Balanced Bipartite Row-Column Designs

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ABSTRACT. This paper deals with combinatorial aspects of designs for two-way elimination of heterogeneity for making all possible paired comparisons of treatments belonging to two disjoint sets of treatments. Balanced bipartite row-column (BBPRC) designs have been defined which estimate all the elementary contrasts involving two treatments one from each of the two disjoint sets with the same variance. General efficiency balanced row-column designs (GEBRC) are also defined. Some general methods of construction of BBPRC designs have been given using the techniques of reinforcement, deletion (addition) of column or row structures, merging of treatments, balanced bipartite block (BBPB) designs, juxtaposition, etc. Some methods of construction give GEBRC designs also.

1 Introduction

Consider a two-way heterogeneity setting where $1, \ldots, v_1$ test treatments (hereafter called tests) belonging to a set T of cardinality v_1 are to be compared with $v_1 + 1, \ldots, v_1 + v_2 = v$ control treatments (hereafter called controls) belonging to a set U of cardinality v_2; $T \cap U = \Phi$, via n experimental units arranged in R rows and C columns, $n = RC$. Suppose that n_{ijk} denotes the number of times the ith treatment is applied in the jth row and the kth column. Here we assume that $n_{ijk} = 0$ or 1; $1 \leq i \leq v$, $1 \leq j \leq R$, $1 \leq k \leq C$.

Let $\mathbf{L}_{v \times R} = [\mathbf{L}_1' : \mathbf{L}_2']' = ((n_{ij.}))$ and $\mathbf{M}_{v \times C} = [\mathbf{M}_1' : \mathbf{M}_2']' = ((n_{i.k}))$ be the $v \times R$ treatments vs rows and $v \times C$ treatments vs columns incidence matrices respectively, and $n_{ij.} = \sum_{k=1}^{C} n_{ijk}; n_{i.k} = \sum_{j=1}^{R} n_{ijk}$. $\mathbf{L}_1$ ($\mathbf{M}_1$) is tests vs rows (columns) incidence matrix and $\mathbf{L}_2$ ($\mathbf{M}_2$) is the controls vs rows (columns) incidence matrix. Under the usual three-way classified, additive, linear, fixed effects, homoscedastic model and further assuming that from each of the RC cells one observation is obtained, the coefficient matrix of reduced normal equations for estimating the treatment effects is given by

$$\mathbf{F} = \begin{bmatrix} \mathbf{R}_1 - \frac{1}{C}\mathbf{L}_1\mathbf{L}_1' - \frac{1}{R}\mathbf{M}_1\mathbf{M}_1' + \frac{\mathbf{r}_1\mathbf{r}_1'}{RC} & -\frac{1}{C}\mathbf{L}_1\mathbf{L}_2' - \frac{1}{R}\mathbf{M}_1\mathbf{M}_2' + \frac{\mathbf{r}_1\mathbf{r}_2'}{RC} \\ -\frac{1}{C}\mathbf{L}_2\mathbf{L}_1' - \frac{1}{R}\mathbf{M}_2\mathbf{M}_1' + \frac{\mathbf{r}_2\mathbf{r}_1'}{RC} & \mathbf{R}_2 - \frac{1}{C}\mathbf{L}_2\mathbf{L}_2' - \frac{1}{R}\mathbf{M}_2\mathbf{M}_2' + \frac{\mathbf{r}_2\mathbf{r}_2'}{RC} \end{bmatrix} \quad (1.1)$$

where $\mathbf{r}_1' = (r_1, \ldots, r_{v_1})$ is the $1 \times v_1$ vector of replications of tests, $\mathbf{r}_2' = (r_{v_1+1}, \ldots, r_v)$ is the $1 \times v_2$ vector of replications of controls, $\mathbf{R}_1 = \text{diag}(r_1, \ldots, r_{v_1})$ and $\mathbf{R}_2 = \text{diag}(r_{v_1+1}, \ldots, r_v)$. The matrix $\mathbf{F}$ is non negative definite with zero row (column) sums and for a connected design, $\text{Rank}(\mathbf{F}) = v - 1$. Henceforth, we shall consider only connected designs. Further define

$$\begin{aligned} s_{tt'} &= R\lambda_{rtt'} + C\lambda_{ctt'} - r_t r_{t'}, & \forall\, 1 \le t \ne t' \le v_1 \\ s_{tu} = s_{ut} &= R\lambda_{rtu} + C\lambda_{ctu} - r_t r_u, & \forall\, 1 \le t \le v_1; v_1 + 1 \le u \le v \\ s_{uu'} &= R\lambda_{ruu'} + C\lambda_{cuu'} - r_u r_{u'}, & \forall\, v_1 + 1 \le u \ne u' \le v \end{aligned}$$

Here, $\lambda_{rtt'}$ ($\lambda_{ruu'}$), $\lambda_{ctt'}$ ($\lambda_{cuu'}$) are respectively the number of concurrences of t, $t' \in T$ ($u, u' \in U$) in rows and columns of the design; λ_{rtu} (λ_{ctu}) is the number of concurrences of treatment $t \in T$ and treatment $u \in U$ in rows (columns) of the design, r_t (r_u) is the replication number of treatment $t \in T$ ($u \in U$).

