The Edge-Isoperimetric Problem for Regular Planar Tesselations

L.H. Harper
Mathematics Department
University of California
Riverside, CA 92521
email: harper@math.ucr.edu

ABSTRACT. Solutions for the edge-isoperimetric problem on the graphs of the triangular and hexagonal tesselations of the Euclidean plane are given. The proofs are based on the fact that their symmetry group is Coxeter. In each case there is certain nice quotient of the stability order of the graph (which is itself a quotient of the Bruhat order of the Coxeter group by a parabolic subgroup).

1 Introduction

1.1 The Edge-Isoperimetrlc Problem

Given a graph $G = (V, E, \partial)$ having vertex-set V, edge-set E and boundary-function $\partial\colon E \to \binom{V}{2}$ which identifies the pair of vertices incident to each edge, we let

$$\Theta(S) = \{e \in E\colon \partial(e) = \{v, w\}, v \in S \;\&\; w \notin S\}.$$

Then given $k \in \mathbb{Z}^+$, the *edge-isoperimetric problem* (EIP) is to minimize $|\Theta(S)|$ over all $S \subseteq V$ such that $|S| = k$. This author's first paper, [4], written over thirty years ago, presented a solution of the EIP for Q_d, the graph of the d-dimensional cube. Many extensions and variations of it have appeared in the literature (see [8]) and the problem still continues to fascinate. One such variant, which was intriguing but until recently lacked the incentive for serious effort, was the possibility of having G be infinite. The original application, solving a kind of layout problem if G is regarded as representing an electronic circuit, does not seem to make sense if G is infinite. However there is a way to make sense of it: Steiglitz and

Bernstein [5] had noted that in laying out Q_d on a linear chassis, the original problem, which was to minimize the total length of the wires necessary to make the connections, could be generalized to arbitrary spacings between sites, $x_1 < x_2 < \cdots < x_n$. The same holds for any graph, G, and then the wirelength for a layout function $\varphi\colon V \to \{1, 2, \ldots, n\}$, assigning v to $x_{\varphi(v)}$, would just be

$$wl(\varphi) = \sum_{k=0}^{n} (x_{k+1} - x_k)|\Theta(S_k(\varphi))|,$$

where $S_k(\varphi) = \{v \in V : \varphi(v) \leq k\}$. Note that $|S_k(\varphi)| = k$ and $S_k(\varphi) \subset S_{k+1}(\varphi)$. Conversely, if the EIP on G has a nested family of solutions, one for each value of k between 0 and n, which it does for Q_d and many other interesting graphs, then the corresponding layout function is optimal for any choice of the sites, $\{x_k\}$. Even if $n = \infty$ there is then a possibility that the wirelength could be finite if $\{x_k\}$ is bounded above. And if G does have nested solutions for the EIP, the finiteness of its wirelength would just depend on the rate of growth of $\min_{|S|=k} |\Theta(S)|$ as $k \to \infty$ and the rate at which $x_{k+1} - x_k \to 0$.

Another, deeper, motivation for considering the EIP for infinite graphs is that there are some very large, i.e. finite but for all practical purposes infinite, graphs for which we would like to solve the EIP. The graph of the 120-cell, an exceptional regular solid in four dimensions, is one such. It has 600 vertices. Another, also 4-dimensional, is the graph of the 5-permutohedron, which has $5! = 120$ vertices. Solving those problems will require developing better methods than we have now. The exceptional regular tesselations of the Euclidean plane are relatively easy to work with but present some of the same kinds of technical problems as the higher dimensional semiregular and exceptional regular solids.

1.2 Some Variants of the EIP

For $S \subseteq V$, let

$$I(S) = \{e \in E\colon \partial(e) = \{v, w\}, v \in S \,\&\, w \in S\}.$$

The members of $I(S)$ are called *internal edges* of S. If G is regular of degree δ, then

$$|\Theta(S)| = \delta|S| - 2|I(S)|,$$

so

$$|I(S)| = \frac{1}{2}(\delta|S| - |\Theta(S)|),$$

and for $|S| = k$, fixed, minimizing $|\Theta(S)|$ is equivalent to maximizing $|I(S)|$. Also, if

$$E(S) = \{e \in E\colon \partial(e) = \{v, w\}, v \in S \text{ or } w \in S\},$$

then

$$|E(S)| = |I(S)| + |\Theta(S)| = \frac{1}{2}(\delta|S| + |\Theta(S)|),$$

so minimizing $|E(S)|$ for $|S| = k$ gives another equivalent version of the *EIP* (on regular graphs).

2 The Triangular tesselation

The tiling of the Euclidean plane by regular triangles is a familiar one. Its Schläfli symbol is $\{3, 6\}$ (see [1]), reflecting the fact that every face (tile) is bounded by 3 edges and every vertex is incident to 6 edges.

