n-COLOR ANALOGUES OF GAUSSIAN POLYNOMIALS

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Abstract : Various n-color restricted partition functions are studied. Two different n-color analogues of the Gaussian polynomials are given.

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1. INTRODUCTION

Planar analogues of various results of the classical theory of partitions are found in the literature (cf. MacMahon [11], Gordon and Houten [7], Gordon [6] and Stanley [12]). In this paper we study various restricted n-color partition functions and give two different n-color analogues of the Gaussian polynomials defined by (cf. [10, Def. 3.1])

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_r}{(q;q)_k (q;q)_{r-k}} & \text{if } 0 \le k \le r \\ 0 & \textit{Otherwise}, \end{cases} \tag{1.1}$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

We know that

$$\begin{bmatrix} r+k \\ k \end{bmatrix} = \sum_{\nu \ge 0} p(r,k,\nu) q^\nu, \tag{1.2}$$

where $p\,(r,k,\nu)$ denotes the number of ordinary partitions of ν into at most k parts, each $\le$ r.

We first recall the definition of an n-color partition from [1,3].

Definition. An n-color partition (or, a partition with "n copies of n") is a partition in which a part of size n, n≥1 can come in n different colors denoted by subscripts : n_1, n_2,..........,n_n. Thus, for example there are six n-color partitions of 3, viz., 3_1, 3_2, 3_3, 2_1+1_1, 2_2+1_1, $1_1+1_1+1_1$. In [3] it was shown that if $P(\nu)$ denotes the number of n-color partitions of ν then

$$1+\sum_{\nu=1}^{\infty} P(\nu) q^{\nu}=\prod_{n=1}^{\infty}(1-q^{n})^{-n}. \tag{1.3}$$

Since the right-hand side of (1.3) is the generating function for plane partitions, this implies that the number of n-color partitions of ν equals the number of plane partitions of ν. in [1,3] several new Rogers – Ramanujan type identities were found using these partitions. Recently, in [4] a graphical representation for n-color partitions was given and conjugate and self-conjugate n-color partitions were defined. In this paper we shall study various restricted n-color partition functions. In Sections 2 and 3 we study two different n-color analogues of the partition function $p(r, k, \nu)$ appearing in Equation (1.2) above. This enables us to obtain two different n-color analogues of the Gaussian polynomials $\begin{bmatrix} r \\ k \end{bmatrix}$ and their following properties (cf.[10,Th. 3.2])

$$\text{degree} \begin{bmatrix} r \\ k \end{bmatrix} = k(r-k) \tag{1.4}$$

$$\begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} = 1 \qquad 1.5)$$

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r \\ r-k \end{bmatrix} \tag{1.6}$$

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k \end{bmatrix} + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} \tag{1.7}$$

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} + q^{k} \begin{bmatrix} r-1 \\ k \end{bmatrix} \tag{1.8}$$

In Section 4 we obtain generating functions for several other restricted n-color partition functions. In Section 5 we obtain two combinatorial identities using the generating functions of Section 4. One of our identities Theorem 5.1 is a true n-color analogue (very similar in structure) of the following famous identify due to Euler (cf. [10, Cor. 1.2]).

Theorem (*). The number of partitions of ν into distinct parts equals the number of partitions of ν into odd parts.

We conclude in Section 6 by posing three open problems.

The most important tool in this work is the bijection $\psi.\phi$ established recently in [2] between plane partitions of ν, on the one hand and the n-color partitions of ν on the other. For the clarity of our presentation we shall first recall some definitions from [9,12] and reproduce the bijection $\psi.\phi$ here.

A plane partition Π of ν is an array of non-negative integers.

$$\begin{array}{|cccc} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ . & . & . & \\ . & . & . & \\ . & . & . & \end{array}$$

for which $\sum_{i,j} n_{ij} = n$ and the rows and columns are in non-increasing order : $n_{ij} \geq n_{(i+1)j}, n_{ij} \geq n_{i(j+1)}$, for all $i,j \geq 1$. The non-zero entries $n_{ij}>0$ are called the parts of Π . A plane partition is called symmetric if $n_{ij}=n_{ji}$ for all i and j. If there are λ_i parts in the i-th row of Π, so that, for some r,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r \geq \lambda_{r+1} = 0,$$

then we call the partition $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r$ of the integer $p = \lambda_1 + \ldots + \lambda_r$, the shape of Π. If the non-zero entries of Π are strictly decreasing in each column, we say that Π is column strict. And if the non-zero elements of Π are strictly decreasing in each row; we shall call such a partition strict. The conjugate trace of Π is defined to be the number of parts n_{ij} of Π satisfying $n_{ij} \geq i$. In $\psi.\phi$, ϕ is due to Knuth [8.Th.2] and is the 1-1 correspondence of the following theorem:

