Isomorphic Antidirected Path Decompositions of Complete Symmetric Graphs

Tay-Woei Shyu*
Department of Banking and Finance
Kai Nan University
Lu-Chu, Tao-Yuan, Taiwan 338, R.O.C.
e-mail: twhsu@mail.knu.edu.tw

Chiang Lin
Department of Mathematics
National Central University
Chung-Li, Taiwan 320, R.O.C.

ABSTRACT. A path in a digraph is antidirected if the two adjacent edges of the path have opposing orientations. In this paper we give a necessary and sufficient condition for the edges of the complete symmetric graph to be decomposed into isomorphic antidirected paths.

1 Introduction

A $v_0 - v_r$ *walk* of length r in a graph G is a sequence of vertices of the form $v_0, v_1, \cdots, v_r$ where $v_{i-1}v_i \in E(G)$ for $i = 1, 2, \cdots, r$; this walk is denoted by $v_0v_1 \cdots v_r$. A *trail* is a walk in which all edges are distinct. A trail is *closed* if its staring vertex and ending vertex are the same. An *Eulerian trail* of a graph G is a closed trail containing all edges of G. A *path* is a trail in which all vertices are distinct, and a path of length l is denoted by P_l. A path in a digraph is *antidirected* if two adjacent edges of the path have opposing orientations. To describe the edge oriented from the vertex u to the vertex v we shall write $u \rightarrow v$ or $v \leftarrow u$. The antidirected path will be designated

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by listing their vertices and orienting the edges between them. When l is odd, there is a unique nonisomorphic antidirected path of length l, denoted by $\overleftrightarrow{P_l}$. When $l \geq 2$ is even, there are two nonisomorphic antidirected paths of length l. (See Fig.1 for two nonisomorphic antidirected paths of length 2.) We use $\overleftrightarrow{P_l}$ to denote the antidirected path with the edges incident to the end vertices of the path oriented away from the end vertices.

$$\bullet \longrightarrow \bullet \longleftarrow \bullet \qquad\qquad \bullet \longleftarrow \bullet \longrightarrow \bullet$$

Fig. 1

In a digraph G, an antidirected Hamiltonian path is an antidirected path that passes every vertex of G; an antidirected Hamiltonian circuit is an antidirected cycle(i.e., a cycle with adjacent edges having opposing directions) that passes every vertex of G. In [1], [2], [3], [4], [6], the existences of antidirected Hamiltonian paths and antidirected Hamiltonian circuits in tournaments were investigated.

For a simple graph G, we use $D(G)$ to denote the digraph obtained from G by replacing each edge e of G by two oppositely oriented edges with the same ends of e. Let K_n be the complete graph on vertex set $\{1, 2, \cdots, n\}$. The directed graph $D(K_n)$ is abbreviated to DK_n. We call DK_n the *complete symmetric graph* on n vertices. Suppose G and H are graphs (digraphs, respectively). If the edges of graph G can be decomposed into subgraphs isomorphic to H, then we say that G has an *H-decomposition*.

In [5], the following result concerning the isomorphic path decomposition of complete graphs was proved.

Theorem 1 *K_n has a P_l-decomposition if and only if $l \leq n-1$ and $n(n-1) \equiv 0 \pmod{2l}$.* □

In this paper we deal with the isomorphic antidirected path decomposition of complete symmetric graphs, and obtain the following.

Theorem. *DK_n has a $\overleftrightarrow{P_l}$-decomposition if and only if the following conditions are satisfied:*

A. n or l is odd.
B. $l \leq n-1$.
C. $n(n-1) \equiv 0 \pmod{l}$.

2 Main Result

In this section we deal with the $\overleftrightarrow{P_l}$-decomposition of DK_n. We begin with the case where l is odd.

Lemma 2 *Suppose that l is an odd integer. If a graph G has a P_l- decomposition. Then $D(G)$ has a $\overleftrightarrow{P_l}$-decomposition.*

Proof. This follows from the fact that if l is an odd integer, then $D(P_l)$ can be decomposed into two antidirected paths of length l. □

Corollary 3 *If l is an odd integer, $l \leq n-1$ and $n(n-1) \equiv 0 \pmod{l}$, then DK_n has a $\overleftrightarrow{P_l}$-decomposition.*

Proof. Since l is odd, the condition $n(n-1) \equiv 0 \pmod{l}$ implies $n(n-1) \equiv 0 \pmod{2l}$. Thus, by Theorem 1, K_n has a P_l-decomposition. Then, by Lemma 2, DK_n has a $\overleftrightarrow{P_l}$-decomposition. □