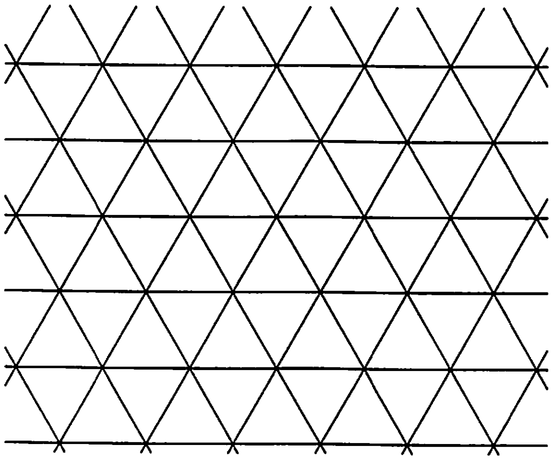

Figure 1. The triangular tesselation

2.1 What is the Solution?

In order to gain some idea of what the solution sets for the edge-isoperimetric problem on the graph, T, of this tesselation are, we begin in the usual fashion, determining them for small values of k.

- $k = 0$: This is trivial since only the null set, $\emptyset$, is of size 0, so it is the unique solution and $\min_{|S|=0} |\Theta(S)| = 0$.
- $k = 1$: There are countably many 1-sets of vertices but they are all equivalent under symmetry so they are all solutions. $\min_{|S|=1} |\Theta(S)| = 6$.
- $k = 2$: There are countably many equivalence classes of 2-sets under symmetries of the triangular tesselation. The problem is solvable though, since every pair of vertices is either connected by an edge

or not. If they are, $|\Theta(S)| = 10$, otherwise $|\Theta(S)| = 12$. Therefore $\min_{|S|=2} |\Theta(S)| = 10$.

- $k = 3$: There is only one type of set with $|S| = 3$ and $|I(S)| \geq 3$, the vertices of a triangle. Therefore $\min_{|S|=3} |\Theta(S)| = 6 \cdot 3 - 2 \cdot 3 = 12$ by the remarks of Section 1.1.1.

The challenge of the problem for $k > 3$ is apparent. There are countably many equivalence classes of k-sets, of increasing complexity. Even if we could characterize them all, we would still need something stronger than symmetry, something which would systematically take the connectivity of k-sets into account. Fortunately we have just such a tool available, the theory of stabilization (See [2] or [3]) which utilizes Coxeter theory. This is not by accident however, the problem was chosen just because T is one of the simplest graphs to which the theory of stabilization applies and whose EIP remains unsolved. It is not difficult to come up with a persuasive conjecture about the solution of the EIP on T, but proving it is a different matter. Isoperimetric theorems are notoriously slippery to prove anyway and the similarity between regular planar tesselations and regular 4-dimensional solids such as the 600-cell, whose EIP does not have nested solutions, indicates, I believe, that proving an isoperimetric theorem for T requires some subtlety.

2.2 The Stability Order of V_T

(See Appendix)

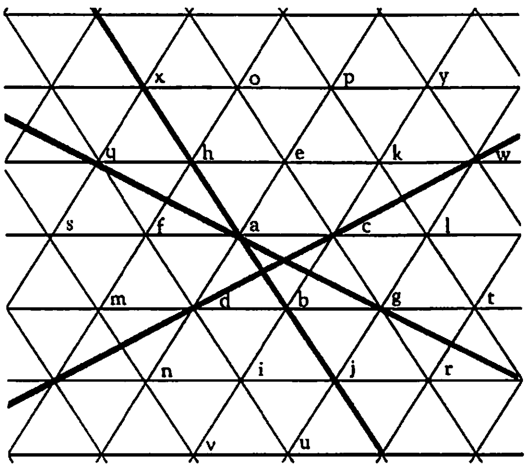

Figure 2. T with the fixed lines of basic reflections darkened

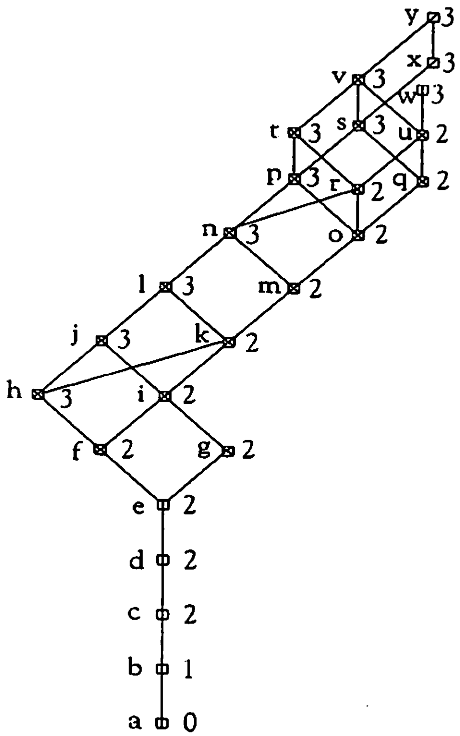

Figure 3. Hasse diagram of $\mathcal{SO}(T)$, the stability order, with $|\iota|$

From this we determine the solutions for $k \leq 19$:

k	0	1	2	3	4	5	6	7	8	9
$\max_{\lvert S\rvert=k} \lvert I(S)\rvert$	0	0	1	3	5	7	9	12	14	16
	10	11	12	13	14	15	16	17	18	19
	19	21	24	26	29	31	34	36	39	42