The treatment contrasts of interest are of the form $\tau_t - \tau_u$, where τ_t and τ_u respectively represent the effect of the treatments $t \in T$ and $u \in U$. These contrasts can be written in the form $\mathbf{P}'\tau$, where

$$\mathbf{P}' = [\mathbf{1}_{v_1} \otimes \mathbf{I}_{v_2} : \; -\mathbf{I}_{v_1} \otimes \mathbf{1}_{v_2}] \quad (1.2)$$

is a $v_1 v_2 \times v$ matrix and τ is a v component vector of the effects of tests and controls given by $\tau = (\tau_1, \tau_{v_1+1}, \cdots +, \tau_v)'$. Here $\mathbf{I}_n$ is an identity matrix of order n, $\mathbf{1}_n$ is a $n \times 1$ vector of ones, and $\otimes$ denotes kronecker product. The problem of obtaining suitable designs for making comparisons of the type $\tau_t - \tau_u$ in row-column setting has been considered by Freeman (1972, 1975). Freeman introduced type B (bipartite) row-column designs as an extension of supplemented balanced (type S) block designs of Pearce (1960). These designs estimate all elementary contrasts of treatments belonging to pth set of treatments ($p = 1, 2$) with same variance $\sigma^2 V_p$ (say), and all elementary contrasts of treatments from two different sets with variance $\sigma^2 V_{12} = \sigma^2 V_{21} = \sigma^2 V_3$ (say), where σ^2 is the common variance of the observations.

In general block design set up, construction and analysis of such designs has been studied by Nair and Rao (1942), Corsten (1962), Adhikary (1965a, 1965b), Federer and Raghavarao (1975) and Kageyama and Sinha

(1988). Federer and Raghavarao (1975) have considered the row-column settings as well, but the designs allow only single replication of tests. In this investigation, we give the methods of construction of these designs for the situations where we can afford more than one replication of the tests. Keeping in conformity with the definition of balanced bipartite block designs of Kageyama and Sinha (1988), we shall call such designs as balanced bipartite row-column (BBPRC) designs. The BBPRC designs are defined below:

Definition 1.1. *An arrangement of $v(= v_1 + v_2)$ treatments in R rows and C columns is said to be a BBPRC design if*

$$\begin{array}{lll} a)\ s_{tt'} = s_1 \text{ (constant)} & \forall\, t \neq t' = 1,\ldots,v_1, & \\ b)\ s_{uu'} = s_2 \text{ (constant)} & \forall\, u \neq u' = v_1+1,\ldots,v, & (1.3) \\ c)\ s_{tu} = s_{ut} = s_0 > 0 \text{ (constant)} & \forall\, t = 1,\ldots,v_1,\ u = v_1+1,\ldots,v. & \end{array}$$

The BBPRC design as defined above is also called a general efficiency balanced row-column (GEBRC) design for comparing two disjoint sets of treatments if $s_0^2 = s_1 s_2$. This is an extension of GEB block designs of Das and Ghosh (1985) and Kageyama and Mukerjee (1986).

The information matrix as in (1.1) for a BBPRC design is given by

$$\mathbf{F} = \frac{1}{RC}\begin{bmatrix} (v_1s_1 + v_2s_0)\mathbf{I}_{v_1} - s_1\mathbf{1}_{v_1}\mathbf{1}'_{v_1} & -s_0\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ -s_0\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & (v_2s_2 + v_1s_0)\mathbf{I}_{v_2} - s_2\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix} \quad (1.4)$$

Here $\sigma^2 V_1 = \sigma^2\{2RC/(v_1s_1 + v_2s_0)\}$, $\sigma^2 V_2 = \sigma^2\{2RC/(v_2s_2 + v_1s_0)\}$ and

$$\begin{aligned} \sigma^2 V_{12} = \sigma^2 V_{21} = \sigma^2 V_3 &= \sigma^2\left\{RC\left[\frac{v_2s_0 + s_1}{v_2s_0(v_1s_1 + v_2s_0)} + \frac{v_2 - 1}{v_2(v_2s_2 + v_1s_0)}\right]\right\} \\ &= \sigma^2\left\{RC\left[\frac{v_1s_0 + s_2}{v_1s_0(v_2s_2 + v_1s_0)} + \frac{v_1 - 1}{v_1(v_1s_1 + v_2s_0)}\right]\right\} \quad (1.5) \end{aligned}$$

