DOUBLE ORDERINGS OF (0,1)-MATRICES

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ABSTRACT. There is a lexicographic ordering of $(0,1)$-tuples. Thus the rows of a $(0,1)$-matrix can be ordered lexicographically decreasing from the top by permutations, or analogously the columns from the left. It is shown that $(0,1)$-matrices allow a simultaneous ordering of the rows and the columns. Those matrices are called *doubly ordered*, and their structure is determined. An answer is given to the question, whether a $(0,1)$-matrix can be transformed into a block diagonal matrix by permutations of the rows and the columns; in fact, the double ordering of a $(0,1)$-matrix already displays the finest block diagonal structure. Moreover, fast algorithms are presented that double order a $(0,1)$-matrix.

1. INTRODUCTION

This paper deals with $(0,1)$-matrices, i. e., matrices whose entries are either 0 or 1. Two algorithms will be established that transform $(0,1)$-matrices into a doubly ordered form which means that its rows, considered as dyadic numbers, are in decreasing order from top to bottom, and simultaneously its columns, also considered as dyadic numbers, are in decreasing order from left to right. It is shown that a doubly ordered matrix displays its block diagonal structure. As an application the block structure of an arbitrary matrix can be found since for this purpose it only matters whether an entry of a given matrix is zero or not, and this reduces the question to $(0,1)$-matrices. Our approach is an alternative to the graph theoretic method that interprets the $(0,1)$–matrix as an encoding of a bipartite graph whose connected components correspond to the blocks of the matrix. There are well–known algorithms in graph theory that determine the connected components of a bipartite graph (see, for example, [2]). The double ordering may be applied to systems of linear equations. But there are several areas that might profit from this method. In particular, this topic is relevant for combinatorial matrix theory where $(0,1)$-matrices are also considered, cf. [1] and [4].

2. Double Ordering

All occurring matrices are $(0,1)$-matrices. A *line* of a matrix denotes either a row or a column. The set of $(0,1)$-rows allows a lexicographic ordering induced by $1 > 0$ from the left. Analogously $(0,1)$-columns allow a lexicographic ordering induced by $1 > 0$ from the top. This is equivalent to considering each of the rows of a $(0,1)$-matrix as dyadic representation. Then the lexicographic order of rows is just the usual linear order on natural numbers and an analogous statement is true for the lexicographic order of the columns. With this in mind we can talk about larger and smaller lines. A matrix is said to be *doubly ordered* if the set of the rows from top to bottom and the set of the columns from left to right simultaneously form descending sequences. Note that the transposed of a doubly ordered matrix is also doubly ordered. Two matrices A, B are called *permutation equivalent* if there are permutation matrices P, Q such that $B = PAQ$.

Every $(0,1)$-matrix allows line permutations such that a double ordering is obtained. The proof of this fact is given in the form of an explicit algorithm. For this we need some preparation. We define the *degree of order* $\operatorname{dgo}(M)$ of a $(0,1)$-matrix $M = [m_{ij}]$ of size m by n by setting

$$\begin{aligned}\operatorname{dgo}(M) &= \sum_{(i,j)} m_{ij}2^{m+n-i-j} = \sum_{i=1}^{m} 2^{m-i}\left(\sum_{j=1}^{n} m_{ij}2^{n-j}\right)\\ &= \sum_{j=1}^{n} 2^{n-j}\left(\sum_{i=1}^{m} m_{ij}2^{m-i}\right).\end{aligned}$$

We will verify in the following that exchanging a later row with an earlier row that is smaller results in an increase of the degree of order by at least one, and the same is true for columns.

Lemma 1. *The degree of order of a $(0,1)$-matrix increases under transpositions of lines if either a bigger row is permuted towards the top or a bigger column is permuted towards the left.*

