Some Families of Elegant and Harmonious Graphs

Parag K. Deb
Department of Mathematics, Cotton College, Guwahati, Assam, 781001, India
N. B. Limaye
Department of Mathematics, University of Mumbai, 400098, India

Abstract: In this paper we give the following labelings:
(1) Elegant labelings of triangular snakes $\Delta_n, n \equiv 0, 1, 2 \bmod 4$.
(2) Near elegant labeling of triangular snakes Δ_n, when $n \equiv 3 \bmod 4$, which are not elegant.
(3) Elegant and Near-elegant labelings of some of the theta graphs $\theta_{\alpha,\beta,\gamma}, \alpha = 1, 2, 3$.
(4) Harmonious labelings of helms H_n when n is even.
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1 INTRODUCTION

Let G be a finite simple graph on p vertices and m edges. When an edge e of the graph G has end points x and y, we simply refer to the edge e as the edge xy.

A **harmonious** labeling of a graph G is a 1-1 function ϕ on the vertex set $V(G)$ to the set $\{0, 1, \ldots, m-1\}$ such that the induced edge labeling given by $\phi(xy) = \phi(x) + \phi(y)(\bmod m)$ for every edge $xy \in E(G)$, is also 1-1. Here, if $m = p - 1$, then exactly one label may be used on two vertices. This additive version of a graceful labeling of a graph was introduced by R. J. Graham and N. J. A. Sloane [5]. Since all the values from 0 to $m-1$ have to be taken by the edges, many simple graphs are easily observed to be non-harmonius.

An **elegant** labeling of a graph G is an injective(1-1) function ϕ from the vertex set $V(G)$ to the set $\{0, 1, \ldots, m\}$ such that the induced edge labeling given by $\phi(xy) = \phi(x) + \phi(y)(\text{mod}(m+1))$ for every edge $xy \in E(G)$, is also injective and $\phi(xy)$ is not congruent to 0 mod $(m+1)$ for every edge xy in the edge set $E(G)$.

The concept was introduced by G. J. Chang, D. F. Hsu and D. G. Rogers [3]. They proved that the cycles C_{4p}, C_{4p+3} and the paths $P_{4p+i}, 1 \le i \le 3$, are elegant whereas the cycles C_{4p+1} are not elegant for all natural numbers p. Cahit [2] proved that the path P_{4p} is elegant for all $p > 1$. M. Mollard and C. Payan [6] proved that the cycle C_{4p+2} is elegant for all $p \ge 1$. For elegant labelings the edge labelings are non-zero. That means all the values from 1 to m are taken by the edges. This does not leave much freedom and hence a simple graph like C_{4p+1} is non-elegant.

We will say that a graph is **near-elegant** if there exists an injective (1-1) function ϕ from the vertex set $V(G)$ to the set $\{0, 1, \dots, m\}$ such that the induced edge labeling given by $\phi(xy) = \phi(x) + \phi(y)(\text{mod}(m+1))$ for every edge $xy \in E(G)$, is also injective and $\phi(xy)$ is not congruent to m mod $(m+1)$ for every edge xy in the edge set $E(G)$. This means that instead of zero being the missing value in the edge labelings, the integer m is the missing value. In [7] D. Moulton has proved that all triangular snakes are graceful. In [1] L. Bolian and Z. Xiankum attempted to show that all triangular snakes Δ_n are harmonious for odd values of n and not harmonious when $n \equiv 2$ mod 4. For the triangular snake Δ_n, the number of vertices is $2n+1$ and the number of edges is $3n$, so that $m > p-1$. However, all the labelings given in [1] have one repeated value. This means that the labeling is not 1-1. Hence the question whether triangular snakes are harmonious or not remains unanswered and it makes sense to ask whether triangular snakes are at least elegant. We show that the triangular snake Δ_n is elegant when $n \equiv 0, 1, 2($ mod 4$)$. However, for $n \equiv 3($ mod 4$)$ the triangular snake Δ_n is not elegant but it is near-elegant. We also give elegant labelings of some of the theta graphs and harmonious labelings of helms H_n when n is even.

2 TRIANGULAR SNAKES

We will first introduce some notations.

Definition: A triangular snake Δ_n is a graph whose vertex set and edge set are as follows:

$$V(\Delta_n) = \{x_0, \cdots, x_n\} \bigcup \{y_1, \cdots, y_n\},$$
$$E(\Delta_n) = \{x_{i-1}x_i, x_{i-1}y_i, x_iy_i : 1 \leq i \leq n\}.$$

One can see that the induced graph on the set $\{x_{i-1}, x_i, y_i\}$ is a triangle for every $1 \leq i \leq n$. This triangle is denoted by T_i. We refer to the edge $x_{i-1}x_i$ by h_i, the edge $x_{i-1}y_i$ by l_i and the edge x_iy_i by r_i.

In the begining we shall prove a general result about elegant graphs employing a technique used by L. Bolian and Z. Xiankun in [1, Theorem 2.3]. As a corollary, we prove that the triangukar snakes $\Delta_n, n \equiv 3$ mod 4 are not elegant.

Theorem 1: Let $\{d_1, \dots, d_p\}$ be the degree sequence of a (p, m)-graph G with vertex set $\{v_1, \dots, v_p\}$. If G is elegant, then

$$\sum_{i=1}^{p} d_i\ \phi(v_i) \equiv \binom{m+1}{2} \text{ mod } (m+1).$$

Proof: If ϕ is an elegant labeling of the vertex set $V(G)$ of G then the induced labeling $\phi(xy) = \phi(x) + \phi(y)$ takes all the values $\{1, 2, \ldots, m\}$ modulo $(m+1)$. Hence for some non-negative integer t,

$$\begin{aligned}\sum_{i=1}^{p} d_i\ \phi(v_i) &= \sum_{v_iv_j \in E(G)} \phi(v_iv_j) + (m+1)\ t \\ &\equiv 1 + \ldots + m \bmod (m+1), \\ &\equiv \binom{m+1}{2} \bmod (m+1).\end{aligned}$$