Remark 1.1. For a variance balanced design in a two-way heterogeneity setting we have $s_0 = s_1 = s_2 = s$ (say) and $\mathbf{F} = \theta\ (\mathbf{I}_v - \mathbf{1}_v\mathbf{1}'_v/v)$, where $\theta = sv/RC$ is the unique positive eigenvalue of $\mathbf{F}$ with multiplicity $v - 1$.

For $v_2 = 1$, all the BBPRC designs defined above reduce to balanced treatment row-column designs given by Ture (1994) and generalized efficiency balanced row-column (GEBRC) designs of Gupta and Prasad (1990). Notz (1985) and Ture (1994) considered the A-optimality aspects of row-column designs for making test treatments-control comparisons. Majumdar (1986) has considered the optimality aspects of row-column designs for comparing two disjoint sets of treatments. Mandeli (1991) has shown that A-optimal row-column designs suggested by Notz (1985) and Majumdar (1986) are particular cases of F-squares. Although A-optimality aspects of these designs can be considered by minimization of trace $(\mathbf{P}'\mathbf{C}^-\mathbf{P})$ over a class of competing designs, we give in the sequel only the methods of construction of BBPRC designs which are the candidate designs for studying the A-optimality aspects of row-column designs for comparing two disjoint sets of treatments.

2 Construction of BBPRC designs

The purpose of this section is to give some general methods of construction of BBPRC designs through the technique of deletion (addition) of column structures in Youden Squares, reinforcement of balanced row-column designs [like Generalized Youden Designs (GYD), Pseudo Youden Designs (PYD), Latin Square Designs (LSD), Youden Square Designs (YSD), etc.], merging of treatments in a balanced row-column design, mutually orthogonal latin squares and juxtaposition. We shall use definition 1.1 and expression (1.4) to get BBPRC designs with parameters as v_1, v_2, R, C, r_1, r_2, s_1, s_0, s_2, where r_1 (r_2) is the replication number of tests (controls).

Method 2.1: Consider a YSD $D(v, R = k, C = b, r, \lambda)$. Without any loss of generality, assume that the last column has treatment labels $1, \ldots, k$. The abridged YSD obtained by deleting one column from D is a BBPRC design with parameters $v_1 = k$, $v_2 = v - k$, $R^* = k$, $C^* = b - 1$, $r_1 = r - 1$, $r_2 = r$, $s_1 = k(r-2)+(b-1)(\lambda-1)-(r-1)^2$, $s_0 = k(r-1)+(b-1)\lambda-r(r-1)$, $s_2 = rk + (b-1)\lambda - r^2$, and information matrix as

$$\mathbf{F}^* = \frac{1}{k(b-1)} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{A}_{v_1 \times v_1} &= [k(b-1)(r-1) - k - (b-1)(r-\lambda)]\mathbf{I}_{v_1} \\ &\quad - [k(r-2) + (b-1)(\lambda-1) - (r-1)^2]\mathbf{1}_{v_1}\mathbf{1}'_{v_1} \\ \mathbf{B}_{v_1 \times v_2} &= -[k(r-1) + (b-1)\lambda - r(r-1)]\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ \mathbf{D}_{v_2 \times v_2} &= [rk(b-1) - (b-1)(r-\lambda)]\mathbf{I}_{v_2} - [kr + (b-1)\lambda - r^2]\mathbf{1}_{v_2}\mathbf{1}'_{v_2}. \end{aligned}$$

Method 2.2: Consider a YSD $D(v, R = k, C = b, r, \lambda)$. Extend the columns of the YSD by one column. Without any loss of generality assume that the added column contains treatment labels $v - k + 1, \ldots, v$. The extended YSD is a BBPRC design with parameters $v_1^* = v - k$, $v_2 = k$, $R^* = k$, $C^* = b + 1$, $r_1 = r$, $r_2 = r + 1$, $s_1 = kr + (b+1)\lambda - r^2$, $s_0 = k(r+1)+(b+1)\lambda-r(r+1)$ and $s_2 = k(r+2)+(b+1)(\lambda+1)-(r+1)^2$, and has information matrix as

$$\mathbf{F}^* = \frac{1}{k(b+1)} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{bmatrix},$$

where

$$\begin{aligned}
\mathbf{A} &= [k(b+1)r-(b+1)(r-\lambda)]\mathbf{I}_{v_1} - [kr+(b+1)\lambda - r^2]\mathbf{1}_{v_1}\mathbf{1}'_{v_1} \\
\mathbf{B} &= -[k(r+1)+(b+1)\lambda - r(r+1)]\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\
\mathbf{D} &= [k(b+1)(r+1)-k-(b+1)(r-\lambda)]\mathbf{I}_{v_2} \\
&\quad - [k(r+2)+(b+1)(\lambda+1)-(r+1)^2]\mathbf{1}_{v_2}\mathbf{1}'_{v_2}.
\end{aligned}$$

