{10,12,4\}), (\{11,9,3\},\{11,13,5\})\}.$$

$$\mathcal{B}_5 = \{(\{2,1,5\},\{2,3,6\}), (\{4,3,7\},\{4,5,8\}), (\{6,5,9\},\{6,7,10\}), (\{8,7,11\},\{8,9,12\}), (\{10,9,13\},\{10,11,1\}), (\{12,11,2\},\{12,1,6\}), (\{13,12,3\},\{13,1,4\}), (\{3,1,8\},\{3,5,10\}), (\{2,4,9\},\{2,7,13\}), (\{6,4,11\},\{6,8,13\}), (\{7,5,12\},\{7,9,1\}), (\{10,8,2\},\{10,12,4\}), (\{11,9,3\},\{11,13,5\})\}.$$

$$\mathcal{B}_6 = \{(\{2,1,5\},\{2,3,6\}), (\{4,3,7\},\{4,5,8\}), (\{6,5,9\},\{6,7,10\}), (\{8,7,11\},\{8,9,12\}), (\{10,9,13\},\{10,11,1\}), (\{12,11,2\},\{12,13,3\}), (\{1,13,4\},\{1,3,8\}), (\{4,2,9\},\{4,6,11\}), (\{8,6,13\},\{8,10,2\}), (\{12,10,4\},\{12,1,6\}), (\{7,9,1\},\{7,13,2\}), (\{3,5,10\},\{3,9,11\}), (\{5,7,12\},\{5,11,13\})\}.$$

Then it is easy to check that

$$|\mathcal{B}_1 \cap \mathcal{B}_2| = 0, \ |\mathcal{B}_1 \cap \mathcal{B}_3| = 1, \ |\mathcal{B}_1 \cap \mathcal{B}_4| = 3,$$
$$|\mathcal{B}_1 \cap \mathcal{B}_5| = 5, \ |\mathcal{B}_1 \cap \mathcal{B}_6| = 10.$$

So $J_B(13) = I_B(13)$.

# 5 Conclusion

We now have our required result.

**Main Theorem.** *$J_B(v) = I_B(v)$ and $J_N(v) = I_N(v)$.*

*Proof.* For $v = 3, 7, 9, 13$ our statement follows from Section 4. Assume therefore $v \geq 15$, and assume that for all $w < v$, $J_N(w) = I_N(w)$ if $w \equiv 3, 7 \pmod{12}$ and $J_B(w) = I_B(w)$ if $w \equiv 1, 9 \pmod{12}$. If $v \equiv 1, 9 \pmod{12}$, by Corollaries 4 and 7 $J_B(v) = I_B(v)$. If $v \equiv 3, 7 \pmod{12}$, by Corollary 8 $J_N(v) = I_N(v)$. This completes the proof of the theorem. □


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