Now we treat the case where l is even. We introduce some definitions. For integers $n \geq k \geq 1$, The *crown* $C_{n,k}$ is defined to be the graph with vertex set $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$ and edge set $\{a_ib_j : 1 \leq i \leq n, \ j = i+1, i+2, \cdots, i+k \pmod{n}\}$. Crowns $C_{n,k}$ are bipartite graphs with regular degree k. In this paper we consider the crown $C_{n,n-1}$, which is just the graph obtained from the complete bipartite graph with bipartition $(\{a_1, a_2, \cdots, a_n\}, \{b_1, b_2, \cdots, b_n\})$ by taking away the perfect matching $\{a_1b_1, a_2b_2, \cdots, a_nb_n\}$. A path P in the crown $C_{n,n-1}$ is said to be *subscript distinct*, if $S \cap T = \phi$ where $S = \{i : a_i \text{ is a vertex of } P\}$ and $T = \{i : b_i \text{ is a vertex of } P\}$. For example, in $C_{6,5}$ the path $a_5b_2a_3b_4a_6$ is subscript distinct, but the path $a_1b_2a_6b_5a_2b_4$ is not. If a subscript distinct path in $C_{n,n-1}$ has end vertices in $\{a_1, a_2, \cdots, a_n\}$, we call it an *S.E. path*; here, S stands for subscript, E stands for end vertex. Note that an S.E. path has even length. A P_l-decomposition of $C_{n,n-1}$ is called an *S.E. P_l-decomposition*, if each member in the decomposition is an S.E. path.

Lemma 4 *Suppose that l is an even integer, and that the crown $C_{n,n-1}$ has an S.E. P_l-decomposition. Then DK_n has a $\overleftrightarrow{P_l}$-decomposition.*

Proof. Suppose $l = 2t$. Let $\Re$ be an S.E. P_l-decomposition of $C_{n,n-1}$. For each S.E. path P : $a_{i_1}b_{i_2}a_{i_3}b_{i_4}\cdots b_{i_{2t}}a_{i_{2t+1}}$ in $\Re$, we associate an antidirected path $\overleftrightarrow{P}$: $i_1 \to i_2 \leftarrow i_3 \to i_4 \leftarrow \cdots \to i_{2t} \leftarrow i_{2t+1}$ of DK_n. It is easy to see that $\{\overleftrightarrow{P}: P \in \Re\}$ is a $\overleftrightarrow{P_l}$- decomposition of DK_n. □

We use $< a_{k_j}b_{l_j} >_{j=1}^{t}$ to denote the following sequence of vertices: $a_{k_1}b_{l_1}a_{k_2}b_{l_2}\cdots a_{k_t}b_{l_t}$, which is a walk in $C_{n,n-1}$. And $< a_{2+j}b_{n-j} >_{j=1}^{2} < a_{n-j}b_j >_{j=1}^{3} < a_5 >$ is the walk $a_3b_{n-1}a_4b_{n-2}a_{n-1}b_1a_{n-2}b_2a_{n-3}b_3a_5$.
Suppose W_1 is the walk $x_1x_2\cdots x_t$, W_2 is the walk $y_1y_2\cdots y_s$ in a graph such that $x_t = y_1$. Then we use $W_1 + W_2$ to denote the walk

$x_1x_2\cdots x_t y_2 y_3 \cdots y_s$. For walks $W_1, W_2, \cdots, W_v$ in a graph, $W_1+W_2+\cdots+W_v$ is similarly defined. For an integer t and a walk $W: a_{i_1}b_{i_2}a_{i_3}b_{i_4}\cdots a_{i_{2s+1}}$ in the crown $C_{n,n-1}$, we use $W+t$ to denote the walk $a_{i_1+t}b_{i_2+t}a_{i_3+t}b_{i_4+t}\cdots a_{i_{2s+1}+t}$; here and in the sequel, the subscripts of $a_i's$ and $b_i's$ are taken modulo n. Suppose G is a subgraph of $C_{n,n-1}$ and H is a subgraph of G such that the edges of G can be decomposed into subgraphs $H, H+1, H+2, \cdots, H+k$ for some integer k. Then H is called a *base graph* of this decomposition.

Let Q be a trail in a graph. Suppose x, y are two vertices on Q. We use $d_Q(x,y)$ to denote the number of edges on Q between x and y. Suppose $Q: x_1x_2x_3\cdots x_n$ is a trail in a graph. Then the trail $T: x_kx_{k+1}x_{k+2}\cdots x_s$ $(1 \le k \le s \le n)$ is called a *subtrail* of Q. For two vertices x, y on T, we have $d_T(x,y) = d_Q(x,y)$. In the following, for an integer i, $3 \le i \le n$, we use Q_i to denote the trail $a_1b_ia_2b_{i+1}a_3b_{i+2}\cdots a_nb_{i+(n-1)}a_1$ in $C_{n,n-1}$.