The above procedure has been given by Jacroux (1982) in Theorem 3.2 for obtaining E-optimal row-column designs for making all possible paired comparisons among v treatments. It is also type 9 design of Freeman (1975).

Method 2.3: Suppose that a variance balanced row-column design with parameters $v+\alpha\beta$, R, C, $\mathbf{r}' = (\mathbf{r}'_1, \mathbf{r}'_2)$; $\mathbf{r}'_1 = (r_1, \ldots, r_v)$; $\mathbf{r}'_2 = (r_{v+1}, \ldots, r_{v+\alpha\beta})$, $s = \frac{RC\theta}{v+\alpha\beta}$, where α, β are any unique positive integers and θ is the unique non-zero eigenvalue of the information matrix of the variance balanced row-column design, exists. Then by merging $v+1, \ldots, v+\alpha$ treatments as the $(v+1)$th treatment, $v+\alpha+1, \ldots, v+2\alpha$ treatments as the $(v+2)$th treatment, and so on, and $v+\alpha(\beta-1)+1, \ldots, v+\alpha\beta$ treatments as the $(v+\beta)$th treatment, we get a BBPRC design with parameters $v_1 = v$, $v_2 = \beta$, $R^* = R$, $C^* = C$, $\mathbf{r}_1^* = \mathbf{r}_1$, $\mathbf{r}_2^* = ((r_{v+1} + \cdots + r_{v+\alpha}), \ldots, (r_{v+\alpha(\beta-1)+1} + \cdots + r_{v+\alpha\beta}))'$, $s_1 = \frac{RC\theta}{v+\alpha\beta}$, $s_0 = \alpha s_1$, $s_2 = \alpha^2 s_1$.

Remark 2.3.1: Consider a Latin square design as a variance balanced row-column design, with parameters $v+\alpha\beta$ $(v = pq)$, R, C, r, s, where p, q, α, β satisfy some conditions specified in corollary 4.1 of Majumdar (1986). Merge the first pq treatments to q treatments as follows: $1, \ldots, p$ treatments to 1st treatment, $p+1, \ldots, 2p$ treatments to 2nd treatment, and so on, and $p(q-1)+1, \ldots, pq$ treatments to qth treatment. Further merge $\alpha\beta$ treatments to β treatments as in Method 2.3. The resulting design is an A-optimal design for comparing two disjoint sets of treatments over $D(q, \beta, pq+\alpha\beta, pq+\alpha\beta)$, the class of connected row-column designs in which q test treatments and β control treatments are arranged in $pq+\alpha\beta$ rows and $pq+\alpha\beta$ columns. Further, for $\beta = 1$, and a latin square of order $v+\alpha$, in the above method such that $\alpha = \sqrt{v}$, and merging $v+1, \ldots, v+\alpha$ treatments to $(v+1)$th treatment, called control treatment, we get a Balanced Treatment Row-Column Design (BTRC Design) which is A-optimal for comparing several test treatments with a control treatment over $D(\alpha^2, 1, \alpha^2+\alpha, \alpha^2+\alpha)$, the class of connected designs in which α^2 test treatments and a control are arranged in $\alpha^2+\alpha$ rows and $\alpha^2+\alpha$ columns as a corollary 2.1 of Notz (1985).

Remark 2.3.2: The BBPRC designs obtained by method 2.3 are also general efficiency balanced row column designs for comparing two disjoint sets of treatments.

Remark 2.3.3: Consider a BBPRC design with parameters $v_1 = pq$ and $v_2 = \alpha\beta$, $R = C$, $\mathbf{r}_1$, $\mathbf{r}_2$, s_1, s_0, s_2.. Following the procedure of merging of treatments as in method 2.3, merge v_1 treatments to p treatments and v_2 treatments to α treatments. The resulting design is a BBPRC design with parameters $v_1^* = p$, $v_2^* = \alpha$, R, C, $q\mathrm{r}_1$, $\beta\mathrm{r}_2$, qs_1, $q\alpha s_0$, αs_2.

