On Zero Sum Subsequences of Restricted Size III

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November 10, 2000

Abstract

Let $p > 2$ be a prime, and $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_k}}$ $(1 \le e_1 \le \cdots \le e_k)$ a finite abelian p group. We prove that $1 + 2\sum_{i=1}^{k}(p^{e_i} - 1)$ is the smallest integer t such that, every sequence of t elements in G contains a zero-sum subsequence of odd length. As a consequence, we derive that, if $p^{e_k} \ge 1 + \sum_{i=1}^{k-1}(p^{e_i} - 1)$ then every sequence of $4p^{e_k} - 3 + 2\sum_{i=1}^{k-1}(p^{e_i} - 1)$ elements in G contains a zero-sum subsequence of length p^{e_k}.

1 Introduction

In 1961, Erdös, Ginzburg and Ziv proved that if $a_1, a_2, \cdots, a_{2n-1}$ is a sequence of $2n - 1$ elements in a finite abelian group of order n (written additively) then 0 can be expressed as $0 = a_{i_1} + \cdots + a_{i_n}$ with $1 \le i_1 < \cdots < i_n \le 2n - 1$. This result is known as the Erdös-Ginzburg-Giv theorem and has been generalized in several directions ([1-7], [10-14], [18-21], [23]). Let G be a finite abelian group with exponent m, by $r(G)$ we denote the smallest integer t such that every sequence of t elements in G contains a zero-sum subsequence of length m. It is well known that $G = C_{n_1} \oplus \cdots \oplus C_{n_k}$ with $1 < n_1 | \cdots | n_k$, k is called the rank of G. Denote $1 + \sum_{i=1}^{k}(n_i - 1)$ by $M(G)$. Let C_m denote the cyclic group of order m, and C_m^k the product of k copies of C_m. $r(C_m^k)$ was first studied by Harborth [21]. Other than its own intersting, $r(G)$ has been used in the study on non-unique factorizations [17]. So far, we have the following results about $r(G)$.

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Proposition 1.1 *Let p be a prime, and G a finite abelian group of exponent m. Then,*
1) $r(C_m^2) = 4m - 3$ for $m = 2^a 3^b 5^c 7^d$. [22]
2) $r(C_m^2) = 4m - 3$ if $m = 2^a 3^b 5^c 7^d n$ and $n \le (2^{a+2} 3^{b-1} 5^c 7^d)^{1/3}$. [11]
3) $r(C_m^2) \le 6m - 5$ [1].
4) $r(G) \le (ck \log_2^k)^k m$, where k is the rank of G and c is a absolute constant. [2]
5) $r(G) \le |G| + m - 1$. [18]

In this paper we prove that

Theorem 1.2 *Let p be a prime, and $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_k}}$ $(1 \le e_1 \le \cdots \le e_k)$ a finite abelian p-group. Suppose that $p^{e_k} \ge 1 + \sum_{i=1}^{k-1}(p^{e_i} - 1)$. Then $r(G) \le 4p^{e_k} - 3 + 2\sum_{i=1}^{r-1}(p^{e_i} - 1)$.*

Let G be a finite abelian group of exponent m. For every positive integer $k \not| m$, by $E_k(G)$ we denote the samllest integer l such that every sequence of l elements in G contains a zero-sum subsequence T with $k \not| |T|$.

We shall derive the following main result of which Theorem 1.2 is an easy consequence.

Theorem 1.3 *Let $p > 2$ be a prime, and G a finite abelian p-group. Then, $E_2(G) = 2M(G) - 1$.*