Let $Q_k{}' = a_1b_ka_2b_{k+1}a_3b_{k+2}\cdots a_nb_{k+(n-1)}$, i.e., we remove the last vertex of Q_k from Q_k.

Lemma 5 *Suppose n is odd. In the crown $C_{n,n-1}$, let Q be the trail $Q_4+Q_6+Q_8+\cdots+Q_{n-1}$.*

(1) *Suppose $4 \le k \le n-2$. For $t = 1, 2, \cdots, n$, let $x = a_t$ be on Q'_k, and let $y = a_t$ be on Q'_{k+2}. Then $d_Q(x,y) = 2n$.*

(2) *Suppose $4 \le k \le n-2$. Let $x = b_t$ be on Q'_k, and let $y = b_t$ be on Q'_{k+2}. Then* $d_Q(x,y) = \begin{cases} 4n-4, & t = k, k+1 \\ 2n-4, & t = k+2, k+3, \cdots, k+n-1. \end{cases}$

(3) *Suppose $4 \le k \le n-1$. Let $x = a_t$ be on Q'_k, and let $y = b_t$ be on Q such that y is after x and closest to x. Then for $t = 1, 2, \cdots, k-1$, we have $y \in Q'_k$, and $d_Q(x,y) = 2n-2k+3$, for $t = k, k+1$, we have $y \in Q'_{k+2}$, and $d_Q(x,y) = 4n-2k-1$, for $t = k+2, k+3, \cdots, n$, we have $y \in Q'_{k+2}$, and $d_Q(x,y) = 2n-2k-1$.*

(4) *Suppose $4 \le k \le n-1$. Let $x = b_t$ be on Q'_k, and let $y = a_t$ be on Q such that y is after x and closest to x. Then $d_Q(x,y) = 2k-3$ for $t = 1, 2, \cdots, n$.*

Proof. (1) Trivial.

(2)First consider $t = k$. Since $x = b_k$ is the second vertex on Q_k, and $y = b_k$ is the $(2n-3)$-th vertex on Q_{k+2}, it is easy to see that there are $4n-4$ edges on Q which are between $x = b_k$ and $y = b_k$. The proof for $t = k+1$ is similar. Now consider $t = k+2$. Since $x = b_{k+2}$ is the sixth vertex on Q_k, and $y = b_{k+2}$ is the second vertex on Q_{k+2}, it is easy to see that there are $2n-4$ edges between $x=b_{k+2}$ and $y=b_{k+2}$. The proof for $t=k+3, k+4, \cdots, n+k-1$ are similar.

(3) Note that for $t=1,2,\cdots,k-1$, a_t is before b_t on Q'_k. Thus $y=b_t \in Q'_k$. Now consider $t=1$. Since b_1 is the last $2(k-1)$ vertices on Q_k, there are $2n+1-2(k-1)+1=2n-2k+4$ vertices on Q which are between $x=a_1$ and $y=b_1$ (inclusively). Thus $d_Q(x,y)=2n-2k+3$. The results for $t=2,3,\cdots,k-1$ follow easily. Next consider $t=k,k+1$. First consider $t=k$. Note that b_k is before a_k on Q_k. Thus the b_k, which is after $x=a_k$ and closest to x, is on Q_{k+2}. We see that a_k is the $(2k-1)$-th vertices on Q_k, and the last four vertices on Q_{k+2} are b_k, a_n, b_{k+1}, a_1. Thus there $4n+1-(2k-2+3)=4n-2k$ vertices on Q between $x=a_k$ and $y=b_k$. Hence $d_Q(x,y)=4n-2k-1$. The result for $t=k+1$ follows easily.

Now consider $t=k+2,k+3,\cdots,n$. Now $t=k+2$. Since b_{k+2} is before a_{k+2} on Q_k, the b_{k+2}, which is after $x=a_{k+2}$ and closest to a_{k+2}, is on Q_{k+2}. We see that $x=a_{k+2}$ is the $(2k+3)$-th vertex on Q_k, and $y=b_{k+2}$ is the second vertex on Q_{k+2}. Thus there are $2n-(2k+2)+2=2n-2k$ vertices between $x=a_{k+2}$ and $y=b_{k+2}$(inclusively). Thus $d_Q(x,y)=2n-2k-1$. The results for $t=k+3,k+4,\cdots,n$ follow immediately.