Proof. Let M be a $(0,1)$-matrix of size m by n. Let r_i denote the number whose dyadic expansion is row i of M, i. e., $r_i = \sum_{j=1}^{n} m_{ij}2^{n-j}$. Suppose that $i_2 > i_1$ and $r_{i_2} > r_{i_1}$. Let M' be the matrix obtained by exchanging row i_1 and i_2 in M. Then

$$\begin{aligned}
\text{dgo}(M') - \text{dgo}(M) &= 2^{m-i_1} r_{i_2} + 2^{m-i_2} r_{i_1} - \left(2^{m-i_1} r_{i_1} + 2^{m-i_2} r_{i_2}\right) \\
&= r_{i_1}\left(2^{m-i_2} - 2^{m-i_1}\right) + r_{i_2}\left(2^{m-i_1} - 2^{m-i_2}\right) \\
&= 2^{m-i_2}\left[r_{i_1}(1 - 2^{i_2-i_1}) + r_{i_2}(2^{i_2-i_1} - 1)\right] \\
&= 2^{m-i_2}(r_{i_2} - r_{i_1})(2^{i_2-i_1} - 1) \geq 1.
\end{aligned}$$

By symmetry the same is true for columns. □

We show next that every $(0,1)$–matrix can be doubly ordered. The process is algorithmic and the algorithm has been programmed in C++.

Theorem 2. *Every* $(0,1)$*-matrix is permutation equivalent to a doubly ordered matrix. The doubly ordered matrix is obtained by interchangeably sorting rows and columns.*

Proof. By interchangeably sorting rows and columns the degree of order increases by Lemma 1 until the matrix is doubly ordered. This must happen in finitely many steps since the degree of order is bounded. □

Remark. The degrees of order of two permutation equivalent doubly ordered matrices need not be the same. The following is an example. Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

By first sorting the columns of M and then the rows of the resulting matrix we arrive at

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

which is doubly ordered and has $\text{dgo}(M_1) = 1668$. By first sorting the rows of M and then sorting the columns of the resulting matrix we obtain

$$M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

which is again doubly ordered and $\text{dgo}(M_2) = 1660 < \text{dgo}(M_1)$. Both matrices show that the first row of a doubly ordered matrix need not have the maximal rowsum. Another matrix that is permutation equivalent with M is

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

It was obtained by maximizing row and column sums and has degree of order 1753 that may be the largest possible in the periodic equivalence class.

We now investigate the structure of doubly ordered matrices using self-explanatory interval notation for intervals of integers.

Theorem 3. *Let M be a $(0,1)$-matrix of size m by n. Then M is doubly ordered if and only if there are two sequences of partitions*

(1) $\quad J_\ell = (j_0^\ell, j_1^\ell] \cup (j_1^\ell, j_2^\ell] \cup \cdots \cup (j_{2^\ell-1}^\ell, j_{2^\ell}^\ell] = [1,n], \quad 1 \le \ell \le m,$

where $0 = j_0^\ell \le j_1^\ell \le \cdots \le j_k^\ell \le j_{k+1}^\ell \le \cdots \le j_{2^\ell}^\ell = n$, *and*

(2) $\quad I_h = (i_0^h, i_1^h] \cup (i_1^h, i_2^h] \cup \cdots \cup (i_{2^h-1}^h, i_{2^h}^h] = [1,m], \quad 1 \le h \le n,$

where $0 = i_0^h \le i_1^h \le \cdots \le i_k^h \le i_{k+1}^h \le \cdots \le i_{2^h}^h = m$ *such that*

(3) $$j_{2k}^\ell = j_k^{\ell-1}, \quad \text{and} \quad i_{2k}^h = i_k^{h-1},$$

(4) $$M[\ell, j] = \begin{cases} 1 & \text{if } (\exists k)\ j \in (j_{2k}^\ell, j_{2k+1}^\ell] \\ 0 & \text{if } (\exists k)\ j \in (j_{2k+1}^\ell, j_{2k+2}^\ell] \end{cases}$$

and

(5) $$M[i, h] = \begin{cases} 1 & \text{if } (\exists k)\ i \in (i_{2k}^h, i_{2k+1}^h] \\ 0 & \text{if } (\exists k)\ i \in (i_{2k+1}^h, i_{2k+2}^h] \end{cases}.$$

Note that $j_k^\ell = j_{k+1}^\ell$ and $i_k^h = i_{k+1}^h$ are allowed so that parts of the partitions may be empty.