Method 2.4: Consider a row-column design which is balanced with respect to row classification as well as column classification separately in v_1 treatments arranged in p rows and q columns such that each treatment is replicated r times and number of concurrences of any two treatments in rows (columns) is λ_r (λ_c) (e.g., a latin square design, a Youden Square design and a Generalized Youden Design).

Reinforce the row-column design to get another row-column design in $p + v_2$ rows and $q + v_2$ columns with v_2 control treatments as follows:

i) the symbols in the ith row and $(q+j)$th column for $i = 1, \ldots, p$ and $j = 1, \ldots, v_2$ will have each of control treatments exactly once in each of the p-rows in a systematic order viz. $1, 2, \ldots, v_2$,

ii) Similarly the symbols in the $(p+i)$th row and jth column for $i = 1, \ldots, v_2$; $j = 1, \ldots, q$ will have each of v_2 control treatments exactly once in each column in a systematic order viz. $1, 2, \ldots, v_2$.

(iii) The arrangement of v_2 controls in the $(p+i)$th row and $(q+j)$th column for i, $j = 1, \ldots, v_2$ will form a latin square of order v_2. For the given row column design let

$$\mathbf{LL}' = (a - \lambda_r)\mathbf{I}_{v_1} + \lambda_r \mathbf{1}_{v_1}\mathbf{1}'_{v_1} \quad \mathbf{MM}' = (b - \lambda_c)\mathbf{I}_{v_1} + \lambda_c \mathbf{1}_{v_1}\mathbf{1}'_{v_1},$$

where $a(b) = \sum_{j=1}^{R} n_{ij.}^2 \left(\sum_{k=1}^{C} n_{i.k}^2\right)$ $\forall\ i = 1, \ldots, v_1$, $j = 1, \ldots, p$, $k = 1, \ldots, q$.

The corresponding matrices for the resulting row-column design are

$$\mathbf{L}^*\mathbf{L}^{*\prime} = \begin{bmatrix} \mathbf{LL}' & r\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ r\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & q^2\mathbf{I} + (2q + p + v_2)\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

$$= \begin{bmatrix} (a - \lambda_r)\mathbf{I} + \lambda_r \mathbf{1}_{v_1}\mathbf{1}'_{v_2} & r\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ r\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & q^2\mathbf{I} + (2q + p + v_2)\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

$$\mathbf{M}^*\mathbf{M}^{*\prime} = \begin{bmatrix} (b - \lambda_c)\mathbf{I} + \lambda_c \mathbf{1}_{v_1}\mathbf{1}'_{v_1} & r\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ r\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & p^2\mathbf{I} + (2p + q + v_2)\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

and

$$\mathbf{R}^* = \begin{bmatrix} r\mathbf{I}_{v_1} & \mathbf{0} \\ \mathbf{0} & (p + q + v_2)\mathbf{I}_{v_2} \end{bmatrix};$$

$$\mathbf{r}^*\mathbf{r}^{*\prime} = \begin{bmatrix} r^2\mathbf{1}_{v_1}\mathbf{1}'_{v_1} & r(p + q + v_2)\mathbf{1}_{v_1}\mathbf{1}'_{v_2} \\ r(p + q + v_2)\mathbf{1}_{v_2}\mathbf{1}'_{v_1} & (p + q + v_2)^2\mathbf{1}_{v_2}\mathbf{1}'_{v_2} \end{bmatrix}$$

The information matrix is

$$\mathbf{F}^* = \frac{1}{(p+v_2)(q+v_2)}$$
$$\left[(p+v_2)(q+v_2)\mathbf{R}^* - (p+v_2)\mathbf{L}^*\mathbf{L}^{*'} - (q+v_2)\mathbf{M}^*\mathbf{M}^{*'} + \mathbf{r}^*\mathbf{r}^{*'}\right]$$
$$= \frac{1}{(p+v_2)(q+v_2)}\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}' & \mathbf{D}\end{bmatrix}$$

where

$$\mathbf{A} = [(p+v_2)(q+v_2)r - (p+v_2)(a-\lambda_r) - (q+v_2)(b-\lambda_c)]\,\mathbf{I}$$
$$- \left[(p+v_2)\lambda_r + (q+v_2)\lambda_c - r^2)\right]\mathbf{1}_{v_1}\mathbf{1}'_{v_1}$$
$$\mathbf{B} = -\left[\{(p+v_2)r + (q+v_2)r - r(p+q+v_2)\}\mathbf{1}_{v_1}\mathbf{1}'_{v_2}\right]$$
$$= -rv_2\mathbf{1}_{v_1}\mathbf{1}'_{v_2}$$
$$\mathbf{D} = \left[(p+v_2)(q+v_2)(p+q+v_2) - (p+v_2)q^2 - (q+v_2)p^2\right]\mathbf{I}_{v_2}$$
$$- \left[(p+v_2)(2q+p+v_2) + (q+v_2)(2p+q+v_2) - (p+q+v_2)^2\right]\mathbf{1}_{v_2}\mathbf{1}'_{v_2}$$