□

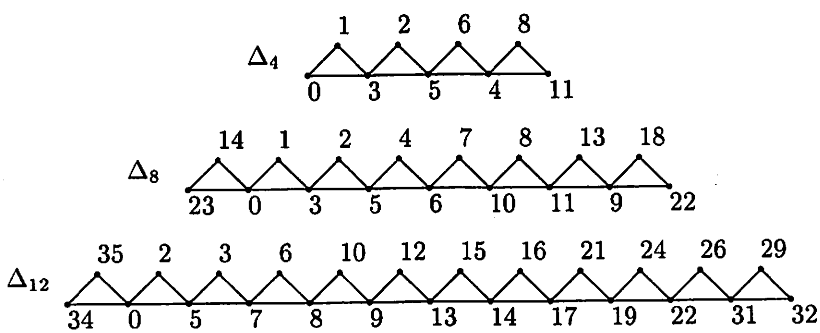
Corollary: If $n \equiv 3 \bmod 4$, then the triangular snake Δ_n is not elegant.
Proof: If $n \equiv 3 \bmod 4$, then $n = 4t+3$ for some non-negative integer t and $m = 12t + 9$. For a triangular snake degree of every vertex is either 2 or 4 and so $\sum d_i\ \phi(v_i)$ is an even number 2α. If Δ_n is elegant then by Theorem 1, we get

$$\binom{m+1}{2} = 2\alpha + \beta(12t+10).$$

for some integers α, β. This means $(6t+5)(12t+9)$ is divisible by 2, a contradiction. □

The following three results show that for every $n \equiv 0, 1, 2 \bmod 4$, Δ_n is elegant.

Theorem 2: If $n \equiv 0 \bmod 4$, then the triangular snake Δ_n is elegant.
Proof: Let $n = 4t$. The following figures give elegant labelings of Δ_n for $t = 1, 2, 3$.



For $t \geq 4$, we give the labeling as follows:
$\phi(x_0) = m - 2, \phi(x_1) = 0, \phi(x_2) = 5, \phi(x_3) = 7, \phi(x_4) = 8,$
$\phi(x_5) = 9, \phi(x_6) = 13, \phi(x_7) = 14, \phi(x_8) = 17, \phi(x_9) = 18.$
The next $2t - 7$ vertices are labeled as follows:

$$\phi(x_{(9+2j-1)}) = 18 + 6j - 1, \quad 1 \le j \le (t-3)$$
$$\phi(x_{(9+2j)}) = 18 + 6j, \quad 1 \le j \le (t-4).$$

The last x_r labeled so far is $x_{(2t+2)}$ and $\phi(x_{(2t+2)}) = 6t - 1$. The next three values are sporadic. Let $\phi(x_{(2t+3)}) = 6t+1, \phi(x_{(2t+4)}) = 6t+4, \phi(x_{(2t+5)}) = 6t + 13$. Now, if $q = 2t + 5$, let,

$$\phi(x_{(q+2j)}) = \phi(x_q) + 6j = 6t + 6j + 13, \quad 1 \le j \le (t-3),$$
$$\phi(x_{(q+2j-1)}) = \phi(x_q) + 6j - 5 = 6t + 6j + 8, \quad 1 \le j \le (t-2).$$

This means $\phi(x_{(2t+6)}) = 6t + 14$ and the last vertex to be labled now is $x_{(q+2t-4-1)} = x_{4t}$. This finishes labeling of $x_r, 0 \le r \le n$.

The labeling of $y_1, \ldots, y_n$ is given as follows:
$\phi(y_1) = m - 1, \phi(y_2) = 2, \phi(y_3) = 3, \phi(y_4) = 6, \phi(y_5) = 10,$
$\phi(y_6) = 12, \phi(y_7) = 15, \phi(y_8) = 16, \phi(y_9) = 19, \phi(y_{10}) = 22.$
The next $2t - 8$ vertices are labeled as follows: For $\le j \le t - 4$,

$$\phi(y_{9+2j}) = \phi(x_{9+2j}) + 1 = 19 + 6j \text{ and } \phi(y_{10+2j}) = \phi(x_{10+2j}) - 1 = 22 + 6j$$

The last vertex labeled so far is $y_{(2t+2)}$, for which $\phi(y_{(2t+2)}) = \phi(x_{(2t+2)}) - 1 = 6t - 2$. Again the next five values are sporadic. Let $\phi(y_{(2t+3)}) = \phi(x_{(2t+3)}) + 2 = 6t + 3, \phi(y_{(2t+4)}) = \phi(x_{(2t+4)}) + 2 = 6t + 6, \phi(y_{(2t+5)}) = \phi(x_{(2t+5)}) - 5 = 6t + 8, \phi(y_{(2t+6)}) = \phi(x_{(2t+6)}) - 3 = 6t + 11, \phi(y_{(2t+7)}) = \phi(x_{(2t+7)}) + 2 = 6t + 21 = \phi(y_{(2t+6)}) + 10$. After this, let $\phi(y_{(2t+7+i)}) = 6t + 21 + 3i, 1 \le i \le (2t-7)$. The last vertex labeled so far is $y_{(2t+7+2t-7)} = y_{4t}$. This completes the valuation.

We prove, by induction, that the map ϕ induces a 1-1 map from the edge set $E(G)$ to the set $\{1, \ldots, m\}$. By the triangle T_i, we mean the triangle $\{x_{i-1}, x_i, y_i\}$. Let

$$E_1 = E(\bigcup_{i=2}^{2t+2} T_i), \quad E_2 = E(\bigcup_{i=2t+3}^{4t} T_i).$$

For $t > 4$, let $s = t - 1$ and $m' = 12s = |E(\Delta_{4s})|$. The triangles corresponding to s are denoted by T_i' and the corresponding edge sets by E_1', E_2'. To use mathematical induction, assume that Δ_{4s} is elegant and the actual values assigned to the edges are

$$\phi(T_1') = \{(m' + 1 + m' - 4, m' - 2, m' - 1\},$$
$$\phi(h_{2s+3}) = m', \phi(E_1') \subset \{1, \ldots, (m' - 3)\},$$
$$\phi(E_2') = \{m'\} \bigcup \{m' + 1 + z : 1 \le z \le m', z \notin \phi(E_1')\}.$$

This means, though all the actual values are mentioned here, they are all distinct modulo $(m^{'}+1)$. It is easy to check this for Δ_{16}. Now, for $t = s+1$, we note that the actual values assigned to the edges are

$$\phi(T_1) = \{(m+1)+m-4, m-2, m-1\},$$
$$\phi(E_1) = \phi(E_1^{'}) \bigcup \phi(T_{2t+1}) \bigcup \phi(T_{2t+2}),$$
$$\phi(E_2) = \{m\} \bigcup \{m^{'}+1+z+12 : 1 \le z \le m^{'}, z \notin \phi(E_1^{'})\} \bigcup \phi(T_{4t-1} \bigcup T_{4t}).$$