Proof. Let M be a doubly ordered $(0,1)$-matrix of size m by n. Thus ℓ ranges from 1 to m. We induct on ℓ. Let $\ell = 1$. The lexicographic order of the columns requires that in row 1 the entries equal to 1 (if any) must come before the entries equal to 0 (if any). Let $j_0^1 = 0$, $j_2^1 = n$ and let j_1^1 be such that

$$M[1, j] = \begin{cases} 1 & \text{if } j_0^1 < j \le j_1^1 \\ 0 & \text{if } j_1^1 < j \le j_2^1 \end{cases}.$$

In this case the relevant part of the requirement (3) is void and (1) and (4) are satisfied.

Now suppose that a partition (1) has been found such that the relevant part of (3) and (4) hold for $1 \leq \ell \leq L < m$. Let $j_{2k}^{L+1} = j_k^L$ so that the relevant part of (3) holds for $\ell = L + 1$. If $j_k^L = j_{k+1}^L$, then set $j_{2k}^{L+1} = j_{2k+1}^{L+1} = j_{2(k+1)}^{L+1}$. If $j_k^L < j_{k+1}^L$ and $j, j' \in (j_k^L, j_{k+1}^L]$ for some k, then $M[i,j] = M[i,j']$ for $i \leq L$. In this case the lexicographic order of columns requires that the entries of $M[L+1,j]$ for $j \in (j_k^L, j_{k+1}^L]$ are in decreasing order, i.e., the ones precede the zeros. Set $j_{2k+1}^{L+1} \in [j_{2k}^{L+1}, j_{2k+2}^{L+1}]$ equal to the column index for which

$$M[L+1,j] = \begin{cases} 1 & \text{if } j_{2k}^{L+1} = j_k^L < j \leq j_{2k+1}^{L+1} \\ 0 & \text{if } j_{2k+1}^{L+1} < j \leq j_{k+1}^L = j_{2(k+1)}^{L+1} \end{cases} .$$

Having done this for $k = 0, \ldots, 2^L$ we have a partition (1) for $\ell = L+1$ satisfying the relevant part of (3) and (4). The same procedure, mutatis mutandis, will produce I_h satisfying (3) and (5), and one direction of the claim is established.

Conversely, assume that M is a matrix and J_ℓ, I_h are sequences of partitions as in (1) and (2) such that (3), (4) and (5) hold. If M is of size m by n, then J_m and I_n are the finest of the partitions (1), (2). It suffices to show that an arbitrary row $M[L,*]$ is lexicographically larger than the adjacent row $M[L+1,*]$, and similarly for columns. Let $j \in \{1, \ldots, n\}$, and consider that entries $M[L,j]$ and $M[L+1,j]$. If there is a value k such that $L, L+1 \in (i_k^n, i_{k+1}^n]$, then $M[L,j] = M[L+1,j]$. If there is no such value k, then there is a least integer $h \geq 1$ and some value k such that $L \in (i_{2k}^h, i_{2k+1}^h]$ while $L+1 \in (i_{2k+1}^h, i_{2k+2}^h]$. By choice of h, we have $L, L+1 \in (i_k^{h-1}, i_{k+1}^{h-1}]$. Hence $M[L,j] = 1$ and $M[L+1,j] = 0$. Considering the possibilities in toto we have that $M[L,*]$ precedes $M[L+1,*]$ in the lexicographic order. The column order is established analogously. □

The matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

shows that both sequences J_ℓ and I_h are needed and also in full length.

There is an alternative characterization of double ordering.

Proposition 4. *Let M be a $(0,1)$-matrix of size m by n. Then the following are equivalent:*

(1) *M is doubly ordered.*
(2) *The set of the rows of M from top to bottom forms a descending sequence, and* (1), *the relevant part of* (3) *and* (4) *of Theorem* 3 *hold.*

(3) *The set of the columns of M from left to right forms a descending sequence, and* (2), *the relevant part of* (3) *and* (5) *of Theorem* 3 *hold.*

