Cyclic and Rotational Decompositions of K_n into Stars

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Abstract. We give necessary and sufficient conditions for the existence of a decomposition of the complete graph into stars which admits either a cyclic or a rotational automorphism.

1 Introduction

We denote the complete graph on n vertices by K_n and the star with m edges by S_m. Let $m_1 \geq m_2 \geq \ldots \geq m_l$ be nonnegative integers. Then a $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$*–decomposition of* K_n (or a *star decomposition of* K_n, for short) is a collection of stars such that

$$E(S_{m_i}) \bigcap E(S_{m_j}) = \emptyset \text{ if } i \neq j, \text{ and } \bigcup_{i=1}^{l} E(S_{m_i}) = E(K_n).$$

It was recently shown in [2] that such a decomposition exists if and only if

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (n-i) \text{ for } k = 1, 2, \ldots, n-1, \text{ and } \sum_{i=1}^{l} m_i = \binom{n}{2}.$$

An *automorphism* of a star decomposition is a permutation of $V(K_n)$ which fixes the set $\{S_{m_1}, S_{m_2}, \ldots, S_{m_l}\}$. The *orbit* of a star under an automorphism π is the collection of images of the star under the powers of π. A permutation of $V(K_n)$ which consists of a single cycle of length n is said to be *cyclic.* A permutation of $V(K_n)$ consisting of a fixed point and a cycle of length $n-1$ is said to be *rotational.* Several graph and digraph decompositions have been studied which admit either a cyclic or rotational automorphism. See, for example, [1, 3, 4, 5]. The purpose of this paper is to give necessary and sufficient conditions for the existence of star decompositions of K_n which admit either a cyclic automorphism or a rotational automorphism.

2 Cyclic Star Decompositions of K_n

Throughout this section, we assume the vertex set of K_n is $\{0, 1, \ldots, n-1\}$ and we will construct star decompositions of K_n admitting $\pi = (0, 1, \ldots, n-$

1) as an automorphism.

Lemma 2.1 *If there exists a $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$-decomposition of K_n which admits a cyclic automorphism and if n is even, then $|\{i \mid m_i = 1\}| \equiv n/2 \pmod{n}$.*

Proof. The edge $(0, n/2)$ must lie in some star, say S_{m_s}. Then $\pi^{n/2}((0,n/2)) = (0,n/2)$ and since each edge occurs in exactly one star of the decomposition, it must be that $\pi^{n/2}(S_{m_s}) = S_{m_s}$. Therefore $m_s = 1$. Let $A = \{\pi^i(S_{m_s}) \mid i \in \mathbf{Z}\}$. Then $|A| = n/2$ and if $S_{m_t} \notin A$ then the length of the orbit of S_{m_t} is n. Therefore $|\{i \mid m_i = 1\}| \equiv n/2 \pmod{n}$. ▮

As argued in Lemma 2.1, the length of the orbit of every star in a cyclic star decomposition of K_n is n except for the special "short obit" stars in set A. We therefore have:

Lemma 2.2 *If there exists a $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$-decomposition of K_n which admits a cyclic automorphism, then for $k = 1, 2, \ldots, n-1$, $|\{i \mid m_i = k\}| \equiv 0 \pmod{n}$, except for the case $k = 1$ when n is even.*

We show the necessary conditions of Lemmas 2.1 and 2.2, along with the necessary conditions for the existence of a star decomposition of K_n, are sufficient for the existence of a cyclic star decomposition of K_n.

Theorem 2.1 *Let $m_1 \geq m_2 \geq \cdots \geq m_l$ be nonnegative integers. Then there is a cyclic $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$-decomposition of K_n if and only if*

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k}(n-i) \text{ for } k = 1, 2, \ldots, n-1, \quad \sum_{i=1}^{l} m_i = \binom{n}{2}$$

and

(a) *$|\{i \mid m_i = k\}| \equiv 0 \pmod{n}$ for all $k = 1, 2, \ldots, n-1$ if n is odd, or*

(b) *$|\{i \mid m_1 = 1\}| \equiv n/2 \pmod{n}$ and $|\{i \mid m_i = k\}| \equiv 0 \pmod{n}$ for all $k = 2, 3, \ldots, n-1$ if n is even.*