(4) First consider $t=k$. Since b_k is the second vertex on Q_k, a_k is the $(2k-1)$-th vertex on Q_k, it follows that there are $2k-3$ edges on Q which are between $x=b_k$ and $y=a_k$. Thus $d_Q(x,y)=2k-3$. The results for $t=k+1,k+2,\cdots,n$ follow immediately. Now consider $t=1$. Since b_k is the second vertex on Q_k, and b_1 is after b_n on Q_k, we see that b_1 is the 2(n-k+2)-th vertex on Q_k. The vertex $y=a_1$ is the first vertex on Q_{k+2}. Thus there are $2k-3$ edges on Q which are between $x=b_1 \in Q'_k$ and $y=a_1 \in Q'_{k+2}$. Thus $d_Q(x,y)=2k-3$. The results for $t=2,3,\cdots,k-1$ follow immediately. □

Suppose that T is a trail in $C_{n,n-1}$, we use $d(T)$ to denote the least number of edges between two vertices of T with the same subscript.

Lemma 6 *Suppose that l is an even integer ≥ 2, and n is an odd integer such that $l \leq n-1$, and $n(n-1) \equiv 0 \pmod l$. Then $C_{n,n-1}$ has an S.E. P_l-decomposition.*

Proof. The edges of $C_{n,n-1}$ are labeled as follows. Each edge can be assumed to be $a_j b_k$ with $1 \leq j \leq n$, $j < k < j+n$. We refer to this edge as an s-edge where $s=k-j$. Let G_1 be the spanning subgraph of $C_{n,n-1}$ such that G_1 consists of all the edges with labels $1,2,\cdots,\frac{l}{2}$, $n-\frac{l}{2}, n-\frac{l}{2}+1,\cdots,n-1$. And let $G_2=C_{n,n-1}-E(G_1)$. We will prove the existence of S.E. P_l-decomposition of $C_{n,n-1}$ by showing that both G_1 and G_2 have S.E. P_l-decompositions. Note that in case $l=n-1$, G_2 is an empty graph.

The S.E. P_l-decomposition of G_1 will be achieved by using a base graph Q defined as follows. When $l = 4m$ ($m \in N$), let Q be the path $< a_j b_{n+1-j} >_{j=1}^{\frac{l}{4}} < a_{\frac{l}{4}+j} b_{\frac{3l}{4}+2-j} >_{j=1}^{\frac{l}{4}} < a_{1+\frac{l}{2}} >$, i.e., Q is the path $a_1 b_n a_2 b_{n-1} a_3 b_{n-2} \cdots a_{\frac{l}{4}} b_{n+1-\frac{l}{4}} a_{\frac{l}{4}+1} b_{\frac{3l}{4}+1} a_{\frac{l}{4}+2} b_{\frac{3l}{4}} \cdots a_{\frac{l}{2}} b_{\frac{l}{2}+2} a_{\frac{l}{2}+1}$. When $l = 2$, let Q be the path $a_1 b_n a_{n-1}$. When $l = 4m+2$ ($m \in N$), let Q be the path $< a_j b_{n+1-j} >_{j=1}^{\frac{l+2}{4}} < a_{n-\frac{3l+2}{4}+j} b_{n-\frac{l-2}{4}-j} >_{j=1}^{\frac{l-2}{4}} < a_{n-\frac{l}{2}} >$, i.e., Q is the path $a_1 b_n a_2 b_{n-1} a_3 b_{n-2} \cdots a_{\frac{l+2}{4}} b_{n-\frac{l-2}{4}} a_{n-\frac{3l-2}{4}} b_{n-\frac{l+2}{4}} a_{n+1-\frac{3l-2}{4}} b_{n-1-\frac{l+2}{4}} \cdots a_{n-\frac{l}{2}-1} b_{n-\frac{l}{2}+1} a_{n-\frac{l}{2}}$. It is easy to see that Q is an S.E. path having length l and consisting of edges with labels in order of $n-1, n-2, \cdots, n-\frac{l}{2}+1, n-\frac{l}{2}, \frac{l}{2}, \frac{l}{2}-1, \cdots, 2, 1$. We also see that G_1 can be decomposed into $Q, Q+1, Q+2, \cdots, Q+(n-1)$, and each of them is an S.E. path of length l. Thus G_1 has an S.E. P_l-decomposition.