Proof. Theorem 3 shows that (1) implies (2). To prove that (2) implies (1) it remains to show that the columns of M form a descending sequence. Consider the columns j and $j+1$ of M. If j and $j+1$ belong to the same interval of the partition J_m, cf. Theorem 3 (1), then $M[i,j] = M[i,j+1]$ for $i = 1,\ldots,m$ by Theorem 3 (3) and (4), and $M[*,j] = M[*,j+1]$. Now assume that j and $j+1$ are not in the same interval of J_m. Let μ be minimal relative to the property that j and $j+1$ belong to different intervals of the partition J_μ. Note that $1 \le \mu \le m$. Then j and $j+1$ belong to the same interval of the partition $J_{\mu-1}$. Thus, by Theorem 3 (3) and (4), the columns $M[*,j]$ and $M[*,j+1]$ are equal up to row $\mu-1$. By Theorem 3 (4) they differ in the μ^{th} row. Moreover, if in some step from line to line an interval is decomposed into two sections, then in the first section all entries are 1 and in the second section all entries are 0, hence $M[\mu,j] = 1$ and $M[\mu,j+1] = 0$, i. e., $M[*,j] > M[*,j+1]$. This shows that M is doubly ordered.

The equivalence of (1) and (3) is shown analogously. □

In view of Proposition 4 it is not surprising that there is another algorithm that double orders a $(0,1)$-matrix row by row or column by column. In preparation it is convenient to establish notation. We need to consider partitions

$$(6) \qquad \begin{aligned} J &= (j_0,j_1] \cup \cdots \cup (j_{t-1},j_t] = [1,n], \\ 0 &= j_0 \le j_1 \le \cdots \le j_{t-1} \le j_t = n, \end{aligned}$$

of $[1,n] = \{1,\ldots,n\}$ that allow empty intervals $(j_{k-1},j_k]$, $j_{k-1} = j_k$. A permutation Q of $\{1,\ldots,n\}$ is said to *respect the partition* J if $(j_{k-1},j_k]Q = (j_{k-1},j_k]$ for each k, so that Q is in effect a product of permutations Q_k of the individual subintervals $(j_{k-1},j_k]$ with Q_k omitted if $(j_{k-1},j_k]$ is void. Let $u = [u_1,\ldots,u_n]$ be a $(0,1)$-row of length n. We introduce the row section $u(j_{k-1},j_k] = [u_{j_{k-1}+1},\ldots,u_{j_k}]$ and a permutation of $[1,n]$ in this context is understood to be a permutation matrix so that the matrix product uQ makes sense. Such a permutation matrix *respects the partition* J if $Q = \operatorname{diag}(Q_1,\ldots,Q_t)$ where each permutation matrix Q_k acts on the piece $u(j_{k-1},j_k]$ and is absent if $(j_{k-1},j_k]$ is empty.

We also need the following concept. A row v of a matrix is said to *cover properly* another row u if for every column permutation Q the row vQ is larger than the row uQ. Note that, in this case, the row sum of v is necessarily larger than that of u, and that v has a 1 in each column in which u has a 1. E.g., while the first row of

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is lexicographically larger than the second, the first does not cover the second since a permutation of the columns produces the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

in which the second row is lexicographically larger.

The following observation will be used later.

Lemma 5. *Let u, v be $(0,1)$-rows and that v does not properly cover u. If $u = [1, \dots, 1, 0, \dots, 0]$ (all ones before all zeros), then u is lexicographically larger than or equal to vQ for any permutation matrix Q.*

Proof. If vQ were greater than $u = [1, \dots, 1, 0, \dots, 0]$, then vQ, and hence v, would cover u contrary to hypothesis. □

Theorem 6. *There is an algorithm doubly ordering a $(0,1)$-matrix that works line by line.*

Proof. We describe this algorithm and show that it leads to the desired result. The algorithm starts with a certain row or column of the matrix. Obviously we have to start with a row or a column that has no proper cover. Here we start with a row. Let M be a $(0,1)$-matrix of size m by n. We associate with this matrix a recursively determined sequence of pairs $(M_\ell, J_\ell)_{0 \le \ell \le m}$ with the following properties.