Theorem (Knuth). There is a one-to-one correspondence between ordered pairs (Π_1, Π_2) of column-strict plane partitions of the same shape and matrices (a_{ij}) of non-negative integers. In this correspondence

(i) k appears in Π_1 exactly $\sum_i a_{ik}$ times, and

(ii) k appears in Π_2 exactly $\sum_i a_{ki}$ times.

A different version of this theorem known as Bender and Knuth Theorm is also found in literature (cf. Bender and Knuth [9], Nijenhuis and Wiff [5]).

Theorem (Bender and Knuth). There is a one-to-one correspondence between plane partitions of ν, on the one hand, and infinite matrices $a_{ij}\,(i, j \geq 1)$ of non-negative integer entries which satisfy.

$$\sum_{r\geq 1} r\left\{\sum_{i+j=r+1} a_{ij}\right\} = \nu$$

on the other.

In the sequal we shall call images $\phi\,(\Pi)$ K_ν -matrices (K for Knuth). Although these matrices are infinite matrices, but we will represent them by largest possible square matrices containing at least one non-zero entry in the last row (or last column). Thus, for example, we will represent six K_3-matrices by

$$3, \quad \begin{matrix}1 & 0,\\ 1 & 0\end{matrix} \quad \begin{matrix}1 & 1,\\ 0 & 0\end{matrix} \quad \begin{matrix}0 & 0,\\ 0 & 1\end{matrix} \quad \begin{matrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{matrix} \quad \begin{matrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{matrix}$$

Before we establish the bijection ψ between K_ν-matrices and the n-color partitions of ν, we give a definition.

Definition 1.1. We define a matrix $E_{i,j}$ as an infinite matrix whose (I,j)-th entry is 1 and the other entries are all zeros. We call $E_{i,j}$ distinct part of a K_ν-matrix.

Now we define the mapping ψ as follows:

Let $\Delta = a_{1,1}E_{1,1} + a_{1,2}E_{1,2} + \ldots + a_{2,1}E_{2,1} + a_{2,2}E_{2,2} + \ldots$

$+ a_{3,1}E_{3,1} + a_{3,2}E_{3,2} + \ldots$

be a K_ν-matrix where $a_{i,j}$ are non-negative integers which denote the multiplicities of $E_{i,j}$. We map each part $E_{p,q}$ of Δ to a single part m_i of an n-color partition of ν. The mapping ψ is

$$\psi : E_{p,q} \to (p+q-1)_p, \qquad (1.9)$$

and the inverse mapping ψ^{-1} is easily seen to be

$$\psi^{-1} : m_i \to E_{i,m-i+1}. \qquad (1.10)$$

Under this mapping we see that each K_ν-matrix uniquely corresponds to an n-color partition of ν and vice-versa. The composite of the two mappings ϕ and ψ denoted by $\psi.\phi$ is clearly a bijection between plane partitions of ν, on the one hand, and the n-color partitions of ν on the other.

To illustrate this bijection we consider the case for $\nu=3$ (Table 1).

TABLE1

Plane partitions of 3 Π	$\phi\ (\Pi)$	$\psi.\phi(\Pi)$
1 1 1	3 $=3E_{1,1}$	$3 1_1 = 1_1+1_1+1_1$
2 1	1 1 0 0 $=E_{1,1}+E_{1,2}$	1_1+2_1
1 1 1	1 0 1 o $=E_{1,1}+E_{2,1}$	1_1+2_2
3	0 0 1 0 0 0 0 0 0 $=E_{1,3}$	3_1
2 1	0 0 0 1 $=E_{2,2}$	3_2
1 1 1	0 0 0 0 0 0 1 0 0 $=E_{3,1}$	3_3

2. FIRST ANALOGUE OF THE GAUSSIAN POLYNOMIALS

In the ordinary restricted partition function $p(r,k,\nu)$ there are two restrictions : one is on the number of parts which is $\leq k$ and the other is on the size of the parts as each part $\leq$ r. if we look at an n-color partition, say, $\Pi=(a_1)_{b_1}+(a_2)_{b_2}+...+(a_m)_{b_m}$, we see that three restrictions can be considered : first on the number of parts, second on the size of each subscript b_i, and the third on the size of each part a_i. Considering these three restrictions we define two different restricted n-color partition functions as follows :

Definition 2.1. Let $P_1(r, k, m, \nu)$ denote the number of n-color partitions of ν into exactly m parts such that each subscript $b_i \leq r$ and each part $a_i \leq k+b_i-1$.