Since earlier values are distinct modulo $(m^{'}+1)$ and $m = m^{'}+12$, all the values, except $\{m^{'}-1, m^{'}-2, m^{'}-4, m^{'}, m^{'}+1, \ldots, m^{'}+11\}$ in the set $\{1, \ldots, m\}$ have appeared precisely once modulo $m+1$.. It remains to see that these values appear in $\phi(E(\Delta_{4t}))$. We note the following: When taken modulo $m+1$,

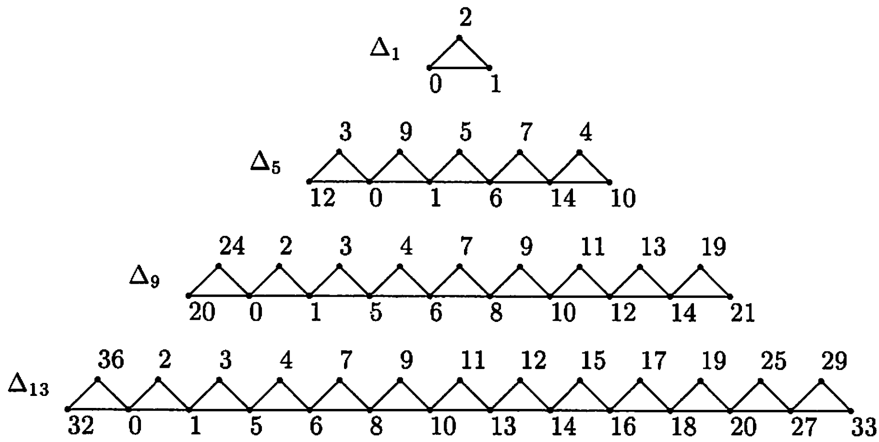
$$\phi(T_{2t+1}) = \{m-13, m-12, m-11\} = \{m^{'}-1, m^{'}, m^{'}+1\},$$
$$\phi(T_{2t+2}) = \{m-8, m-7, m-3\} = \{m^{'}+4, m^{'}+5, m^{'}+9\},$$
$$\phi(T_{4t-1}) = \{m-16, m-14, m-9\} = \{m^{'}-4, m^{'}-2, m^{'}+3\},$$
$$\phi(T_{4t}) = \{m-10, m-6, m-5\} = \{m^{'}+2, m^{'}+6, m^{'}+7\},$$
$$\phi(T_1) = \{m-4, m-2, m-1\} = \{m^{'}+8, m^{'}+10, m^{'}+11\}.$$

These are the missing values. Using mathematical induction, one can show that the map ϕ is 1-1 on the vertex set also. Thus Δ_{4t} is elegant for every positive integer t. □

Remark: In the labeling given in the proof of Theorem 2, one can observe the following. Consider the sequences of edges $\{h_{10}, h_{11}, \cdots, h_{(2t+2)}\}$ and $\{h_{(2t+6)}, h_{(2t+7)}, \cdots, h_{(4t)}\}$. The labelings of these edges are given by following sequences $\{41, 47, \cdots, m-7\}$ and $\{26, 32, 38, 44, \cdots, m-10\}$, respctively. Both these sequences are in an arithmatic progression with the same common difference. The elements of the second sequence are placed exactly half way between those of the first. This itself shows that all these elements are distinct.

The proofs of next three theorems are simmilar to that of Theorem 2 and hence we shall just give the labelings.

Theorem 3: If $n \equiv 1 \bmod 4$, then the triangular snake Δ_n is elegant.
Proof: Let $n = 4t+1$. The following figures give elegant labelings of Δ_n for $t = 0, 1, 2, 3$.



For $t \geq 4$, we give the labeling as follows:
$\phi(x_0) = m - 7, \phi(x_1) = 0, \phi(x_2) = 1, \phi(x_3) = 5, \phi(x_4) = 6,$
$\phi(x_5) = 8, \phi(x_6) = 10, \phi(x_7) = 13, \phi(x_8) = 14, \phi(x_9) = 16.$
The next $3t - 9$ vertices are labeled as follows:

$$\phi(x_{(3j+10)}) = 19 + 6j, \quad \phi(x_{(3j+11)}) = 20 + 6j, \quad \phi(x_{(3j+12)}) = 22 + 6j,$$

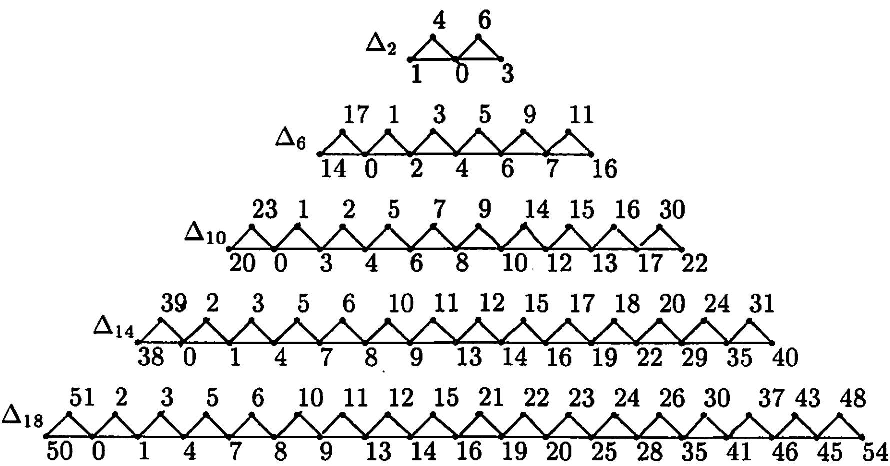
where $0 \leq j \leq t-4$. The last vertex labeled so far is x_{3t} and $\phi(x_{3t}) = 6t-2$. The next three values are sporadic. Let $\phi(x_{(3t+1)}) = 6t, \phi(x_{(3t+2)}) = 6t + 2, \phi(x_{(3t+3)}) = 6t+9$. If $q = 3t+3$, let $\phi(x_{(q+j)}) = \phi(x_q)+6j = 6(t+j)+9$ for $1 \leq j \leq t-2$. The last vertex labeled so far is $x_{q+t-2} = x_{3t+3+t-2} = x_{4t+1}$. This finishes labeling of $x_r, 0 \leq r \leq n$.