2 Proof of Theorem 1.2 and Theorem 1.3

To prove Theorem 1.3 we need some preliminaries.
Let G be a finite abelian group and $S = (a_1, \cdots, a_l)$ a sequence of elements in G. By $\iota(S)$ we denote the sum $\sum_{i=1}^{l} a_i$. We say S a zero-sum sequence if $\iota(S) = 0$. A subsequence is a sequence $T = (a_{i_1}, \cdots, a_{i_t})$ with $1 \le i_1 < \cdots < i_t \le l$, we denote the index set $\{i_1, \cdots, i_t\}$ by I_T. We say subsequences $S_1, \cdots, S_u$ of S disjoint means that $I_{S_1}, \cdots, I_{S_u}$ disjoint. If subsequences $S_1, \cdots, S_u$ of S disjoint, then we define $S_1 + \cdots + S_u$ to be the subsequence X of S with $I_X = I_{S_1} \cup \cdots \cup I_{S_u}$, and define $S - S_1 - \cdots - S_u$ to be the sequence with index set $I_S - I_{S_1} - \cdots - I_{S_u}$. Sometimes, we denote $S_1 + \cdots + S_u$ also by $S_1 \cdots S_u$. Davenport's constant $D(G)$ is the smallest integer d such that every sequence of d elements in G contains a nonempty zero sum subsequence.

Lemma 2.1 *If p is a prime and G is a finite abelian p-group then $D(G) = M(G)$. [23]*

The following lemma is crucial.

Lemma 2.2 *[11] Let p be a prime, H a finite abelian p-group, and $G = H \oplus C_{p^n m}$. Suppose that $p^n \geq M(H)$ and suppose that*

$$m \geq \frac{p^n(p^n-2)|H|(r(C_{p^n} \oplus H) - 2p^n - M(H) + 2) - M(H) + 3}{2p^n}$$

Then, $r(G) \leq 2p^n m + p^n + M(H) - 3$.

Lemma 2.3 *Let p be a prime, H a finite abelian p-group, and $G = H \oplus C_{p^n m}$. Then, $2p^n m + E_2(H) - 2 \leq r(G)$.*

Proof. Let $l = E_2(H) - 1$, and let $a_1, \cdots, a_l$ be a sequence of l elements in H such that the sequence contains no zero-sum subsequence T with $2 \not| |T|$. Put

$$S = \left(\underbrace{\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}}_{p^n m - 1}, \underbrace{\begin{pmatrix} -1 \\ \mathbf{0} \end{pmatrix}, \cdots, \begin{pmatrix} -1 \\ \mathbf{0} \end{pmatrix}}_{p^n m - 1}, \begin{pmatrix} 0 \\ a_1 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ a_l \end{pmatrix} \right)$$

with $\mathbf{0}$ is the identify element in H.
Clearly, S contains no zero-sum subsequence of length $p^n m$. Therefore, $2p^n m + E_2(H) - 3 = |S| \leq r(G) - 1$. Hence, $2p^n m + E_2(H) - 2 \leq r(G)$. □

Lemma 2.4 *Let p be an odd prime, H a finite abelian p-group with $M(H) = p^n$ for some positive integer n. Then, $E_2(H) = 2M(H) - 1$.*

Proof. Choose positive integer m so that

$$m \geq \frac{p^n(p^n-2)|H|(r(C_{p^n} \oplus H) - 2p^n - M(H) + 2) - M(H) + 3}{2p^n}$$

Set $G = H \oplus C_{p^n m}$. By Lemma 2.2, $r(G) \leq 2p^n m + p^n + M(H) - 3$. It follows from Lemma 2.3 that $2p^n m + E_2(H) - 2 \leq 2p^n m + p^n + M(H) - 3$, therefore, $E_2(H) \leq p^n + M(H) - 1 = 2M(H) - 1$.

To prove the lower bound we consider the following example. Assume $H = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_k}}$. Let

$$x_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in H$$

with the ith coordinate is 1 and the other's is 0.
Put

$$S = (\underbrace{x_1, \cdots, x_1}_{p^{e_1}-1}, \underbrace{-x_1, \cdots, -x_1}_{p^{e_1}-1}, \cdots, \underbrace{x_k, \cdots, x_k}_{p^{e_k}-1}, \underbrace{-x_k, \cdots, -x_k}_{p^{e_k}-1})$$

Clearly, S contains no zero-sum subsequence T with $2 \not| |T|$. This implies $E_2(H) > |S| = 2M(H) - 2$. Hence, $E_2(H) = 2M(H) - 1$. □
From the proof of Lemma 2.4 we see that

Lemma 2.5 *If H is a finite abelian group of odd order. Then, $E_2(H) \geq 2M(H) - 1$.*