Table 1

2.3 Solutions for all k

Theorem 1. *T has nested solutions for the Edge-Isoperimetric Problem, i.e. there exists a total order $\mathcal{TO}$ on V_T such that for all $k \in \mathbb{Z}^+$ the initial k-set of $\mathcal{TO}$ minimizes $|\Theta(S)|$ over all $S \subseteq V_T$ with $|S| = k$.*

For any $v, w \in V_T$ let $d\ (v, w)$ denote the minimum length of any path from v to w in T and for any $r \in \mathbb{Z}^+$ let

$$B_r = \{v \in V_T \colon d(v_0, v) \leq r\},$$

the ball of radius r centered at v_0, v_0 being the unique vertex of T which is in the fundamental chamber (labeled a in Figure 2). The sides of B_r for $r > 0$ lie in 6 straight lines, i.e. B_r has the shape of a regular hexagon. From this it is easy to see that $|B_r| = 1 + 3r(r+1)$.

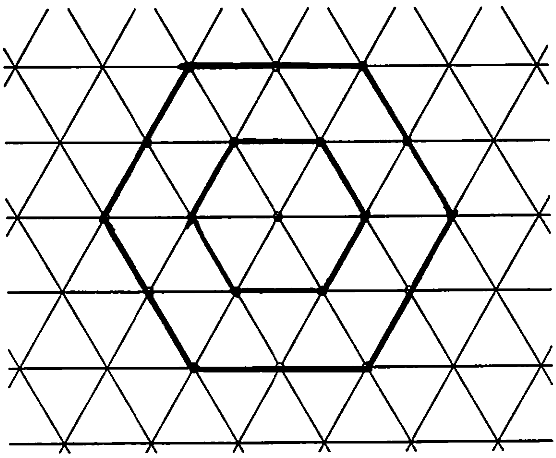

Figure 4. T with B_0, B_1 and B_2 darkened

All the edges of T lie on 3 families of parallel lines, which we denote altogether as $\mathcal{L}$, and each vertex lies on 3 of these lines, one from each family. V_3, the group of symmetries of T (see Table IV of [1]), acts transitively on $\mathcal{L}$. The theory of stabilization summarized in the Appendix is for vertices (points in $\mathbb{R}^d$) but applies equally well to geometric objects, such as lines, which are closed under the action of a Coxeter group, such as V_3. If $\mathcal{R}$ is a reflection in V_3 we let

$$\mathcal{SO}(\mathcal{L};\mathcal{R};p) = \{(L,\mathcal{R}(L))\colon \|L-p| < \|\mathcal{R}(L)-p\|\}.$$

Then the *stability order of $\mathcal{L}$ with respect to V_3 and p* is the transitive closure of

$$\bigcup_{\mathcal{R}\in V_3} \mathcal{SO}(\mathcal{L};\mathcal{R};p).$$

Compare this to Definition 5 of the Appendix. We denote this stability order on $\mathcal{L}$ by $\mathcal{LO}$ for short. The symmetries of B_r constitute a dihedral group, D_6, a subgroup of V_3, which act transitively on the 6 lines bounding B_r. The stability order of any D_n acting on the sides of a regular n-gon is total (see [2]) so the relative order (in $\mathcal{LO}$) of the 6 lines bounding B_r is total. Thus we may denote them as $L_{r,i}$, $0 \le i \le 5$ with $L_{r,i} <_{\mathcal{LO}} L_{r,j}$ if $i < j$.

Lemma 1. *Lines $L_{r,i}$ and $L_{s,j}$ are incomparaole in $\mathcal{LO}$ iff $s = r+1$, $i = 5$ and $j = 0$.*

Proof: $L_{r,5}$ intersects $L_{r+1,1}$ and the lines which bisect the angles between them are lines of symmetry for T, so $L_{r,5} <_{\mathcal{LO}} L_{r+1,1}$. Similarly, $L_{r,4} <_{\mathcal{L}} L_{r+1,0}$. However, $L_{r,5}$ and $L_{r+1,0}$ are parallel and the bisector of any perpendicular which connects them is not a line of symmetry for T. If we could show that $L_{r,5} <_{\mathcal{LO}} L_{r+1,0}$ it would have to be because there is a line L such that $L_{r,5} <_{\mathcal{LO}} L <_{\mathcal{LO}} L_{r+1,0}$ but $\forall\, L \in \mathcal{L} - \{L_{r,5}, L_{r+1,0}\}$, $L <_{\mathcal{LO}} L_{r,5}$ or $L_{r+1,0} <_{\mathcal{LO}} L$, so we are done. □

We have

$$V_T = \{v_0\} \cup \bigcup_{r=1}^{\infty} \bigcup_{i=0}^{5} (V_T \cap L_{r,i})$$