The labeling of $y_1, \ldots, y_n$ is given as follows:
$\phi(y_1) = m - 3, \phi(y_2) = 2, \phi(y_3) = 3, \phi(y_4) = 4, \phi(y_5) = 7,$
$\phi(y_6) = 9, \phi(y_7) = 11, \phi(y_8) = 12, \phi(y_9) = 15, \phi(y_{10}) = 17.$
The next $(3t - 9)$ vertices are labeled as follows:

$$\phi(y_{(3j+11)}) = 18 + 6j, \quad \phi(y_{(3j+12)}) = 21 + 6j, \quad \phi(y_{(3j+13)}) = 23 + 6j,$$

where $0 \leq j \leq t-4$. The last vertex labeled so far is $y_{(3t+1)}$ and $\phi(y_{(3t+1)}) = 6t - 1$. Again, the next three values are sporadic. Let $\phi(y_{(3t+2)}) = 6t + 1, \phi(y_{(3t+3)}) = 6t + 7, \phi(x_{(3t+4)}) = 6t + 11$. If $r = 3t + 4$, let $\phi(y_{(r+j)}) = 6(t + j) + 11$ for $1 \leq j \leq t - 3$. The last vertex labeled is $y_{(r+t-3)} = y_{(3t+4+t-3)} = y_{(4t+1)}$. This finishes the labeling of the vertices. □

Theorem 4: If $n \equiv 2 \bmod 4$, then the triangular snake Δ_n is elegant.
Proof: Let $n = 4t + 2$. The following figures give elegant labelings of Δ_n for $t = 0, 1, 2, 3, 4$.



For $t \geq 5$, we give the labeling as follows:
$\phi(x_0) = m - 4, \phi(x_1) = 0, \phi(x_2) = 1, \phi(x_3) = 4, \phi(x_4) = 7,$
$\phi(x_5) = 8, \phi(x_6) = 9, \phi(x_7) = 13, \phi(x_8) = 14, \phi(x_9) = 16, \phi(x_{10}) = 19.$
The next $2(t-3)$ vertices are labeled as: $\phi(x_{(2j+10)}) = 19+6j$, $\phi(x_{(2j+9)}) = 19 + 6j - 5 = 14 + 6j$, where $1 \leq j \leq t - 3$.

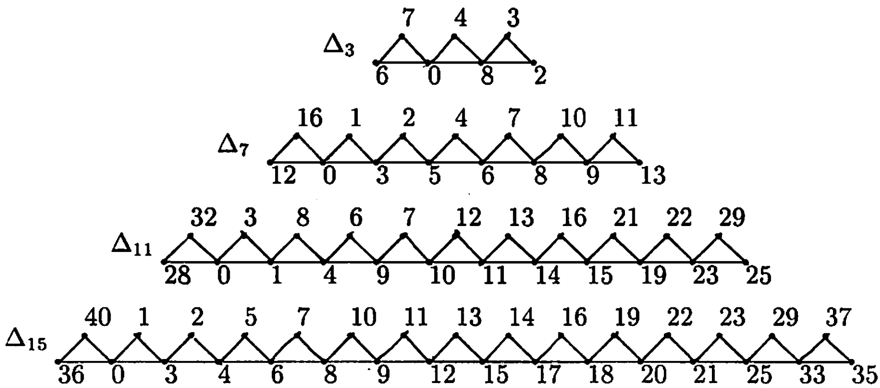
The last vertex labeled so far is x_{4+2t} and $\phi(x_{4+2t}) = 6t+1$. The next six values are sporadic. Let $\phi(x_{5+2t}) = 6t + 4, \phi(x_{6+2t}) = 6t + 11, \phi(x_{7+2t}) = 6t + 17, \phi(x_{8+2t}) = 6t + 22, \phi(x_{9+2t}) = 6t + 21, \phi(x_{10+2t}) = 6t + 29$. If $q = 10+2t$, let $\phi(x_{q+2i+1}) = 6t+29+(i+1)+5i = 6t+30+6i, 0 \leq i \leq t-5$ and $\phi(x_{q+2i}) = 6t + 29 + 6i, 1 \leq i \leq (t - 5)$. The last vertex labeled so far is x_{4t+1} and $\phi(x_{4t+1}) = 12t$. Let $\phi(x_{4t+2}) = 12t + 6 = m$. This completes labelling of $x_i, 0 \leq i \leq n$.

The labeling of $y_1, \ldots, y_n$ is given as follows:
$\phi(y_1) = m - 3, \phi(y_2) = 2, \phi(y_3) = 3, \phi(y_4) = 5, \phi(y_5) = 6, \phi(y_6) = 10,$
$\phi(y_7) = 11, \phi(y_8) = 12, \phi(y_9) = 15, \phi(y_{10}) = 21, \phi(y_{11}) = 22$. After this, let $\phi(y_{11+2i-1}) = 22+6i-4$, $\phi(y_{11+2i}) = 22+6i, 1 \leq i \leq t-4$. The last vertex labeled so far is $y_{11+2t-8} = y_{3+2t}$ and $\phi(y_{3+2t}) = 6t-2$. Again the next eight values are sporadic. Let $\phi(y_{2t+4}) = 6t - 1, \phi(y_{2t+5}) = 6t, \phi(y_{2t+6}) = 6t + 2,$ $\phi(y_{2t+7}) = 6t+6, \phi(y_{2t+8}) = 6t+13, \phi(y_{2t+9}) = 6t+19, \phi(y_{2t+10}) = 6t+24,$ $\phi(y_{2t+11}) = 6t+25$. For $r = 2t+11$, let $\phi(y_{r+2i-1}) = 6t+25+2i+4(i-1) = 6t + 21 + 6i, 1 \leq i \leq t - 4$ and $\phi(y_{r+2i}) = 6t + 25 + 6i, 1 \leq i \leq t - 5$. The last vertex labeled so far is $y_{r+2t-8-1} = y_{4t+2} = y_n$ and $\phi(y_n) = m - 9$. □

3 NEAR ELEGANCE OF Δ_{4t+3}

We have seen that Δ_{4t+3} is not elegant. We now give a near elegant labeling of Δ_{4t+3}.

Theorem 5: If $n \equiv 3 \bmod 4$, then the triangular snake Δ_n is near-elegant.
Proof: Let $n = 4t + 3$. The following figures give near-elegant labelings of Δ_n for $t = 0, 1, 2, 3$.