(1) M_ℓ is permutation equivalent to M,

(2) $J_\ell = (j_0^\ell, j_1^\ell] \cup (j_1^\ell, j_2^\ell] \cup \cdots \cup (j_{2\ell-1}^\ell, j_{2\ell}^\ell] = [1, n]$ with $0 = j_0^\ell \le j_1^\ell \le \cdots \le j_{2\ell}^\ell = n$ is a partition of $[1, n]$ into (possibly void) intervals,

(3) for $0 \le L < \ell$, $j_{2k}^{L+1} = j_k^L$,

(4) for $1 \le i \le \ell$

$$M_\ell[i, j] = \begin{cases} 1 & \text{if} \quad (\exists k)\ j \in (j_{2k}^i, j_{2k+1}^i] \\ 0 & \text{if} \quad (\exists k)\ j \in (j_{2k+1}^i, j_{2k+2}^i] \end{cases},$$

(5) for $L = 1, \dots, \ell$ the row $M_\ell[L, *]$ is lexicographically larger than or equal to any permuted row $M_\ell[i, *]Q$ with $i > L$ provided that Q respects J_L.

Let $M_0 = M$, $J_0 = (0, n]$ and, given (M_ℓ, J_ℓ) determine $(M_{\ell+1}, J_{\ell+1})$ recursively as follows:

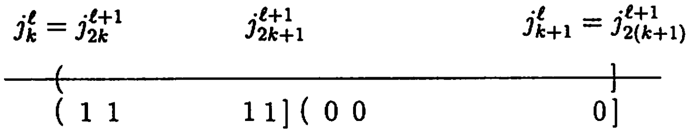
(i) The rows $1, \dots, \ell$ are unchanged. Choose any row, say u, from the set $\{M_\ell[\ell+1, *], \dots, M_\ell[m, *]\}$ of the last $m - \ell$ rows of M_ℓ such that the leading parts $u(0, j_k^\ell]$ have no proper cover among the leading parts $M_\ell[i, *](0, j_k^\ell]$ for $i = \ell+1, \dots, m$. Choose a permutation matrix P such that PM_ℓ has the rows $M_\ell[1, *], \dots, M_\ell[\ell, *], u, \dots$.

(ii) There is a column permutation $Q = \operatorname{diag}(Q_1, \dots, Q_{2^\ell})$ respecting J_ℓ, such that the new row uQ has entries $[1, \dots, 1, 0, \dots, 0]$ (all ones before all zeros) in each interval of J_ℓ. Define (with the permutation P in (i))

$$M_{l+1} = PM_lQ.$$

(iii) By choice of Q an interval $(j_k^\ell, j_{k+1}^\ell]$ of the partition J_ℓ is occupied in row uQ by a list $(uQ)[j_k^\ell + 1, \dots, j_{k+1}^\ell] = [1, \dots, 1, 0, \dots, 0]$. Let $j_{2k}^{\ell+1} = j_k^\ell$, and let $j_{2k+1}^{\ell+1} \in [j_{2k}^{\ell+1}, j_{2k+2}^{\ell+1}$ be such that

$$(uQ)[j] = \begin{cases} 1 & \text{if } j \in (j_{2k}^{\ell+1}, j_{2k+1}^{\ell+1}] \\ 0 & \text{if } j \in (j_{2k+1}^{\ell+1}, j_{2k+2}^{\ell+1}] \end{cases}.$$



Done for all intervals of J_ℓ this leads to a refinement $J_{\ell+1}$ of the interval partition J_ℓ.

The pair $(M_{\ell+1}, J_{\ell+1})$ clearly satisfies condition (1) through (4) by construction. Condition (5) is true by construction and Lemma 5.

After m steps this algorithm leads to a matrix M_m and an interval partition J_m. The rows of M_m are lexicographically non-increasing by (5).

By Conditions (2), (3) and (4) above the hypothesis of Proposition 4 (2) is satisfied, hence M_m is doubly ordered. □

Remark. By (i) the algorithm of Theorem 6 can be started with any row of the matrix that has no proper cover.

A row with maximal row sum never has a proper cover, so we may always replace the relation "cover" by "maximal row sum" in the above algorithm.

3. Block Diagonal Form

The purpose of this section is to prove that a doubly ordered (0,1)-matrix M displays the block diagonal structure of M, i.e., if M is permutation equivalent to a block diagonal matrix A with indecomposable diagonal blocks, then M itself has block diagonal structure with indecomposable blocks of the same size as A.