Each vertex, except v_0, is contained in multiple $L_{r,i}$'s but if we let

$$L'_{r,i} = V_T \cap L_{r,i} - \left[\bigcup_{j=i+1}^{5} (V_T \cap L_{r,j}) \cup \bigcup_{s=r+1}^{\infty} \bigcup_{i=0}^{5} (V_T \cap L_{s,i}) \right]$$

then $\{v_0\}$ and the $L'_{r,i}$'s partition V_T. Also, $B_r = \{v_0\} \cup \bigcup_{s=1}^{r} \bigcup_{i=0}^{5} L'_{s,i}$ and

$$|L'_{r,i}| = \begin{cases} r-1 & \text{if } i = 0, \\ r & \text{if } 0 < i < 5, \\ r+1 & \text{if } i = 5. \end{cases}$$

Note that $V_T \cap L_{r,i}$ is totally ordered by $\mathcal{SO}$, the vertex nearest p being its least element of course and this lies at or near the midpoint of $B_r \cap L_{r,i}$. The others follow in increasing order of their distance from p so that they alternate from side to side. $L'_{r,i}$ is an initial segment in this order. Note also that for $v \in L'_{r,i}$,

$$|\iota(v)| = \begin{cases} 0 & \text{if } v = v_0, \\ 1 & \text{if } v = v_1 \in L'_{1,1}, \\ 2 & \text{if } v \neq v_0, v_1 \text{ \& minimal in } L'_{r,i}, \\ 3 & \text{otherwise.} \end{cases}$$

(See Appendix for the definition of ι).

Proof (of Theorem 1): We define a total order, $\mathcal{TO}$, on V_T by $v <_{\mathcal{TO}} w$ if $v \in L'_{r,i}$, $w \in L'_{s,j}$ with $r < s$ or $r = s$ & $i < j$ or $r = s$ & $i = j$ & $v <_{\mathcal{SO}} w$. Note that $v <_{\mathcal{SO}} w$ implies $v <_{\mathcal{TO}} w$, i.e. $\mathcal{TO}$ is an extension of $\mathcal{SO}$. By the theory of stabilization, we need only show that if $S \subseteq V_T$, is stable, i.e.

a lower set in the stability order, $\mathcal{SO}$, $|S| = k$ and S_k is the initial segment of $\mathcal{TO}$ of the same cardinality, then

$$|I(S_k)| \geq |I(S)|.$$

If $S \neq S_k$, then $\exists$ a minimal element, a, with respect to $\mathcal{TO}$, in $S_k - S$ and a maximal element, b, in $S - S_k$. Note that $a <_{\mathcal{TO}} b$ but they must be incomparable with respect to $\mathcal{SO}$. Having already proved the theorem for $k = 0, 1, 2$ we may assume $k > 2$ so $|\iota(a)|$, $|\iota(b)| = 2$ or 3. If $|\iota(a)| \geq |\iota(b)|$ then $|S + \{a\} - \{b\}| = k$ and $|I(S + \{a\} - \{b\})| \geq |I(S)|$ and a finite series of such switches will achieve our goal. The only way $|\iota(a)| < |\iota(b)|$ is if a is the minimal element of $L'_{r,5}$, for some r, and $b \in L'_{r+1,0}$ but not minimal. Then switching all of $S \cap L'_{r+1,0}$ for the initial segent of $L'_{r,5}$ of the same size will do the job. This is possible because $|L'_{r+1,0}| = (r+1) - i = r < r+1 = |L'_{r,5}|$. □

Corollary 1. *If $k = 1 + 3r(r+1)$ then the only stable solution is B_r*

Corollary 2. *$\mathcal{TO}$ is the only total extension of $\mathcal{SO}$ whose initial segments are solutions of the EIP.*

There is another, in some ways more natural, total ordering of V_T whose initial segments are solutions of the EIP: Begin with $v_0 < v_1$ and having chosen $v_0 < v_1 < \cdots < v_n$ choose v_{n+1} to be the furthest clockwise neighbor of v_n which has not been chosen yet. The initial segments of this total order are not stable but the sequence of marginal contributions of the vertices is the same as the sequence of weights with respect to $\mathcal{TO}$. The counterclockwise spiral works equally well, of course.

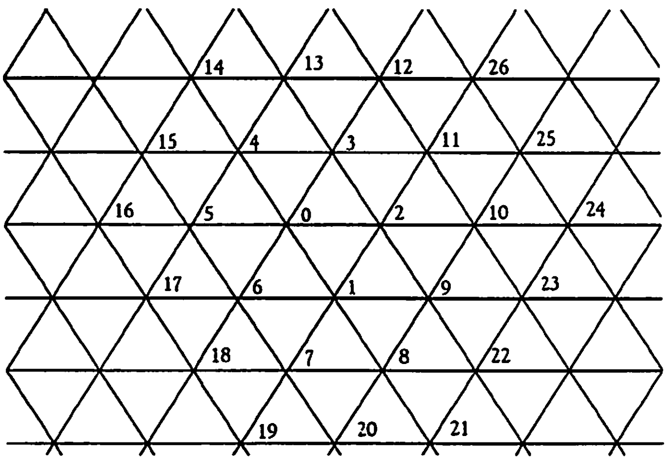

Figure 5. T with an optimal spiral numbering

3 The Hexagonal Tesselation

The tesselation of the Euclidean plane by regular hexagons is also familiar. Alias "the honeycomb", it is the dual of the triangular tesselation and has Schläfli symbol $\{6,3\}$ (see [1]) meaning that every face (tile) is bounded by 6 edges and every vertex is incident to 3 edges. Let H denote the graph of this tesselation.