For $t \geq 4$, we give the labeling as follows:
$\phi(x_0) = m - 9, \phi(x_1) = 0, \phi(x_2) = 3, \phi(x_3) = 4, \phi(x_4) = 6,$
$\phi(x_5) = 8, \phi(x_6) = 9, \phi(x_7) = 12, \phi(x_8) = 15, \phi(x_9) = 17.$
The next $3(t - 3)$ vertices are labeled as follows:

$$\phi(x_{(9+3j-2)}) = 17 + 2j + 4(j - 1) = 13 + 6j,$$
$$\phi(x_{(9+3j-1)}) = 17 + 6j - 3 = 14 + 6j, \quad \phi(x_{(9+3j)}) = 17 + 6j.$$

where $1 \leq j \leq t - 3$. The last vertex labeled so far is $x_{9+3(t-3)} = x_{3t}$ and $\phi(x_{3t}) = 6t - 1$. The next six values are sporadic. Let $\phi(x_{1+3t}) = 6t, \phi(x_{2+3t}) = 6t + 2, \phi(x_{3+3t}) = 6t + 3, \phi(x_{4+3t}) = 6t + 7, \phi(x_{5+3t}) = 6t + 15, \phi(x_{6+3t}) = 6t + 17$. If $q = 6 + 3t$, let $\phi(x_{q+i}) = 6t + 17 + 6i, 1 \leq i \leq t - 3$. The last vertex labeled so far is $x_{6+3t+t-3} \doteq x_n$ and $\phi(x_n) = 6t+17+6t-18 = 12t-1 = m-10$. This completes labelling of $x_i, 0 \leq i \leq n$.

The labeling of $y_1, \ldots, y_n$ is given as follows:
$\phi(y_1) = m - 5, \phi(y_2) = 1, \phi(y_3) = 2, \phi(y_4) = 5, \phi(y_5) = 7,$
$\phi(y_6) = 10, \phi(y_7) = 11, \phi(y_8) = 13, \phi(y_9) = 14, \phi(y_{10}) = 16.$
After this, for $1 \leq i \leq t - 3$, let

$$\phi(y_{(10+3i-2)}) = 16 + 2i + 4(i - 1) = 12 + 6i,$$
$$\phi(y_{(10+3i-1)}) = 15 + 6i,$$
$$\phi(y_{(10+3i)}) = 16 + 6i.$$

The last vertex labeled so far is $y_{10+3t-9} = y_{3t+1}$ and $\phi(y_{3t+1}) = 16+6(t-3) = 6t-2$. Again the next six values are sporadic. Let

$\phi(y_{3t+2}) = 6t+1$, $\phi(y_{3t+3}) = 6t+4$, $\phi(y_{3t+4}) = 6t+5$,

$\phi(y_{3t+5}) = 6t+11$, $\phi(y_{3t+6}) = 6t+19$, $\phi(y_{3t+7}) = 6t+27$.

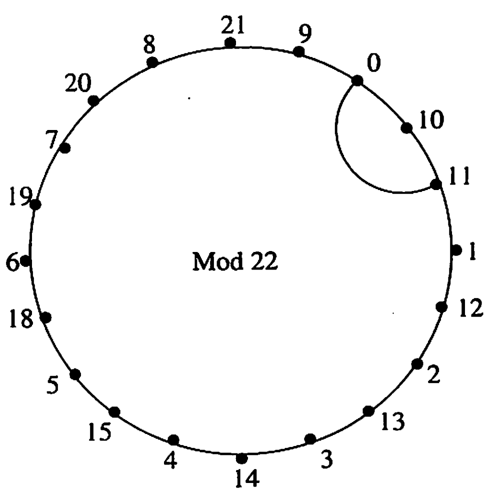
For $r = 3t+7$, let $\phi(y_{r+i}) = 6t+27+6i, 1 \leq i \leq t-4$. The last vertex labeled so far is $y_{r+t-4} = y_{4t+3} = y_n$ and $\phi(y_n) = 6t+27+6t-24 = m-6$. □

4 THETA GRAPHS

Definition: For three natural numbers $\alpha \leq \beta \leq \gamma$ the **theta graph** $\theta_{\alpha,\beta,\gamma}$ consists of three vertex disjoint paths of length α, β, γ having same end points.

We give here elegant or near elegant labelings of some of the theta graphs:

Theorem 6: Let n be a positive integer, $n \equiv 0 \bmod 4$ and let $\theta_{1,\beta,\gamma}$ be a theta graph on n points.
(a) If β and γ are both congruent to 2 modulo 4, then $\theta_{1,\beta,\gamma}$ is elegant.
(b) If β and γ are both odd then $\theta_{1,\beta,\gamma}$ is near-elegant.
Proof: (a) First consider $\alpha = 1, \beta = 2$ and $\gamma = 4t-2$. This graph has $n = 4t$ vertices and $m = 4t+1$ edges. Thus $m+1 = 4t+2$. Let the vertices be $\{a_0, b_0, a_1, b_1, \ldots, a_{2t-1}, b_{2t-1}\}$, connected in a cyclic manner. The extra edge being $b_{2t-2}b_{2t-1}$.



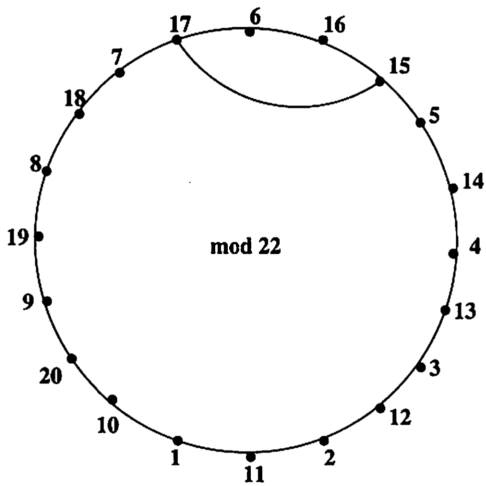
Label the vertices as follows:
$\phi(a_i) = i+1, 0 \leq i \leq 2t-1$, $\phi(b_{2t-2}) = 0$, $\phi(b_{2t-1}) = 2t+1$,

$\phi(b_i) = 2t+2+i, 0 \le i \le t-2$, $\phi(b_i) = 2t+i+4, t-1 \le i \le 2t-3$,
We have given here elegant labeling of $\theta_{1,2,18}$.