Let $M = [m_{ij}]_{1\le i\le m,\ 1\le j\le n}$ be a matrix of size m by n. A *submatrix* of M is a matrix obtained be deleting lines of M. A submatrix of M of the form

$$M[m_1..m_2, n_1..n_2] = [m_{ij}]_{m_1\le i\le m_2,\ n_1\le j\le n_2}$$

is called a *block* of M. If $m = m_1 = m_2$, then we write $M[m, n_1..n_2]$. A block of the form $M[1..r, 1..s]$ is called a *main block*. Note that a main block of a doubly ordered matrix is also doubly ordered.

A matrix M is said to be in *block diagonal form* or a *block diagonal matrix* if $M = \operatorname{diag}(M_1, \dots, M_k)$ where the blocks M_i are rectangular and all entries outside of the blocks are 0. A matrix M, not of size 1 by 1, without 0-lines, is said to be *indecomposed* if it has only the trivial block diagonal form, i. e., $M = \operatorname{diag}(M)$. A matrix of size 1 by 1 is indecomposed. Matrices with 0-lines are decomposed.

A matrix is called *indecomposable* if it is not permutation equivalent to a decomposed matrix. The *support* M^* of the matrix M is the submatrix obtained by deleting all 0-lines of M. A matrix M is called *totally decomposed* if its support $M^* = \operatorname{diag}(M_1, \dots, M_k)$ is a block diagonal matrix with all blocks M_i indecomposable.

There are *total decompositions* of matrices relative to periodic equivalence, i. e., block diagonal forms of the support such that the diagonal blocks are indecomposable. We will show that the size of the indecomposable diagonal blocks of those total decompositions is unique up to rearrangements of the diagonal blocks.

Note that a block diagonal matrix $A = \operatorname{diag}(A_1, \dots, A_k)$ is doubly ordered if and only if all its diagonal blocks A_i are doubly ordered.

The following technical lemma settles a special situation that occurs in the proof of the next theorem.

Lemma 7. *Let M and A be doubly ordered* (0,1)*-matrices of size m by n, that are permutation equivalent, i. e., $PMQ = A$. Let $A = \operatorname{diag}(A_1, \dots)$ be in block diagonal form and let $A_1 \ne A$ be indecomposed of size l by k. Let $P = \operatorname{diag}(P', P'')$, $Q = \operatorname{diag}(Q', Q'')$ where P', Q'*

are of size $m' \leq l$, $n' \leq k$ *respectively, and* $m' + n' < l + k$. *Then*

$$\begin{aligned} P \cdot M[1..m, 1..n'] \cdot Q' &= A[1..m, 1..n'], \\ P' \cdot M[1..m', 1..n] \cdot Q &= A[1..m', 1..n], \end{aligned}$$

and if $M[1..m', 1..n']$ *has no* 0*-line, then there exist permutation matrices* P_1, Q_1 *of size* m_1 *and* n_1 *respectively, where*

$$m' \leq m_1 \leq l, \quad n' \leq n_1 \leq k, \quad m' + n' < m_1 + n_1,$$

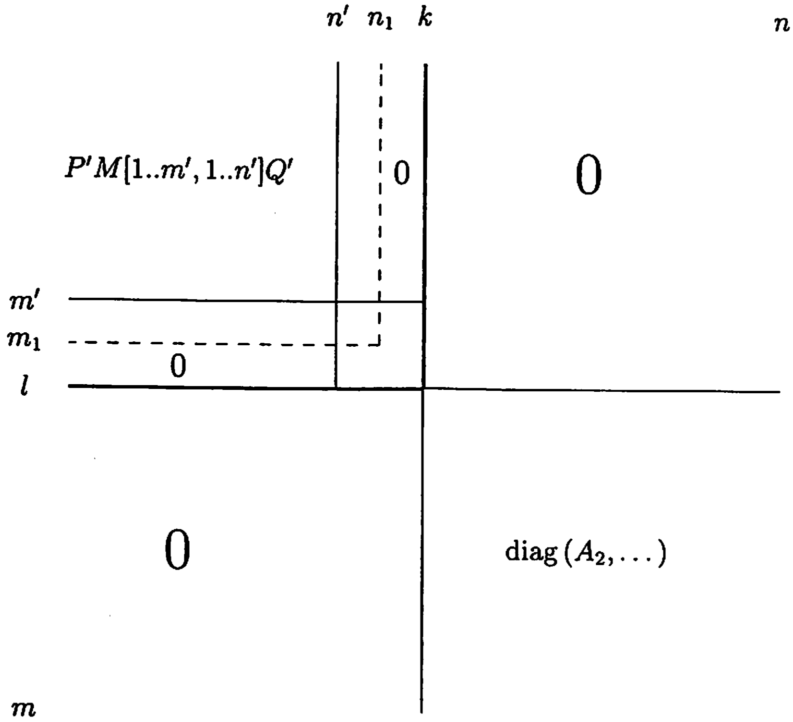
such that $M[1..m_1, 1..n_1]$ *has no* 0*-line and*