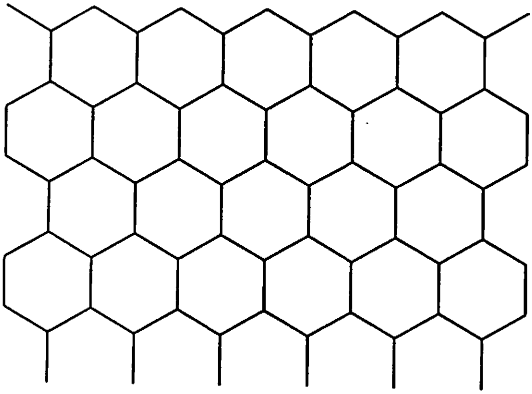

Figure 6. The hexagonal tesselation

The steps which we took to find and prove the solution for T also suffice for H. There are some minor complications since the solution sets are no longer balls in an intrinsic metric and their boundary vertices do not lie on straight lines but zig-zag a bit, however, the same program does work and we recommend it as an exercise for the reader. As a check and for later reference, we list the solutions for $k \leq 24$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12
$\max_{\lvert S \rvert = k} \lvert I(S) \rvert$	0	0	1	2	3	4	6	7	8	9	11	12	13
	13	14	15	16	17	18	19	20	21	22	23	24	
	15	16	17	19	20	21	23	24	25	27	28	30	

Table 2

Since our purpose here is to develop methods as well as to solve problems, we shall proceed a little differently. In the proof of Theorem 1 we decomposed V_T into "lines". In order for such a decomposition to work, the blocks of the partition (the $L'_{r,i}$'s in that case) must be highly connected. With that requirement in mind, the first sets one would think of, would be the vertices of faces (i.e. triangles in T or hexagons in H). They are the sets of highest connectivity in some sense. And this decomposition by faces does produce a proof for T and H, just not as simple a proof as we found in Section 2. The additional difficulty is due to the reduced problem being more complicated than the one in Section 2. In fact the stability order of the triangular faces in T is just the stability order of V_H (the centroid of each face is a dual vertex). Also, the stability order of the hexagonal faces of H is just that of V_T. As we know, these are both fairly complex, so the value of these reductions is not at first evident. It is possible to make a proof from them by inducting on the sizes of subsets of V_H and V_T. We shall not present all the details of such a proof here, just those necessary to get the solution for H from the one we already have for T.

If G is any planar graph (which may be represented on the surface of a sphere if finite) let G^* be its dual. Then $H^* = T$, $T^* = H$ and $G^{**} = G$ in general. If $S \subseteq V_G$, let

$$S^* = \bigcup_{v \in S} \{w \in V_{G^*} : w \text{ lies on the face of } v\}.$$

Also, let $S^{\dot{-}}$ be the inverse of this map, i.e. if $S = U^*$ and U is maximal with respect to this property, then $S^{\dot{-}} = U$.

Lemma 2. *$S \subseteq V_G$ is stable iff $S^* \subseteq V_{G^*}$ is stable.*

Proof: For any reflection, $\mathcal{R}$, it follows from the definition of $Stab_{\mathcal{R},p}(S)$. Being true for each of $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{k-1}$, it holds for stabilization with respect to the whole set. □

One would further expect that the optimality of S and S^* would be closely connected but the following examples show that neither implies the other.

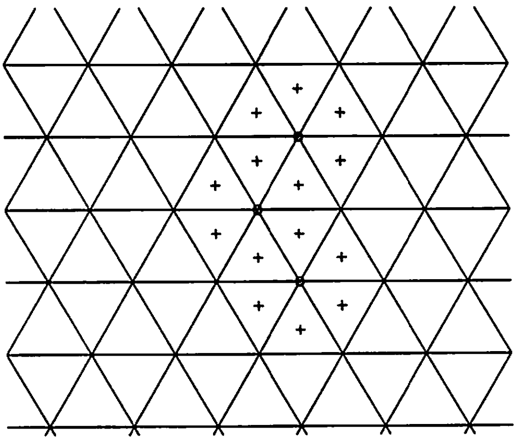

Figure 7. S (circled vertices) is not optimal but S^* (the crosses) is

$S \subsetneq V_T$, $|S| = 3$, $|I(S)| = 2$, is not optimal, but $S^* \subsetneq V_H$, $|S^*| = 14$, $|I(S^*)| = 16$ so S^* is optimal (see Table 2).

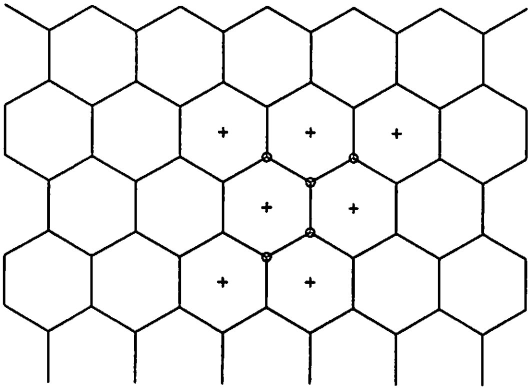

Figure 8. S (circled vertices) is optimal but S^* (the crosses) is not

$S \subsetneq V_H$, $|S| = 5$, $|I(S)| = 4$, is optimal, but $S^* \subsetneq V_T$, $|S^*| = 7$, $|I(S^*)| = 11$ so S^* is not optimal (see Table 1).