which is information matrix of a BBPRC design with parameters v_1, v_2, $R = p+v_2$, $C = q+v_2$, $r_1 = r$, $r_2 = (p+q+v_2)$, $s_1 = (p+v_2)\lambda_r+(q+v_2)\lambda_c-r^2$, $s_0 = rv_2$; $s_2 = (p+v_2)(2q+p+v_2) + (q+v_2)(2p+q+v_2) - (p+q+v_2)^2$.

It may be noted that the designs constructed using this method may not perform well from an efficiency perspective as these are not generalized binary, neither with respect to rows nor with respect to columns.

Remark 2.4.1: When the design considered is a Latin square (Youden square) design then the parameters of BBPRC design are v_1, v_2, $R = C = v_1 + v_2$, $r_1 = v_1$, $r_2 = 2v_1 + v_2$, $s_1 = (v_1 + 2v_2)v_1$, $s_0 = v_1v_2$, $s_2 = 2(v_1 + v_2)(3v_1 + v_2) - (2v_1 + v_2)^2$; $[v_1, v_2, R = p + v_2, C = v_1 + v_2, r_1 = p, r_2 = p + v_1 + v_2, s_1 = p(p + v_2) + (v_1 + v_2)\lambda_c - p^2]$; $s_0 = pv_2$, $s_2 = (p+v_2)(2v_1+p+v_2) + (v_1+v_2)(2p+v_1+v_2) - (p+v_1+v_2)^2$.

Remark 2.4.2: When the design considered is a latin square design of order v_1 and we add $v_2 \times v_2$ latin square in the diagonally opposite corner and two $(v_1 \times v_2)$ Youden squares on v_1 treatments, this method yields designs of Type 17 of Freeman (1975).

Remark 2.4.3: In above method we have considered a row-column design which is balanced with respect to rows as well as columns classifications, but when such a design does not exist, then we can make use of a Pseudo Youden Design in v_1 treatments arranged in p rows and p columns such that each of the treatments is equally replicated and the sum of the concurrences of the treatments in rows (λ_r) and in columns (λ_c) i.e., $\lambda_r + \lambda_c$ is constant say λ. Then following the procedure of method 2.4, we get a BBPRC design with parameters v_1, v_2, $R = p + v_2 = C$, $r_1 = r$, $r_2 = (2p + v_2)$, $s_1 = (p+v_2)\lambda - r^2$, $s_0 = rv_2$ and $s_2 = 2(p+v_2)(3p+v_2)^2 - (2p+v_2)^2$.

Remark 2.4.4: For $v_2 = 1$ and row-column design to be YSD or GYD design, the method reduces to the method 1 of construction of GEBRC designs of Gupta and Prasad (1990) and for $v_2 = 1$ and row-column design to be PYD, it reduces to method 3 of Gupta and Prasad (1990).

Method 2.5: Consider a balanced bipartite block (BBPB) design with parameters v_1, v_2, b, r_1, r_2, k, λ_1, λ_0 and λ_2 such that $k \mid r_1$ and $k \mid r_2$, where $x \mid y$ means x divides y. Arrange the contents of each of the blocks of BBPB design in the form of $k \times b$ array, such that each of the v_1 treatments occur $r_1 k^{-1}$ times in each of the rows of the array and each of the v_2 control treatments occur $r_2 k^{-1}$ times in each of the rows of the array. We get a BBPRC design with parameters v_1, v_2, $R = k$, $C = b$, r_1, r_2, $s_1 = bk\lambda_1$, $s_0 = bk\lambda_0$, $s_2 = bk\lambda_2$.

Various methods of construction of BBPB designs are available in literature see e.g. Kageyama and Sinha (1988), Sinha and Kageyama (1990), Jaggi, Gupta and Parsad (1996). A BBPB design obtainable from the above methods satisfying the conditions of method 2.5 can be used to obtain BBPBRC designs. For $v_2 = 1$, this method reduces to the method of Gupta and Prasad (1990) for constructing GEBRC design.