$$\begin{aligned} P \cdot M[1..m, 1..n_1] \cdot Q_1 &= A[1..m, 1..n_1], \\ P_1 \cdot M[1..m_1, 1..n] \cdot Q &= A[1..m_1, 1..n]. \end{aligned}$$

Proof. Since permutation equivalent matrices have the same number of 0-rows and the same number of 0-columns, and since doubly ordered matrices have 0-lines either all at the right or all at the bottom, we may restrict to the case that the matrices M and A coincide with their supports, respectively, i. e., they have no 0-lines. Hence

$$k < n \quad \text{and} \quad l < m,$$

since otherwise either there is a 0-line or $A = A_1$. By hypothesis the main blocks of size m by n' of M and of A are permutation equivalent and the main blocks of size m' by n are permutation equivalent. Since all of these main blocks are doubly ordered, we obtain that their lower or their right 0-blocks, respectively, have the same size. Since A has block diagonal form and $M[1..m', 1..n']$ has no 0-line, there are minimal n_1, m_1, with $n' \leq n_1 \leq k$ and $m' \leq m_1 \leq l$, such that $M[m_1 + 1..m, 1..n'] = 0$ and $M[1..m', n_1 + 1..n] = 0$. Thus the permutation matrices P and Q have the form $P = \operatorname{diag}(P', P'_1, P'_2)$ and $Q = \operatorname{diag}(Q', Q'_1, Q'_2)$, where P'_1 is of size $m_1 - m'$ and Q'_1 is of size $n_1 - n'$. If $m_1 = m'$ or $n_1 = n'$ then P'_1 or Q'_1 do not exist, respectively.



It remains to show that $m' + n' = m_1 + n_1$ is impossible. But in this case $m' = m_1$ and $n' = n_1$. If $m' < l$ and $n' < k$ then

$$M[1..l, 1..k] = \text{diag}\,(M[1..m', 1..n'], M[m'+1..l, n'+1..k]),$$

a block diagonal matrix. Since $m' < l$ and $n' < k$, this is a proper decomposition of M which, because of $PMQ = A$, leads to a proper decomposition of A_1, but A_1 was indecomposed. By hypothesis $m' = l$ and $n' = k$ cannot hold simultaneously. So assume that either $m' = l$ or $n' = k$. In the first case $m' = l$ and $n' = n_1 < k$, we obtain $M[1..m, n_1 + 1..k] = 0$ and M contains a 0-column, contradicting the fact that we took M to coincide with its support. The second case $m' = m_1 < l$ and $n' = k$ leads to a 0-row. This completes the proof. □

We are now ready to show that one can recognize whether a matrix is decomposable by inspecting any doubly ordered permutation equivalent matrix.

Theorem 8. *A decomposable doubly ordered* $(0,1)$*-matrix is decomposed and totally decomposed. The diagonal block structure of this total decomposition is an invariant of its periodic equivalence class, i. e., the size of the indecomposable diagonal blocks is unique up to rearrangement of the blocks.*

Proof. Let M be a doubly ordered $(0,1)$-matrix of size m by n. We may assume that M coincides with its support. Suppose that M is permutation equivalent to some proper block diagonal matrix $A = \operatorname{diag}(A_1, \dots)$, where $A_1 \neq A$ is indecomposable of size l by k.

First we will show that M is decomposed with a diagonal block permutation equivalent to A_1. The periodic equivalence of M and A is preserved if the blocks of A are arranged in a different order or if the individual blocks are doubly ordered according to the algorithm of Theorem 6. We will take advantage of this fact later.