Definition 1. *$k \in \mathbb{Z}^+$ is called a critical cardinal if*

$$\max_{|S|=k} |I(S)| - \max_{|S|=k-1} |I(S)| > 1.$$

Note that if S is optimal and has a pendant vertex, then $S - \{v\}$ must also be optimal since $|I(S)| = |I(S - \{v\})| + 1$, the least that $|I|$ can increase. Thus such an S is not critically optimal. Therefore, if S is critically optimal it can have no pendant vertices and must be a union of faces. So for a critically optimal S, $S^{\dot{-}}$ is defined.

Lemma 3. *If $S \subsetneq V_G$ is optimal and $|S| = k$, a critical cardinal, then $S^{\dot{-}} \subsetneq V_{G^*}$ is optimal.*

Proof: Euler's relation, $v + f = e + 2$, holds for the subgraph of G induced by S with $v = |S|$, $f = |S^{\dot{-}}|$ and $e = |I(S)| = |E(S^{\dot{-}})|$. If $S^{\dot{-}}$ is not optimal then $\exists\ U \subsetneq V_{G^*}$ such that $|U| = |S^{\dot{-}}|$ and $|(U)| < |E(S^{\dot{-}})$. The Euler relation then implies that $|U^*| < |S|$. In fact $|S| - |U^*| = |E(S^{\dot{-}})| - |E(U)| = |I(S)| - |I(U^*)|$. Adding $|S| - |U^*|$ vertices to U^* optimally will give a set S' such that $|S'| = |S|$ and $|I(S')| \geq |I(S)|$. This contradicts the optimality of S if $|I(S')| > |I(S)|$ or its criticality if $|I(S')| = |I(S)|$. □

Theorem 2. *H has nested solutions for the Edge-Isoperimetric Problem, i.e. there exists a total order on V_H whose initial k-set minimizes $|\Theta(S)|$ over all $S \subsetneq V_H$ with $|S| = k$ for all $k \in \mathbb{Z}^+$.*

Proof: If k is a critical cardinal for H, and S is a stable optimal set of size k, then by Lemma 3 $S^{\dot{-}}$ is optimal in $H^* = T$. By Lemma 2 it is also stable in T so must be as described in the proof of Theorem 1. For

$$1 + 3r(r+1) \leq k' \leq 1 + 3r(r+1) + r + 4(r+1)$$

this set is uniquely determined and $S = (S^{\dot{-}})^*$ is also. If

$$1 + 3r(r+1) + r + 4(r+1) < k' < 1 + 3(r+1)(r+2)$$

then there are two possibilities but since $|I|$ is the same for both, they are both optimal. In particular, $S^*_{k'}$ is optimal for $k' = 1, 2, \ldots$. For the non-critical cardinals we need only interpolate the vertices in $(S_{k'+1})^* - S^*_{k'}$, in the order determined by stabilization, to prove the theorem. □

Corollary 3. *If* $k = 6(r+1)^2$ *then the only stable solution is* B^*_r.

There are also optimal spiral (clockwise and counterclockwise) orderings of V_H.

3.1 Variations and Extensions

3.1.1. The Square Tesselation. A solution of the EIP on the graph of the square tesselation was given by Harary & Harborth [7] and may easily be reproved with the methods of this paper. Solution sets for $k = (2r+1)^2$ are $2r \times 2r$ squares which are balls of radius r in the sup norm. The square tesselation is not exceptional, being the beginning of the infinite family of tesselations of $\mathbb{R}^d$ by d-cubes, $d = 2, 3, \ldots$. The graphs of these tesselations are products, $\mathbb{Z}^d$. The theory of compression applies to product graphs, assuming that they have nested solutions, and makes it relatively easy to solve their EIP. This was first carried out by Bollobas and Leader [9] and later by Ahlswede and Bezrukov [10]. Up to now compression has been the most powerful tool for solving isoperimetric problems. Our hope is to develop something more powerful. Is it possible to give a proof of the solution to the EIP for $\mathbb{Z}^d$ with the methods of this paper?

3.1.2. Powers of T and H. $\mathbb{Z}$ is a tesselation of $\mathbb{R}$ and, as noted above, the EIP on $\mathbb{Z}^d$ has nested solutions, $d = 1, 2, \ldots$. What about T^d for $d \geq 2$, does it have nested solutions? The answer is, unfortunately, no. For $d = 2$ and $k = 3$ an optimal set must be a triangle and therefore be contained in $T \times \{i\}$ or $\{i\} \times T$ for some i. Adding a vertex in any other copy of T thereafter would mean that its marginal contribution would be 1 whereas staying in that copy would give a marginal contribution of at least 2. However for $k = 14$, $I(B_2 \times \{v_0, v_1\}) = 2 \cdot 12 + 7 = 31 > 29$, since v_0 and v_1 are connected by an edge. The answer is also no for H^d, $d \geq 2$. The minimum cycles of H^2 are 4-cycles, and optimizing locally will build up squares of side $s \leq 6$ until a whole torus, $\mathbb{Z}_6 \times \mathbb{Z}_6$ has been taken. For $k = 12$ though, $I(\mathbb{Z}_6 \times \{v_0, v_1\}) = 18 > 17$, the number of internal edges in a 3×4 rectangle.