It is interesting to note here that the above method is based on the result of Agarwal (1966) of SDR's in which it was proved that in an equireplicate, binary design with v treatments, b blocks and constant block size $k < v$ such that $b = mv$ i.e. $r = mk$ for some interger $m(\geq 1)$, the treatments can be rearranged in the blocks in such a way that when blocks are written as columns each row in the arrangement contains every treatment m times. This result was generalised for unequireplicate nonbinary block designs by Das (1989). One new method of construction of BBPB designs is given below which can be used for the construction of BBPBRC designs.

Remark 2.5.1: Consider an extended group divisible (EGD) design with parameters $v_1 = mn$, b_1, r_1, k_1, $\lambda_{11} = a$, $\lambda_{12} = a + p$, $\lambda_{13} = a + q + p$. Considering the rows of the association scheme as blocks and taking $(p + q)$ copies of these blocks we get another EGD design with parameters as $v_1 = mn$, $b_2 = m(p + q)$, $r_2 = (p + q)$, $k_2 = n$, $\lambda_{21} = p + q$, $\lambda_{22} = 0$, $\lambda_{23} = 0$. Similarly considering columns of the association scheme as blocks and taking q copies of these blocks we get the third EGD design with parameters $v_1 = mn$, $b_3 = nq$, $r_3 = q$, $k_3 = m$, $\lambda_{31} = 0$, $\lambda_{32} = q$, $\lambda_{33} = 0$.

Suppose that for non-negative integers i_1, i_2, i_3, v_2 and v_3, $k_1 + i_1 v_2 = n + i_2 v_2 = m + i_3 v_3 = k$, $r_1 + p + 2q$ and $i_1 b_1 + i_2 m(p + q) + i_3 nq$ is divisible by k. Take the union of all the blocks of the three EGD designs to get $b_1 + m(p + q) + nq$ blocks. Add v_2 control treatments to each of the b_1 blocks i_1 times, to each of the $m(p + q)$ blocks i_2 times each and to each of the nq blocks i_3 times each. Now arrange the contents of these blocks in the form of a $k \times \{(b_1 + m(p + q) + nq)\}$ array such that each row of the array contains each of the v_1 treatments $(r_1 + p + 2q)k^{-1}$ times and each of the v_2 treatments $[i_1b_1 + i_2m(p+q) + i_3nq]k^{-1}$ times. We get a BBPRC design with parameters $v_1 = mn$, v_2, $R = k$, $C = b_1+m(p+q)+nq$, $r_1^* = r_1+p+2q$, $r_2^* = i_1b_1+i_2m(p+q)+i_3nq$, $s_1 = k(b_1+m(p+q)+nq)\lambda_{13}$, $s_0 = [i_1r_1 + i_2(p+q) + i_3q]RC$; $s_2 = [i_1b_1 + i_2m(p+q) + i_3nq]RC$.

Remark 2.5.2: If we consider a general efficiency balanced block design for comparing two disjoint sets of treatments and follow the procedure of method 2.5, we get a GEBRC design.

Note: Following the procedure of the above method, we can get partially balanced row-column designs. Consider a partially balanced incomplete block design with α-associate classes and parameters v, b, r, k, λ_i ($i = 1,\ldots,\alpha$) (for notations, see Dey (1986)) such that r divides k. Arrange the contents of blocks of m-associate class PBIB design in the form of a $k \times b$ array such that each of v treatments occur rk^{-1} times in each of the row of the array. Then, we get a row-column design with parameters v, $R = k$, $C = b$, r, λ_i ($i = 1,\ldots,\alpha$). For $\alpha = 1$, we get a variance balanced row-column design. For $\alpha = 2$, we get a two-associate class partially balanced row-column design. For $v = mn$ and $\alpha = 2$, and PBIB design based on GD association scheme we get a group divisible row-column design.

If in a GD row-column design, $v = 2n$, then it is also considered as a BBPRC design with parameters $v_1 = n$, $v_2 = n$, $R = k$, $C = b$, $r_1 = r$, $r_2 = r$, $s_1 = \lambda_1$, $s_0 = \lambda_2$, $s_2 = \lambda_1$.

Method 2.6: Consider a YSD (v, p, v) such that the number of concurrences of any two treatments in v columns is λ. Then append it by a row regular GYD with p rows and $a\alpha$ columns based on the first α treatments, where a is the number of times each of the α treatments occurs in each of the p rows of the GYD. Then corresponding matrices are given by $\mathbf{L}^* = \begin{bmatrix} \mathbf{1}_{v-\alpha}\mathbf{1}'_{v-\alpha} \\ (a+1)\mathbf{1}_\alpha\mathbf{1}'_p \end{bmatrix}$, $\mathbf{M}^* = \begin{bmatrix} \mathbf{M}_{1(v-\alpha)\times q} & \mathbf{0} \\ \mathbf{M}_{2(\alpha\times v)} & \mathbf{M}_3 \end{bmatrix}$, where $\mathbf{M}_1$ and $\mathbf{M}_2$ are bifurcation of treatments vs columns incidence matrix of the YSD and $\mathbf{M}_3$ is the corresponding matrix for the GYD design such that diagonal (off diagonal) elements of $\mathbf{M}_3\mathbf{M}_3'$ are $\delta(\eta)$. Hence