Proof. We need only establish sufficiency. Without loss of generality, we may assume $m_l \geq 1$. If n is odd, consider the collection of stars with edge sets

$$E(S_{m_{l-kn-i}}) = \{(i, i + r + \sum_{j=1}^{k} m_{l-(j-1)n}) \mid r = 1, 2, \ldots, m_{l-kn}\}$$

for $i = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, l/n - 1$. If n is even, consider the collection of stars with edge sets

$$E(S_{m_{l-i}}) = \{(i, i + n/2)\}$$

for $i = 0, 1, \ldots, n/2 - 1$, and

$$E(S_{m_{l-n/2-kn-i}}) = \{(i, i+r+\sum_{j=1}^{k} m_{l-n/2-(j-1)n}) \mid r = 1, 2, \ldots, m_{l-n/2-kn}\}$$

for $i = 0, 1, \ldots, n-1$ and $k = 0, 1, \ldots, (l - n/2)/n - 1$. In each case, the given collection of stars forms a cyclic star decomposition of K_n. ∎

3 Rotational Star Decompositions of K_n

Throughout this section, we assume the vertex set of K_n is $\{\infty, 0, 1, \ldots, n-2\}$ and we will construct star decompositions of K_n admitting $\pi = (\infty)(0, 1, \ldots, n-2)$ as an automorphism.

As in Lemma 2.1, if $n - 1$ is even, then the edge $(0, (n-1)/2)$ must occur in some S_{m_s} where $m_s = 1$. We analogously have:

Lemma 3.1 *If there exists a $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$−decomposition of K_n which admits a rotational automorphism and if n is odd, then $|\{i \mid m_i = 1\}| \equiv (n-1)/2 \pmod{n-1}$.*

The orbit of each star of a rotational star decomposition of K_n is of length $n - 1$, with two possible types of exceptions: **(1)** if n is odd, then the stars S_1 with edge sets $\{(i, i + (n-1)/2)\}$ for some i have orbits of length $(n-1)/2$, and **(2)** if $m \mid (n-1)$, $m \neq 1$, say $(n-1)/m = p$ then the stars S_m with edge sets $\{(\infty, i), (\infty, i+p), \ldots, (\infty, i+n-1-p)\}$ for some i have orbits of length p.

Theorem 3.2 *Let $m_1 \geq m_2 \geq \cdots \geq m_l$ be nonnegative integers. Then there is a rotational $S_{m_1}, S_{m_2}, \ldots, S_{m_l}$−decomposition of K_n if and only if*

$$\sum_{i=1}^{k} m_i \leq \sum_{i=1}^{k} (n-i) \text{ for } k = 1, 2, \ldots, n-1, \quad \sum_{i=1}^{l} m_i = \binom{n}{2}$$

and

(a) $|\{i \mid m_i = k\}| \equiv 0 \pmod{n-1}$ *for all $k = 1, 2, \ldots, n-1$ if n is even, or*

(b) $|\{i \mid m_i = 1\}| \equiv (n-1)/2 \pmod{n-1}$ *and* $|\{i \mid m_i = k\}| \equiv 0 \pmod{n-1}$ *for all* $k = 2, 3, \ldots, n-1$ *if* n *is odd, or*

(c) *if* $m \mid (n-1)$, *say* $(n-1)/m = p$, *for some* $m \in \{m_1, m_2, \ldots, m_l\}$, $m \neq 1$, *then* $|\{i \mid m_i = m\}| \equiv p \pmod{n-1}$ *and* $|\{i \mid m_i = k\}| \equiv 0 \pmod{n-1}$ *for all* $k = 1, 2, \ldots, m-1, m+1, \ldots, n-1$ *if* n *is even, or*

(d) *if* $m \mid (n-1)$, *say* $(n-1)/m = p$, *for some* $m \in \{m_1, m_2, \ldots, m_l\}$, $m \neq 1$, *then* $|\{i \mid m_i = m\}| \equiv p \pmod{n-1}$, $|\{i \mid m_i = 1\}| \equiv (n-1)/2 \pmod{n-1}$ *and* $|\{i \mid m_i = k\}| \equiv 0 \pmod{n-1}$ *for all* $k = 2, 3, \ldots, m-1, m+1, \ldots, n-1$ *if* n *is odd.*

Proof. We need only establish sufficiency. Without Loss of generality, we may assume $m_l \geq 1$. We consider the four cases separately.