Necessarily the first row z_1 of M is of the form

$$z_1 = (\underbrace{1, \dots, 1}_{n'}, 0, \dots, 0)$$

with row sum n'. Since $PMQ = A$ with permutation matrices P, Q, the row z_1 of M is permuted to some row u_1 of A. This row is in one of the blocks of A. Since the rearrangement of the blocks of A is done by a permutation of lines, we may assume that u_1 is in the first block A_1 of A. Moreover, we assume that u_1 is the first row of A_1. Since M is doubly ordered the first row z_1 never has a proper cover among the rows of M, cf. Lemma 5. So u_1 has no proper cover among the rows of A, and by doubly ordering the block A_1 according to the algorithm of Theorem 6 using the Remark, we may assume that the block A_1 is doubly ordered and the first row of A is equal to z_1. Altogether we have

$$PMQ = A \quad \text{and} \quad P = \operatorname{diag}(P', P''), \quad Q = \operatorname{diag}(Q', Q''),$$

where $P' = (1)$ is of size 1, Q' is of size n' where $n' \leq k$. Moreover,

$$M[1, 1..n] \cdot Q = P' \cdot M[1, 1..n] \cdot Q = A[1, 1..n] = z_1$$

and

$$P \cdot M[1..m, 1..n'] \cdot Q' = A[1..m, 1..n'].$$

This is a setting where Lemma 7 applies. Iterating Lemma 7 we end up with $P = \operatorname{diag}(P^*, P^{**}), Q = \operatorname{diag}(Q^*, Q^{**})$, where P^*, Q^* are of size l and k respectively, showing that

$$M = \operatorname{diag}(M[1..l, 1..k], M[l+1..m, k+1..n]),$$

and

$$P^* M[1..l, 1..k] Q^* = A_1.$$

Thus M is decomposed and since the diagonal blocks of a decomposable doubly ordered matrix are also doubly ordered this is a total decomposition. Moreover, since the indecomposable block A_1 was arbitrary among the occuring diagonal blocks of any decomposed matrix A permutation equivalent to M, it is obtained that the diagonal block structure of M is, up to rearrangement of the diagonal blocks, an invariant of the periodic equivalence class of M. □

An immediate consequence of Theorem 8 is the following corollary.

Corollary 9. *The block structure of totally decomposed representatives in a class of permutation equivalent matrices is unique up to rearrangement of the diagonal blocks.*

As representatives of a class of permutation equivalent matrices we can choose doubly ordered and totally decomposed matrices. Note that there can be different doubly ordered indecomposable matrices that are permutation equivalent. For example, the different earlier matrices

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

are doubly ordered and both permutation equivalent to the matrix M, so permutation equivalent to one another. Specifically,

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

But there is always a unique lexicographically maximal doubly ordered representative in a class of permutation equivalent matrices, if we for instance take the sequence of rows to be lexicographically maximal.

4. Open Questions

There are some natural questions in this context.

(1) Find an algorithm to decide whether two indecomposable doubly ordered $(0, 1)$-matrices are permutation equivalent.
(2) Are two invertible and indecomposable doubly ordered and permutation equivalent $(0, 1)$-matrices permutation similar, i. e., exists a permutation matrix P such that $B = PAP^{-1}$?
(3) Determine the periodic similarity classes of doubly ordered matrices.
(4) The degrees of order of all doubly ordered matrices in a periodic equivalence class form a set of invariants of this class. What can be said about those sets?
(5) How many matrices in a periodic equivalence class have maximal degree of order?
(6) Find an algorithm that determines the matrices of maximal degree of order within a periodic equivalence class.
(7) The *term rank* of a $(0, 1)$-matrix is the maximal number of 1's with no two of the 1's on a line. This is an invariant of a periodic equivalence class of matrices. Is the double ordering helpful, to determine the term rank of a $(0, 1)$-matrix?

Remark. There are fast C-programs for both algorithms double ordering $(0, 1)$-matrices, cf. [3]. Double ordering of a square matrix of size 1000 on a PC takes less than one second. These algorithms seem to be quadratic in the size of the matrices, whereas the graph theoretic algorithm to find the block structure is linear. But we would like to emphasize that double ordering remains interesting in itself beyond getting a block decomposition. Sparse matrices, even if they are indecomposable, get a characteristic shape, since double ordering concentrates the entries 1 close to the diagonal.

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