Lemma 2.6 *Let p be a prime, n a positive integer, H a finite abelian p-group, and $G = H \oplus C_{p^n}$. Suppose that, $E_2(G) = 2M(G) - 1$. Then, $E_2(H) = 2M(H) - 1$.*

Proof. Let $l = E_2(H) - 1$, and let $a_1, \cdots, a_l$ be a sequence of l elements in H such that the sequence contains no zero-sum subsequence T with $2 \not| |T|$. Consider the following sequence

$$S = \left(\underbrace{\begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}}_{p^n-1}, \underbrace{\begin{pmatrix} -1 \\ \mathbf{0} \end{pmatrix}, \cdots, \begin{pmatrix} -1 \\ \mathbf{0} \end{pmatrix}}_{p^n-1}, \begin{pmatrix} 0 \\ a_1 \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ a_l \end{pmatrix} \right)$$

with $\mathbf{0}$ is the zero element in H.
Clearly, S contains no zero-sum subsequence T with $2 \not| |T|$. Therefore, $2p^n + E_2(H) - 3 \leq E_2(G) - 1 = 2M(G) - 2 = 2(p^n - 1 + M(H)) - 2$. This gives that $E_2(H) \leq 2M(H) - 1$ and the lemma follows from Lemma 2.5. □

Proof of Theorem 1.3. Assume $G = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_k}}$. Note that

$$\begin{aligned} p^{2e_1e_2\cdots e_k} &= 1 + (p^{e_1e_2\cdots e_k} + 1)(p^{e_1e_2\cdots e_k} - 1) \\ &= 1 + (p^{e_1e_2\cdots e_k} + 1 - k)(p^{e_1e_2\cdots e_k} - 1) + \sum_{i=1}^{k} \frac{p^{e_1e_2\cdots e_k} - 1}{p^{e_i} - 1}(p^{e_i} - 1). \end{aligned}$$

Set $H = C_{p^{e_1e_2\cdots e_k}}^{p^{e_1e_2\cdots e_k}+1-k} \oplus C_{p^{e_1}}^{\frac{(p^{e_1e_2\cdots e_k}-1)}{p^{e_1}-1}} \oplus \cdots \oplus C_{p^{e_k}}^{\frac{(p^{e_1e_2\cdots e_k}-1)}{p^{e_k}-1}}$. Then, $M(H) = p^{2e_1e_2\cdots e_k}$. By Lemma 2.3, $E_2(H) = 2M(H) - 1$. By using Lemma 2.6 repeatedly, one can get $E_2(G) = 2M(G) - 1$. This completes the proof.

Lemma 2.7 *([8]) Let p be a prime, H a finite abelian p group, and $G = C_{p^n} \oplus H$. Suppose that $p^n \geq M(H)$. Then, every sequence of $2p^n + M(H) - 2$ elements in G contains a zero-sum subsequence of length p^n or $2p^n$.*

Proof of Theorem 1.2. Let S be a sequence of $4p^{e_k} - 3 + 2\sum_{i=1}^{k-1}(p^{e_i} - 1) = t$ elements in G. Set $H = C_{p^{e_1}} \oplus \cdots \oplus C_{p^{e_{k-1}}}$. Suppopse $S = (a_1, \cdots, a_t)$. Define

$$b_i = \begin{pmatrix} 1 \\ a_i \end{pmatrix}$$

for $i = 1, \cdots, t$.

Set $T = (b_1, \cdots, b_t)$. By Theorem 1.3, $E_2(C_{p^{e_k}} \oplus G) = 2M(C_{p^{e_k}} \oplus G) - 1 = |S| = |T|$. Therefore, there is a zero-sum subsequence W of T such that $2 \not| |W|$. By the making of T we must have, $p^n | T$. Since, $p^n \geq M(H)$, $|W| \leq |T| = |S| < 6p^n$. Therefore, $|W| = p^n, 3p^n or 5p^n$. Assume

$$W = \begin{pmatrix} 1 \\ a_{i_1} \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ a_{i_l} \end{pmatrix}$$

Put $U = (a_{i_1}, \cdots, a_{i_l})$. Then, $|U| = p^n, 3p^n$ or $5p^n$, and U is a zero-sum subsequence of S. We distinguish three cases.