Let us see the induced labeling ϕ on the edges. It is easy to check that the edges $a_0b_0, b_0a_1, \ldots, a_{t-2}b_{t-2}, b_{t-2}a_{t-1}$ receive the values from $2t+3$ to $4t$ in a sequential manner and the edges $a_{t-1}b_{t-1}, b_{t-1}a_t, \ldots, b_{2t-2}a_{2t-1}$ receive values $1, 2, \ldots, 2t$ in a sequential manner. Finally we see that $\phi(b_{2t-1}a_0) = 2t+2, \phi(a_{2t-1}b_{2t-1}) = 4t+1$ and $\phi(b_{2t-2}b_{2t-1}) = 2t+1$. This shows that ϕ takes all the values of the set $\{1, \ldots, 4t+1\}$ on the edges. Thus the graph $\theta_{1,2,4t-2}$ is elegant.

If in the labeling given above, we keep the labels unchanged but remove the edge $e = b_{2t-2}b_{2t-1}$ and introduce a suitable new edge, we get elegant labeling of some other θ-graph. For example if e is replaced by $b_{(\lambda-1)}b_{(2t-3-\lambda+1)}$ then we get an elegant labeling of $\theta_{1,2+4\lambda,4t-2-4\lambda}, \lambda = 1, 2, 3 \ldots$ This will cover all the cases of $\theta_{1,\beta,\gamma}$ where β and γ are both congruent to 2 modulo 4. There are other possible replacements but they do not give elegant labelings of θ-graphs different from the ones mentioned here.

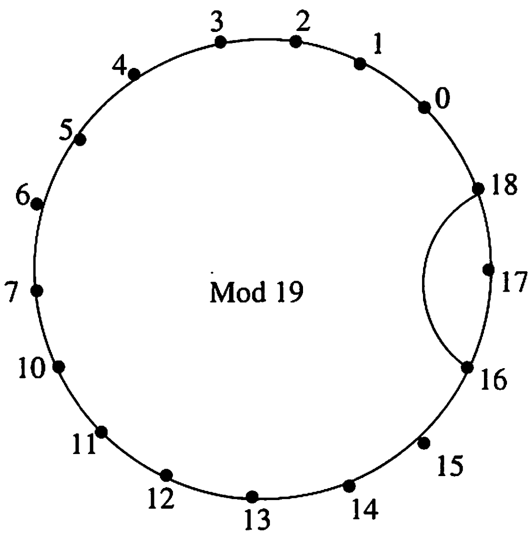
(b) Consider $\theta_{1,3,4t-3}$, that is $\alpha = 1, \beta = 3$ and $\gamma = 4t-3$.
Let the vertices be $\{a_0, b_0, a_1, b_1, \ldots, a_{2t-1}, b_{2t-1}\}$, connected in a cyclic manner. The extra edge being $a_{t+1}b_{t-1}$. Label the vertices as follows:
$\phi(a_i) = i+1, 0 \le i \le t-1$, $\phi(a_i) = 2t+1+i, t \le i \le 2t-1$,
$\phi(b_i) = 2t+1+i, 0 \le i \le t-1$, $\phi(b_i) = i+1, t \le i \le 2t-1$.
We have given here near-elegant labeling of $\theta_{1,3,17}$.



In the labeling given above, if we keep the labels unchanged but remove

the edge $e = a_{t+1}b_{t-1}$ and introduce a suitable new edge, we get elegant labeling of some other θ-graph. For example, if e is replaced by $a_\lambda b_{(2t-2-\lambda)}$, then we get an elegant labeling of $\theta_{1,3+4\lambda,4t-3-4\lambda}$, $0 \le \lambda \le t-2$. If we replace the edge e by the edge $b_{\lambda+1}a_{2t-1-\lambda}$, then we get elegant labeling of $\theta_{1,5+4\lambda,4t-5-4\lambda}$,
$0 \le \lambda \le t-2$. This shows that $\theta_{1,\beta,\gamma}$ is near-elegant when β and γ are both odd. □

Theorem 7: Let n be a positive integer, $n \equiv 1 \mod 4$. If $\theta_{1,\beta,\gamma}$ is a theta graph on n points then it is elegant.
Proof: First consider $\alpha = 1, \beta = 2$ and $\gamma = 4t-1$.



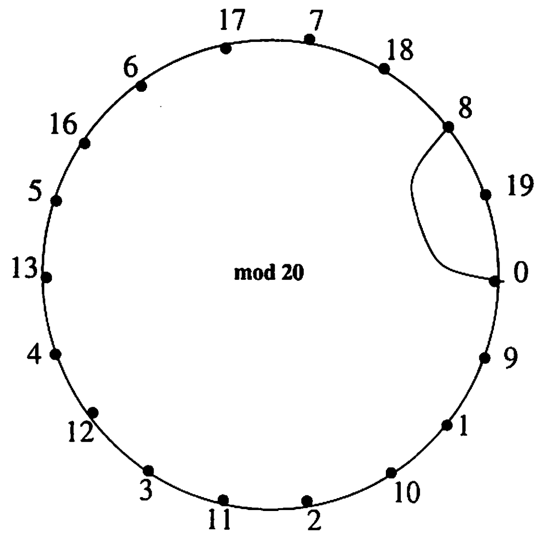
This graph has $4t+1$ vertices and $m = 4t+2$ edges. Thus $m+1 = 4t+3$. Let the vertices be $\{a_0, a_1, \ldots, a_{2t-1}, b_0, b_1, \ldots, b_{2t}\}$, connected in a cyclic manner. The extra edge being $b_{2t-2}b_{2t}$. Label the vertices as follows: $\phi(a_i) = i, 0 \le i \le 2t-1, \quad \phi(b_i) = 2t+2+i, 0 \le i \le 2t,$
We have given here elegant labeling of $\theta_{1,2,15}$.

It is easy to check that the edges $a_0a_1, a_1a_2, \ldots, a_{2t-2}a_{2t-1}$ and $a_{2t-1}b_0$ receive numbers $1, 3, \ldots, 4t-3$ and $4t+1$ sequentially. The edges $b_0b_1, b_1b_2, \ldots, b_{2t-1}b_{2t}, b_{2t}a_0$ receive values $2, 4, \ldots, 4t, 4t+2$ sequentially. Finally, one can check that $\phi(b_{2t-2}b_{2t}) = 4t-1$. This shows that the induced numbering ϕ on the edges takes all the values from the set $\{1, 2, \ldots, 4t+1\}$ and hence the graph $\theta_{1,2,4t-1}$ is elegant for all $t \ge 1$.