$$\mathbf{L}^*\mathbf{L}^{*\prime} = \begin{bmatrix} p\mathbf{1}_{v-\alpha}\mathbf{1}'_{v-\alpha} & p(a+1)\mathbf{1}_{v-\alpha}\mathbf{1}'_{\alpha} \\ p(a+1)\mathbf{1}_{\alpha}\mathbf{1}'_{v-\alpha} & p(a+1)^2\mathbf{1}_{\alpha}\mathbf{1}'_{\alpha} \end{bmatrix}$$

$$\mathbf{M}^*\mathbf{M}^{*\prime} = \begin{bmatrix} \mathbf{M}_1\mathbf{M}'_1 & \mathbf{M}_1\mathbf{M}'_2 \\ \mathbf{M}_2\mathbf{M}'_1 & \mathbf{M}_2\mathbf{M}'_2 + \mathbf{M}_3\mathbf{M}'_3 \end{bmatrix}$$

$$\mathbf{M}^*\mathbf{M}^{*\prime} = \begin{bmatrix} (\alpha\beta-\lambda)\mathbf{I} + \lambda\mathbf{J} & \lambda\mathbf{1}_{v_1}\mathbf{1}'_{\alpha} \\ \lambda\mathbf{1}_{\alpha}\mathbf{1}'_{v_1} & (\alpha\beta-\lambda+\delta)\mathbf{I} + (\lambda+\eta)\mathbf{1}_{\alpha}\mathbf{1}'_{\alpha} \end{bmatrix}$$

$$\mathbf{r}^*\mathbf{r}^{*\prime} = \begin{bmatrix} p^2\mathbf{1}\mathbf{1}' & p(a+p)\mathbf{1}\mathbf{1}' \\ p(a+p)\mathbf{1}\mathbf{1}' & (a+p)^2\mathbf{1}\mathbf{1}' \end{bmatrix},$$

$$\mathbf{R}^* = \begin{bmatrix} p\mathbf{I}_{v-\alpha} & \mathbf{0} \\ \mathbf{0} & (a+p)\mathbf{I}_{\alpha} \end{bmatrix}$$

and

$$\mathbf{F} = \frac{1}{p(v+a\alpha)} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}' & \mathbf{D} \end{bmatrix}$$

where

$$\begin{aligned} \mathbf{A} &= \left[p^2(v+a\alpha) - (\alpha\beta-\lambda)(v+a\alpha)\right]\mathbf{I} - (v+a\alpha)\lambda\mathbf{1}\mathbf{1}' \\ \mathbf{B} &= -\left[ap(p-1) + (v+a\alpha)\lambda\right]\mathbf{1}\mathbf{1}' \\ \mathbf{D} &= \left[(a+p)p(v+a\alpha) - (\alpha\beta-\lambda+\delta)(v+a\alpha)\right]\mathbf{I} \\ &\quad - \left[p^2(a+1)^2 + (v+a\alpha)(\lambda+\eta) - (a+p)^2\right]\mathbf{1}\mathbf{1}' \end{aligned}$$

which is the information matrix of a BBPRC design with parameters $v_1 = v-\alpha$, $v_2 = \alpha$, $R = p$, $C = v + a\alpha$, $r_1 = p$, $r_2 = p(a+1)$, $s_1 = (v+a\alpha)\lambda$, $s_0 = [ap(p+1) + (v+a\alpha)\lambda]$, $s_2 = [p^2(a+1)^2 + (v+a\alpha)(\lambda+\eta) - (a+p)^2]$.

If in this method, we consider a LSD in place of YSD, it is similar to type 2 of Freeman (1975).

Example: Consider a YSD $\{v = 5, p = 4(2\times 2), q = 5, \lambda = 1\}$. Then, for $a = 1$, $\alpha = 2$, following method (2.6), we get the BBPRC design as

1	2	3	4	5	1	2
2	3	4	5	1	2	1
3	4	5	1	2	1	2
4	5	1	2	3	2	1

The parameters of the design are $v_1 = 3$, $v_2 = 2$, $R = 4$, $C = 7$, $r_1 = 4$, $r_2 = 8$, $s_1 = 7$, $s_0 = 7$, $s_2 = 63$.

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