(a) Consider the collection of stars with edge sets

$$E(S_{m_{l-i}}) = \{(\infty, i)\} \bigcup \{(i, i+r) \mid r = 1, 2, \ldots, m_l - 1\}$$

for $i = 0, 1, \ldots, n-2$ and

$$E(S_{m_{l-k(n-1)-i}}) = \{(i, i+r-1+\sum_{j=1}^{k} m_{l-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{l-k(n-1)}\}$$

for $i = 0, 1, \ldots, n-2$ and $k = 1, 2, \ldots, l/(n-1) - 1$.

(b) Consider the collection of stars with edge sets

$$E(S_{m_{l-i}}) = \{(i, i + (n-1)/2)\}$$

for $i = 0, 1, \ldots, (n-1)/2 - 1$,

$$E(S_{m_{l-(n-1)/2-i}}) = \{(\infty, i)\} \bigcup \{(i, i+r) \mid r = 1, 2, \ldots, m_{l-(n-1)/2-k(n-1)} - 1\}$$

for $i = 0, 1, \ldots, n-2$, and

$$E(S_{m_{l-(n-1)/2-k(n-1)-i}}) = \{(i, i+r-1+\sum_{j=1}^{k} m_{l-(n-1)/2-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{l-(n-1)/2-k(n-1)}\}$$

for $i = 0, 1, \ldots, n-2$ and $k = 1, 2, \ldots, (l-(n-1)/2)/(n-1) - 1$.

(c) Let t be the largest index such that $m_t = m$. Consider the collection of stars with edge sets

$$E(S_{m_{l-k(n-1)-i}}) = \{(i, i + r + \sum_{j=1}^{k} m_{l-(j-1)(n-1)})$$
$$\mid r = 1, 2, \ldots, m_{l-k(n-1)}\}$$

for $i = 0, 1, \ldots, n-2$ and $k = 0, 1, \ldots, (l-t)/(n-1) - 1$,

$$E(S_{m_{t-i}}) = \{(\infty, i + rp) \mid r = 0, 1, \ldots, m_t - 1\}$$

for $i = 0, 1, \ldots, p-1$,

$$E(S_{m_{t-p-k(n-1)-i}}) = \{(i, i + r + \sum_{j=1}^{(l-t)/(n-1)} m_{l-(j-1)(n-1)}$$
$$+ \sum_{j=1}^{k} m_{t-p-(j-1)(n-1)}) \mid r = 1, 2, \ldots, m_{t-p-k(n-1)}\}$$

for $i = 0, 1, \ldots, n-2$ and $k = 0, 1, \ldots, (t-p)/(n-1) - 1$.

(d) Let t be the largest index such that $m_t = m$. Consider the collection of stars with edge sets

$$E(S_{m_{l-i}}) = \{(i, i + (n-1)/2)\}$$

for $i = 0, 1, \ldots, (n-1)/2 - 1$,

$$E(S_{m_{l-(n-1)/2-k(n-1)-i}}) = \{(i, i + r + \sum_{j=1}^{k} m_{l-(n-1)/2-(j-1)(n-1)})$$
$$\mid r = 1, 2, \ldots, m_{l-(n-1)/2-k(n-1)}\}$$

for $i = 0, 1, \ldots, n-2$ and $k = 0, 1, \ldots, (l-t)/(n-1) - 1$,

$$E(S_{m_{t-i}}) = \{(\infty, i + rp) \mid r = 0, 1, \ldots, m_t - 1\}$$

for $i = 0, 1, \ldots, p-1$,

$$E(S_{m_{t-p-(n-1)/2-k(n-1)-i}}) = \{(i, i + r$$
$$+ \sum_{j=1}^{(l-t)/(n-1)} m_{l-(n-1)/2-(j-1)(n-1)} + \sum_{j=1}^{k} m_{t-p-(n-1)/2-(j-1)(n-1)})$$
$$\mid r = 1, 2, \ldots, m_{t-p-(n-1)/2-k(n-1)}\}$$

for $i = 0, 1, \ldots, n-2$ and $k = 0, 1, \ldots, (t-p-(n-1)/2)/(n-1) - 1$.

In each case, the given stars form a rotational decomposition of K_n. ∎

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