3.1.3. Regular Tesselations of the Hyperbolic Plane. The connection between Euclidean geometry and combinatorics which seemed implicit in the theory of stabilization has puzzled me since its inception. It now appears possible to penetrate this mystery a bit: From hyperbolic geometry we learn that the hyperbolic plane also has regular tesselations. The symmetry groups of these tesselations are, in an abstract sense, Coxeter, and correspond to solutions of the inequality,

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}, p, q \in \mathbb{Z}^+.$$

They occur in dual pairs whose Schläfli symbols are $\{p, q\}$ and $\{q, p\}$. There are infinitely many such in contrast to the Euclidean condition

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}, p, q \in \mathbb{Z}^+,$$

which only has the solutions $\frac{1}{4} + \frac{1}{4} = \frac{1}{3} + \frac{1}{6} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ giving the three tesselations which we have already treated. I believe that the methods of this paper will produce solutions for all of the regular tesselations of the hyperbolic plane as well.

3.1.4. The *EIP* for Cayley-Coxeter Graphs. The theory of stabilization applies to all Cayley graphs of Coxeter groups with respect to their Coxeter generators. The stability order is just the Bruhat order. This does simplify the *EIP* considerably but even for S_5, the symmetric group on 5 letters, the number of lower sets in the Bruhat order is still too large for the fastest computer. With the methods of this paper a solution of the problem for S_4 can be calculated by hand and it is hoped that they will give a tractable calculation for S_5 also. The catalog of Coxeter groups is essentially "terra incognita" for these methods so one can hope that there are beautiful insights awaiting discovery.

3.1.5. The Vertex-Isoperimetric Problem (VIP). If, in the definition of the *EIP* (Section 1.1), we replace $\Theta(S)$ by

$$\Phi(S) = \{v \in V - S : \exists e \in E, \partial(e) = \{v, w\} \ \&\ w \in S\}$$

and seek to minimize $|\Phi(S)|$ over all $S \subseteq V$ with $|S| = k$, then we have the *VIP*. The analogy between the *EIP* and *VIP* is clear. There are many similarities between their theories but there are also many contrasts (see [3]). The *VIP* for $\mathbb{Z}^d$ has already been solved by Wang & Wang [11]. Can the methods of this paper be used to solve the *VIP* for T and H?

3.1.6. What are "the methods of this paper"? The above references to "the methods of this paper" are intentionally vague but we hope to make them precise soon. What I believe is implicit in the proofs of Theorems 1 & 2 is a notion of morphism for the EIP extending that of "Steiner operation" as defined in [3]. Steiner operations are characterized by certain one-to-one, order-preserving functions and it has been a long-standing personal challenge to extend them to many-to-one functions. One can, of course, define such a thing with ease but there is a responsibility to provide good examples and to prove the utility of a new concept, especially one so complex. The examples of this paper and others, positive and negative, will show what the definition "should" be. The idea is that it will embody a "divide and conquer" strategy for the EIP, but one must understand *how* to divide in order to conquer.

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Appendix. A Summary of the Theory of Stabilization (See [3])

If $G = (V, E, \partial)$ is a finite graph embedded in $\mathbb{R}^d$, d-dimensional Euclidean space, and $\mathcal{R}$ is a reflection which acts as a symmetry of G, then

Definition 2. *$\mathcal{R}$ is called stabilizing if for all $e \in E$, $\partial(e) = \{v, w\}$, if v and w are on opposite sides of the fixed hyperplane of $\mathcal{R}$, then $\mathcal{R}(v) = w$.*

Definition 3. *If $\mathcal{R}$ is stabilizing for G, $p \in \mathbb{R}^d$ is not fixed by $\mathcal{R}$ and $S \subseteq V$ with*

$$\sum = \{v \in S: \|v - p\| > \|\mathcal{R}(v) - p\| \;\&\; \mathcal{R}(v) \notin S\}$$

then

$$Stab_{\mathcal{R},p}(S) = S - \sum + \mathcal{R}\left(\sum\right).$$

Theorem 3. *$|Stab_{\mathcal{R},p}(S)| = |S|$ and $|\Theta(Stab_{\mathcal{R},p}(S))| \leq |\Theta(S)|$. Also if $S \subseteq S' \subseteq V$ then $Stab_{\mathcal{R},p}(S) \subseteq Stab_{\mathcal{R},p}(S')$.*

The first equality follows directly from the definition of $Stab_{\mathcal{R},p}$. The second is based upon the observation that for any edge in $\Theta(Stab_{\mathcal{R},p}(S))$ but not in $\Theta(S)$, there is a corresponding edge (its image under $\mathcal{R}$) in $\Theta(S)$ but not in $\Theta(Stab_{\mathcal{R},p}(S))$. There are two ways this can happen depending on which side of the fixed hyperplane the edge lies. It cannot penetrate the fixed hyperplane because of Definition 2 above.