In the labeling given above, if we keep the labels unchanged but remove the edge $e = b_{2t-2}b_{2t}$ and introduce a suitable new edge to get elegant labeling of some other θ-graph. For example if e is replaced by $a_\lambda b_{(2t-3-\lambda)}$ then we get an elegant labeling of $\theta_{1,2\lambda+4,4t-3-2\lambda}, 0 \leq \lambda \leq 2t-3$. This will cover all the theta graphs $\theta_{1,\beta,\gamma}$ on $4t+1$ points. □

The proofs of the remaining theorems is similar and hence we give only the labelings of more θ-graphs without the proofs.

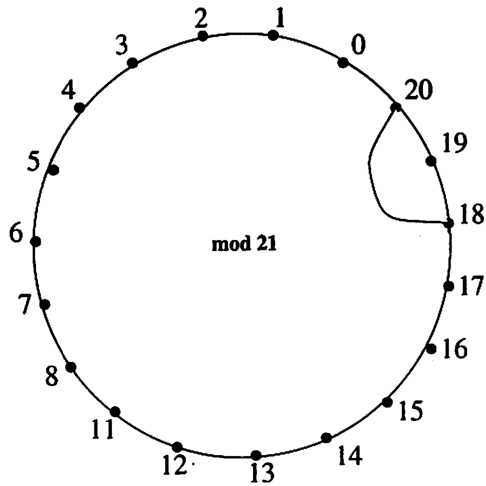
Theorem 8: Let n be a positive integer, $n \equiv 2 \bmod 4$ and let $\theta_{1,\beta,\gamma}$ be a theta graph on n points. If both β and γ are even then $\theta_{1,\beta,\gamma}$ is elegant.
Proof: First consider $\alpha = 1, \beta = .2$ and $\gamma = 4t$.



This graph has $4t+2$ vertices and $m = 4t+3$ edges. Thus $m+1 = 4t+4$. Let the vertices be $\{a_0, b_0, a_1, b_1, \ldots, a_{2t}, b_{2t}\}$, connected in a cyclic manner. The extra edge being a_0a_{2t}. Label the vertices as follows: $\phi(a_i) = i, 0 \leq i \leq 2t$, $\phi(b_i) = 2t+1+i, 0 \leq i \leq t$, $\phi(b_i) = 2t+3+i, t+1 \leq i \leq 2t$. We have given here elegant labeling of $\theta_{1,2,16}$.

In the labeling given above, if we keep the labels unchanged but remove the edge $e = a_0a_{2t}$ and introduce a suitable new edge to get elegant labeling of some other θ-graph. For example if e is replaced by $a_\lambda a_{(2t-\lambda)}$ or $b_\lambda b_{2t-\lambda}$ then we get an elegant labeling of $\theta_{1,2+4\lambda,4t-4\lambda}, 0 \leq \lambda \leq t-1$ for every $t \geq 1$. This will cover all $\theta_{1,\beta,\gamma}$ where β and γ are both even. □

Theorem 9: If $n \equiv 3 \bmod 4$ and if $\theta_{1,\beta,\gamma}$ is a theta graph on n points then it is elegant.
Proof: First consider $\alpha = 1, \beta = 2$ and $\gamma = 4t+1$.



This graph has $4t+3$ vertices and $m = 4t+4$ edges. Thus $m+1 = 4t+5$. Let the vertices be $\{a_0, a_1, \ldots, a_{2t}, b_0, b_1, \ldots, b_{2t+1}\}$, connected in a cyclic manner. The extra edge being $b_{2t-1}b_{2t+1}$. Label the vertices as follows: $\phi(a_i) = i, 0 \leq i \leq 2t, \phi(b_i) = 2t+3+i, 0 \leq i \leq 2t+1$. We have given here elegant labeling of $\theta_{1,2,17}$.

In the above labeling, if we keep the labels unchanged but remove the edge $e = b_{2t-1}b_{2t+1}$ and introduce a suitable new edge to get elegant labeling of some other θ-graph. For example if e is replaced by $a_\lambda b_{(2t-2-\lambda)}$ then we get an elegant labeling of $\theta_{1,4+2\lambda,4t-1-2\lambda}, 0 \leq \lambda \leq 2t-2$ for every $t \geq 1$. This will cover all $\theta_{1,\beta,\gamma}$ on $4t+3$ points. □

We give below elegant labelings of some of the theta graphs when $\alpha > 1$.

(1) $\alpha = 2, \beta = 3$ and $\gamma = 4t-4$.

This graph has $4t$ vertices and $m = 4t+1$ edges. Thus $m+1 = 4t+2$. Let the vertices be $\{a_0, b_0, a_1, b_1, \ldots, a_{2t-2}, b_{2t-2}\}$, connected in a cyclic manner. The extra path being $\{b_t, y, x, b_{t+1}\}$. Label the vertices as follows: $\phi(a_0) = 0$, $\phi(a_{i+1}) = 2t+i, 0 \leq i \leq t-1$, $\phi(a_{i+1}) = 2t+4+i, t \leq i \leq 2t-3$, $\phi(b_i) = 1+i, 0 \leq i \leq 2t-2$, $\phi(x) = 3t+2, \phi(y) = 3t$.

(2) $\alpha = 2, \beta = 3$ and $\gamma = 4t-2, t \geq 2$.

This graph has $4t+2$ vertices and $m = 4t+3$ edges. Thus $m+1 = 4t+4$. Let the vertices be $\{a_0, b_0, a_1, b_1, \ldots, a_{2t-1}, b_{2t-1}\}$, connected in a cyclic manner. The extra path being $\{b_{t+1}, y, x, b_t\}$. Label the vertices as follows: $\phi(a_0) = 0$, $\phi(a_i) = i, 1 \leq i \leq t+1$, $\phi(a_i) = i+1, t+2 \leq i \leq 2t-1$,

$\phi(b_i) = 2t + 1 + i, 0 \le i \le t$, $\phi(b_i) = 2t + i + 4, t + 1 \le i \le 2t - 1$,
$\phi(x) = 3t + 3, \phi(y) = t + 2$.

(3) $\alpha = 3, \beta = 4$ and $\gamma = 4t - 3, t \ge 2$.