Case 1. $|U| = p^n$ and we are done.

Case 2. $|U| = 3p^n$. By Lemma 2.7, there exists a zero-sum subsequence V of U with $|V| = p^n$ or $2p^n$. Therefore, V or $U - V$ is a zero-sum subsequence of length p^n.

Case 3. $|U| = 5p^n$. By Lemma 2.7, there exists a zero-sum subsequence V of U with $|V| = p^n$ or $2p^n$. Therefore, either V is a zero-sum subsequence of length p^n, or $U - V$ is a zero-sum subsequence of length $3p^n$ and it reduces to Case 2. □

3 Remaks and Open Problems

Quite recently, it was proved that $r(C_p^2) \leq 4p-2$ by Rónyai [26] and that $r(C_{p^k}^2) \leq 4p^k - 2$ by the author [16] , respectively. By using $r(C_p^2) \leq 4p-2$ one can prove by induction that $r(C_n^2) \leq 5n-4$ for any positive integer n.

Proposition 3.1 *If n is odd then $E_2(C_n) = 2n-1$.*

Proof. By the Erdös-Ginzburg-Ziv theorem we know that $E_2(C_n) \leq 2n-1$, and by Lemma 2.5 we see that the equality holds. □

Theorem 3.2 *Let n, m be odd positive integer with $m|n$. Then, $E_2(C_m \oplus C_n) = 2m + 2n - 3$.*

Proof. We proceed by induction on m. If $m = 1$ then it follows from Proposition 3.1.
Suppose this theorem is true for $m < k$ we prove it is true also for $m = k$. Let p be a prime divisor of m. Let ϕ be the natural homomorphsim from $C_m \oplus C_n$ onto C_p^2 with $\ker(\phi) = C_{\frac{m}{p}} \oplus C_{\frac{n}{p}}$. Let $S = (a_1, \cdots, a_{2m+2n-3})$ be a sequence of $2m+2n-3$ elements in $C_m \oplus C_n$. Set $\phi(S) = (\phi(a_1), \cdots, \phi(a_{2m+2n-3}))$. By applying $r(C_p^2) \leq 4p-2$ to $\phi(S)$ repeatedly, one can find $2\frac{m}{p} + 2\frac{n}{p} - 4$ disjoint subsequences $S_1, \cdots, S_{2\frac{m}{p}+2\frac{n}{p}-4}$ such that $\iota(S_i) \in C_{\frac{m}{p}} \oplus C_{\frac{n}{p}}$ and such that $|S_i| = p$ for $i = 1, \cdots, 2\frac{m}{p} + 2\frac{n}{p} - 4$. By Theorem 1.3 one can find a subsequence $S_{2\frac{m}{p}+2\frac{n}{p}-3}$ (say) of $S - S_1 - \cdots - S_{2\frac{m}{p}+2\frac{n}{p}-4}$ such that $\iota(S_{2\frac{m}{p}+2\frac{n}{p}-3}) \in C_{\frac{m}{p}} \oplus C_{\frac{n}{p}}$ and $|S_{2\frac{m}{p}+2\frac{n}{p}-3}|$ is odd. Now by the assumpation on the induction there is a subset I of $\{1, \cdots, 2\frac{m}{p} + 2\frac{n}{p} - 3\}$ such that $\sum_{i \in I} \iota(S_i) = 0$ and $|I|$ is odd. Hence, $\sum_{i \in I} S_i$ is a zero-sum subsequence of S and $\sum_{i \in I} |S_i|$ is odd. Therefore, $E_2(C_m \oplus C_n) \leq 2m + 2n - 3$. Now the theorem follows from Lemma 2.5. □

One can use Theorem 1.3 to get some graphic results in a similiar way to [3].

Problem To determine $E_k(G)$.
It seems not easy to determine $E_k(C_n)$ for all n , k with $k < n$ and $k \nmid n$. The readers can find several intersting but difficult open problems in [1] and [2].

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