Given G and stabilizing reflections $\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{k-1}$ and $p \in \mathbb{R}^d$ not fixed by any $\mathcal{R}_i$, define a transformation $T_j: 2^V \to 2^V$, $j = 0, 1, \ldots,$ by

$$T_0 = I, \text{ the identity, and}$$
$$T_{j+1} = Stab_{\mathcal{R}_{j \,(\mathrm{mod}\, k)},p} \circ T_j.$$

Theorem 4. *There exists an integer j_0 such that for all $j \geq j_0$, $T_{j+1} = T_j$.*

Definition 4. *A set $S \subseteq V$ such that $Stab_{\mathcal{R}_i,p}(S) = S$ for $i = 0, 1, \ldots, k-1$ is called stable.*

Theorems 3 and 4 show that in minimizing $|\Theta|$ over 2^V, we need only consider stable sets. But how can we tell which sets are stable and which are not?

Definition 5. Let

$$\mathcal{SO}(V; \mathcal{R}; p) = \{(v, w) \in V \times V: \mathcal{R}(v) = w \;\&\; \|v - p\| < \|w - p\|\}.$$

Then the stability order, $\mathcal{SO}(V; \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{k-1}; p)$, is defined to be the transitive closure of $\bigcup_{i=0}^{k-1} \mathcal{SO}(V; \mathcal{R}_i; p)$.

Theorem 5. *A set $S \subseteq V$ is stable iff it is a lower set in $\mathcal{SO}(V; \mathcal{R}_0, , \mathcal{R}_1, \ldots, \mathcal{R}_{k-1}; p)$.*

Note that every edge of T is perpendicularly bisected by the fixed hyperplane of a reflective symmetry. The ends of the edge are therefore comparable in its stability order. Thus we may define

$$\iota(v) = \{w \in V : \exists e \in E, \partial(e) = \{v, w\} \ \&\ w <_{SO} v\}$$

and then

$$|I(S)| = \sum_{v \in S} |\iota(v)|.$$

This same representation of $|I(S)|$ for stable sets is valid in any graph where every vertex is comparable to its neighbors. This holds for the graphs of all regular solids and tesselations and many others.

The EIP has thus been transformed to maximizing a sum of weights $|\iota|$ over all lower sets in the poses $\mathcal{SO}(V; \mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{k-1}; p)$ but this only constitutes real progress if we can facilitate the calculation of $\mathcal{SO}$. If all of the reflective symmetries of G are stabilizing, as they are for T, then the group they generate is a Coxeter group (see [1] or [6]). As Coxeter showed, the fixed hyperplanes of the reflections in a Coxeter group partition $\mathbb{R}^d$ into connected components, called *chambers*, which are simplices if the group is irreducible. The chamber containing p is called the *fundamental chamber*. Each connected component of $\mathcal{SO}$ will have exactly one vertex in the fundamental chamber, its minimum element. In [6] the connected stability orders of irreducible Coxeter groups are shown to be the same as the quotients of the Bruhat order of that group by its parabolic subgroups.

A reflection whose fixed hyperplane bounds the fundamental chamber is called a *basic reflection*. Coxeter showed that the basic reflections deserve their name by forming a minimal generating set for the group. There are d of them if the group is finite and $d+1$ if it is infinite. The Hasse diagram of the stability order of G with respect to the basic reflections, called the *weak stability order*, is particularly easy to calculate since it is just $\bigcup_{i=0}^{k-1} \mathcal{SO}(V; \mathcal{R}_i; p)$ (with $k = d$ or $d+1$ as noted above). The Matsumoto-Verma Exchange Property implies that the weak and strong stability orders of G have a rank function, ℓ, (called *length*) which is the same for both. Altogether these observations give us a very simple 2-step process for constructing the Hasse diagram of the stability order of G. (Let $\mathcal{SO}_\ell = \{v \in V : \ell(v) = \ell\}$):

1. Generate the Hasse diagram of the weak stability order
   (a) Begin with the unique vertex, v_0, in the fundamental chamber: $\mathcal{SO}_0 = \{v_0\}$,
   (b) Extend from $\mathcal{SO}_\ell$ to $\mathcal{SO}_{\ell+1}$ by applying each basic reflection to each member of $\mathcal{SO}_\ell$. The result will either be in $\mathcal{SO}_{\ell-1}$ or $\mathcal{SO}_{\ell+1}$ so we need only eliminate those we know to be in $\mathcal{SO}_{\ell-1}$ to get those in $\mathcal{SO}_{\ell+1}$.

2. Examine all pairs (v, w), $v \in \mathcal{SO}_\ell$ and $w \in \mathcal{SO}_{\ell+1}$, to see if $v <_{SO} w$, i.e. if there exists a reflective symmetry, $\mathcal{R}$, such that $\mathcal{R}(v) = w$. Those for which it does, complete the Hasse diagram of $\mathcal{SO}$.