Definition 2.2. Let $P_2(r, k, m, \nu)$ denote the number of n-color partitions of ν into exactly m parts such that each subscript

$b_i \leq$ r and each part $a_i \leq$ k.

In this section we shall study the function $P_1(r, k, m, \nu)$ and in the next section we shall study the function $P_2(r, k, m, \nu)$.

Our first object here is to prove the following result :

Theorem 2.1. We have

$$\sum_{\nu=0}^{\infty}\sum_{m=0}^{\infty} P_1(r,k,m,\nu) z^m q^\nu = \prod_{j=1}^{k} \frac{1}{(zq^j;q)_r}. \tag{2.1}$$

Proof. Under the mapping ϕ every plane partition Π of ν corresponds to a K_ν-matrix say, Δ, via an ordered pair (Π_1, Π_2) of column strict plane partitions of the same shape. Stanley [12] pointed out that the number of rows of Π equals the largest part of Π_2, the largest part of Π equals the largest part of Π_1 and the conjugate trace of Π equals the number of parts of Π_1 or Π_2.

Now suppose

$$\Delta = \sum a_{p,q} E_{p,q} \text{ and}$$
$$\psi(\Delta) = \sum a_{p,q}(p+q-1)_p = \sum a_{p,q}(a_i)_{b_i},$$

where a_i=p+q−1 and b_i=p

Under the mapping ψ the largest part of Π_2 is the largest p which is the largest subscript in the corresponding n-color partition under the mapping ψ. Similarly, under the mapping ϕ the largest part of Π_1 is the largest q which is the largest $(p+q-1) - p+1 = a_i - b_i + 1$ under the mapping ψ.

Also, the number of parts of Π_1 or Π_2 equals the number of parts $E_{p,q}$ of Δ under the mapping ϕ which equals the number of parts $(a_i)_{b_i}$ of the corresponding n-color partition under the mapping ψ. We thus conclude that our restricted n-color partition function $P_1(r, k, m, \nu)$ also enumerates plane partitions of ν with $\leq$ r rows, largest part $\leq$ k, and with conjugate trace m. now an appeal to Stanley's Theorem (Section 2.2, p.56, [12]) proves our Theorem 2.1.

To illustrate the method of the proof we have given, we consider the plane partition $\begin{matrix}2&1\\2&\\1&\end{matrix}$ of 6. We see that

$$\begin{matrix}21\\2\\1\end{matrix} \xrightarrow[\text{via}\begin{pmatrix}2&1&3&1\\1&,1&&\end{pmatrix}]{\phi} \begin{matrix}1&1&0\\0&0&0\\1&0&0\end{matrix} = E_{1,1}+E_{1,2}+E_{3,1} \xrightarrow{\psi} 1_1 2_1 3_3$$

Now we define the first analogue $A_1(r,\ k,\ m;\ q)$ of the Gaussian polynomials by

$$A_1(r,k,m;q) = \sum_{v\geq 0} p_1(r,k,m,v)q^v. \tag{2.2}$$

Remark. Since $P_1(r,\ k,\ m,\ v)=0$ if $v > m\ (k+r-1)$ and $P_1\ (r,\ k,\ m,\ m(k+v\ -1))=1$, therefore $A_1\ (r,k,m;q)$ is a polynomial in q of degree $m(k+r-1)$.

Using Theorem 2.1 and the Taylor's theorem, we get the following formula :

$$A_1(r,k,m;q) = \frac{1}{m!}\frac{d^m}{dz^m}\left[\prod_{j=1}^{k}\frac{1}{(zq^j;q)_r}\right]_{z=0}. \tag{2.3}$$

Formula (2.3) is implementable on computer.