Next we consider the decomposition of G_2. As mentioned before, G_2 is an empty graph if $l = n-1$. Thus assume $l \leq n-2$. We will define an Eulerian trail of G_2, and then cut the trail into paths which are needed for the decomposition.

For $i = \frac{l}{2}+3, \frac{l}{2}+5, \frac{l}{2}+7, \cdots, n-\frac{l}{2}$, as defined in the paragraphs preceding Lemma 5, let Q_i be the trail $a_1 b_i a_2 b_{i+1} a_3 b_{i+2} \cdots a_n b_{i+(n-1)} a_1$. We see that each Q_i is in fact a Hamiltonian cycle of $C_{n,n-1}$ and consists of all the edges with labels $i-1$ and $i-2$. Thus $E(G_2) = \bigcup_{i \in A} E(Q_i)$, where $A = \{\frac{l}{2}+3, \frac{l}{2}+5, \cdots, n-\frac{l}{2}\}$. Let T be the trail $Q_{\frac{l}{2}+3} + Q_{\frac{l}{2}+5} + \cdots + Q_{n-\frac{l}{2}}$. Obviously T is an Eulerian trail of G_2. To determine $d(T)$, we need to evaluate the minimum number of edges between two vertices x, y on T with the same subscript. Suppose x is before y on T. We consider four cases: (1)$x = a_t$, $y = a_t$, (2)$x = b_t$, $y = b_t$, (3)$x = a_t$, $y = b_t$, (4)$x = b_t$, $y = a_t$. In each case we will show $d_T(x,y) \geq l+1$.

As defined in Lemma 5, let Q be the trail $Q_4 + Q_6 + Q_8 + \cdots + Q_{n-1}$ in $C_{n,n-1}$. Obviously T is a subtrail of Q in $C_{n,n-1}$. For any two vertices u, v on T, we have $d_T(u,v) = d_Q(u,v)$.

Case 1. $x = a_t$, $y = a_t$

By Lemma 5(1), $d_Q(x,y) = 2n$. Thus $d_T(x,y) = 2n \geq l+1$.

Case 2. $x = b_t$, $y = b_t$

By Lemma 5(2), $d_Q(x,y) = 4n-4$ or $2n-4$, which implies $d_T(x,y) = d_Q(x,y) \geq l+1$.

Case 3. $x = a_t$, $y = b_t$

We use the result of Lemma 5(3), First consider $t = 1, 2, \cdots, k-1, k, k+1$. We have $d_Q(x,y) = 2n-2k+3$ or $4n-2k-1$, which implies $d_Q(x,y) \geq l+1$, since $k \leq n-\frac{l}{2}$. Now consider $t = k+2, k+3, \cdots, n$. Since $x \in Q'_k$,

$y \in Q'_{k+2}$, we have $k+2 \leq n-\frac{l}{2}$. Thus $d_Q(x,y) = 2n-2k-1 \geq l+3 \geq l+1$. Hence $d_T(x,y) \geq l+1$.

Case 4. $x = b_t$, $y = a_t$

By Lemma 5(4), $d_Q(x,y) = 2k-3$. Thus $d_T(x,y) = d_Q(x,y) \geq l+3 \geq l+1$, since $k \geq \frac{l}{2}+3$.

From above, we conclude that $d(T) \geq l+1$. Now from the starting vertex we cut the trail T into subtrails with l edges. From the facts that $d(T) \geq l+1$, the starting vertex of T is in $\{a_1, a_2, \cdots, a_n\}$, and l is even, we see that each subtrail is an S.E. path. Thus G_2 has an S.E. P_l-decomposition. This completes the proof. □

The following corollary follows immediately from Lemma 4 and 6.

Corollary 7 *Suppose that l is an even integer, and n is an odd integer such that $l \leq n-1$, and $n(n-1) \equiv 0 \pmod{l}$. Then DK_n has a $\overleftrightarrow{P_l}$-decomposition.* □

Proof of Theorem. (Necessity) Conditions B and C are obvious. We prove Condition A. Suppose that l is even. We will show that n is odd. Let $\mathfrak{R}$ be a $\overleftrightarrow{P_l}$-decomposition of DK_n. Each antidirected path in $\mathfrak{R}$ contributes 0 or 2 to the indegree of every vertex of DK_n. Thus every vertex of DK_n has even indegree, which implies that n is odd.
(Sufficiency) Consider two cases.

Case 1. l is odd.

By Corollary 3, DK_n has a $\overleftrightarrow{P_l}$-decomposition.

Case 2. l is even.

By Condition A, n is odd. Then, by Corollary 7, DK_n has a $\overleftrightarrow{P_l}$-decomposition. □

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