This graph has $4t+3$ vertices and $m = 4t+4$ edges. Thus $m+1 = 4t+5$. Let the vertices be $\{a_0, a_1, \ldots, a_{2t-2}, b_0, b_1, \ldots, b_{2t+1}\}$, connected in a cyclic manner. The extra path being $\{a_3, y, x, b_{2t+1}\}$. Label the vertices as follows:
$\phi(b_0) = 0$, $\phi(a_i) = 2t + 6 + i, 0 \le i \le 2t - 2$,
$\phi(b_{2i-1}) = 3 + 2(i - 1) = 1 + 2i, 1 \le i \le t + 1$, $\phi(b_{2i}) = 2i, 1 \le i \le t$,
$\phi(x) = 2t + 4, \phi(y) = 2t + 2$.

(4) $\alpha = 3, \beta = 5$ and $\gamma = 4t - 6, t \ge 2$.

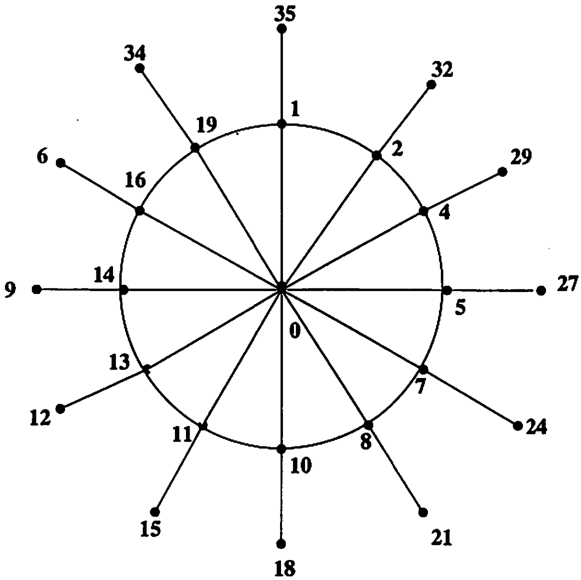
This graph has $4t+1$ vertices and $m = 4t+2$ edges. Thus $m+1 = 4t+3$. Let the vertices be $\{a_0, a_1, \ldots, a_{2t}, b_0, b_1, \ldots, b_{2t-3}\}$, connected in a cyclic manner. The extra path being $\{b_2, y, x, a_{2t-2}\}$. Label the vertices as follows:
$\phi(b_0) = 0$, $\phi(a_i) = 2t + 2 + i, 0 \le i \le 2t$,
$\phi(b_{2i-1}) = 3 + 2(i - 1) = 1 + 2i, 1 \le i \le t - 1$,
$\phi(b_{2i}) = 6 + 2(i - 1) = 2i + 4, 1 \le i \le t - 2$, $\phi(x) = 4, \phi(y) = 1$.

5 HARMONIOUS LABELINGS OF HELMS

In 1993, Bolian and Xiankun [1], conjectured that helms are harmonious and they proved the conjecture for the helms H_t when t is odd. We prove the conjecture for all even values of t.

A **wheel** on $n + 1$ points is obtained by taking a cycle on n points and making all these points adjacent to an extra point at the center of the cycle. The helm H_t is obtained by taking a wheel on $t + 1$ points and attaching a pendant vertex to each of the t vertices on the cycle. Clearly H_t has $2t + 1$ vertices and $3t$ edges.

Theorem 10: The helm H_t is harmonious for every even integer t.
Proof: Let $t \equiv 0, 2 \bmod 4, t \ge 4$. Let us first construct the even cycle of H_t.. Take vertices $\{a_0, b_0, \ldots, a_{(t-2)/2}, b_{(t-2)/2}\}$ and connect them in cyclic manner. Take the central vertex to be c and join it to all the $a_i, b_i, 0 \le i \le (t - 2)/2$. The pendant vertex attached to a_i is x_i and that attached to b_i is $y_i, 0 \le i \le (t - 2)/2$. We give here harmonious labeling of H_{12}.



Label the vertices as follows: $\phi(c) = 0$, $\phi(a_i) = 1 + 3i, 0 \le i \le \frac{t-2}{2}$, $\phi(b_i) = \frac{3t+2}{2}, i = \frac{t-2}{2}$ $\phi(b_i) = 2 + 3i, 0 \le i \le \frac{t-4}{2}$, $\phi(b_{(t-2)/2}) = \frac{3t+2}{2}$.
This covers labeling of the central vertex and the vertices on the cycle. Next label the pendant vertices as follows:
$\phi(x_0) = 3t - 1$,
$\phi(y_0) = 3t - 4$, $\phi(x_1) = 3t - 7$, $\phi(y_1) = 3t - 9$,
$\phi(x_{i+1}) = 3t - 6 - 6i$, $1 \le i \le \frac{t-4}{2}$,
$\phi(y_{i+1}) = 3t - 9 - 6i, 1 \le i \le \frac{t-6}{2}$, $\phi(y_{(t-2)/2}) = 3t - 2$.

It is easy to check that the edges $ca_0, ca_1, \ldots, ca_{(t-2)/2}$ receive values $1, 4, 7, \ldots, \frac{3t}{2} - 2$ in a sequential manner. The edges $cb_0, cb_1, \ldots, cb_{(t-4)/2}$ receive values $2, 5, 8, \ldots, \frac{3t}{2} - 4$ in a sequential manner. The edge $cb_{(t-2)/2}$ receives value $\frac{3t}{2} + 1$. The cyclic edges $a_0b_0, b_0a_1, \ldots, b_{(t-4)/2}a_{(t-2)/2}$ receive values $3, 6, \ldots, 3t-6$ in a sequential manner. The edges $a_{(t-2)/2}b_{(t-2)/2}$ and $b_{(t-2)/2}a_0$ receive values $3t - 1, \frac{3t}{2} + 2$ respectively. Finally, one can see that
$\phi(a_0x_0) = 0, \phi(b_0y_0) = 3t - 2, \phi(a_1x_1) = 3t - 3$,
$\phi(a_2x_2) = 3t - 5, \phi(b_1y_1) = 3t - 4$,

$\phi(b_{(t-2)/2}y_{(t-2)/2}) = \frac{3t}{2} - 1,$
$\phi(a_{i+2}x_{i+2}) = 3t - 5 - 3i, 1 \leq i \leq \frac{t-6}{2},$
$\phi(b_{i+1}y_{i+1}) = 3t - 4 - 3i, 1 \leq i \leq \frac{t-6}{2}.$
It can be easily checked that the numbers less than $3t - 6$, not covered by the earlier three sequences are covered by the last two sets. Hence, H_t is harmonious for all the even values of t. □

We are thankful to the refree for the valuable suggestions.

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