Using this formula on MACSYMA – a symbolic Mathematics software, we obtained the following table of $A_1(r,\ k,\ m;\ q)$ for $1\leq$ r, k$\leq$ 4 and m=2

TABLE –2

	1	2	3	4
1	q^2	$q^2+q^3+q^4$	$q^2+q^3+2q^4+q^5+q^6$	$q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8$
2	$q^2+q^3+q^4$	$q^2+2q^3+4q^4+2q^5+q^6$	$q^2+2q^3+5q^4+5q^5+5q^6+2q^7+q^8$	$q^2+2q^3+5q^4+6q^5+8q^6+6q^7+5q^8+2q^9+q^{10}$
3	$q^2+q^3+2q^4+q^5+q^6$	$q^3+2q^3+5q^4+5q^5+5q^6+2q^7+q^8$	$q^2+2q^3+6q^4+8q^5+11q^6+8q^7+6q^8+2q^9+q^{10}$	$q^2+2q^3+6q^4+9q^5+14q^6+14q^7+14q^8+9q^9+6q^{10}+2q^{11}+q^{12}$
4	$q^2+q^3+2q^4+2q^5+2q^6+q^7+q^8$	$q^2+2q^3+5q^4+6q^5+8q^6+6q^7+5q^8+2q^9+q^{10}$	$q^2+2q^3+6q^4+9q^5+14q^6+14q^7+14q^8+9q^9+6q^{10}+2q^{11}+q^{12}$	$q^2+2q^3+6q^4+10q^5+17q^6+20q^7+24q^8+20q^9+17q^{10}+10q^{11}+6q^{12}+2q^{13}+q^{14}$

Analogues to the properties (1.4) – (1.8) of the Gausssian polynomials we shall now prove properties of the polynomials $A_1(r, k, m, ;q)$.

Theorem 2.2. The polynomials $A_1(r, k, m, ;q)$ satisfy the following relations.

$$\text{degree } A_1(r, k, m ;q)=m (r+k-1), \tag{2.4}$$

$$A_1(r, k, 0 ;q)=1, \tag{2.5}$$

$$A_1(r, k, m ;q).= A_1(k, r, m ;q), \tag{2.6}$$

for $1 \le r \le m$,

$$q^m A_1(r,k,m;q) = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} (-1)^j q^{j(j+1)/2} A_1(r,k+1,m-j;q), \tag{2.7}$$

and

$$A_1(r,k+1,m;q) = q^m \sum_{j=0}^{m} \begin{bmatrix} r+m-j-1 \\ m-j \end{bmatrix} A_1(r,k,j;q). \tag{2.8}$$

Proof. Equation (2.4) follows from the remark given below Equation (2.2). Equation (2.5) is obvious from the Definition (2.2) while Equation (2.6) follows from the Definition (2.2) and the fact that $P_1(r, k, m, \nu)$ is symmetrical in r and k which is clear from Theorem 2.1. To prove (2.7) and (2.8) we set

$$G(r,k;z,q) = \prod_{j=1}^{k} \frac{1}{(zq^j;q)_r} = \sum_{m=0}^{\infty} A_1(r,k,m;q) z^m. \tag{2.9}$$

From the first part of (2.9) we get the q-functional equation

$$G(r, k; zq, q) = (zq; q)_r G(r, k+1,; z, q). \tag{2.10}$$

In (2.10) if we use Euler's identity (cf. [10,eq (3.3.6)])

$$(z;q)_r = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}$$

and then equate the coefficients of z^m we arrive at (2.7).

Similarly, (2.8) is obtained from (2.10) by using another identity of Euler (cf. [10, Eq. (3.3.7)])

$$\frac{1}{(z;q)_r} = \sum_{j=0}^{\infty} \begin{bmatrix} r+j-1 \\ j \end{bmatrix} z^j$$

and then comparing the coefficients of z^m. This completes the proof of Theorem 2.2.

We close this section by proving one more theorem which establishes the fact that our polynomials $A_1(r, k, m; q)$ generalize the Gaussian polynomials.

Theorem 2.3. We have

$$A_1(1,k,m;q) = q^m \begin{bmatrix} k+m-1 \\ m \end{bmatrix} \tag{2.11}$$

and $$\sum_{s=0}^{m} A_1(1,k,s;q) = \begin{bmatrix} k+m \\ m \end{bmatrix}. \tag{2.12}$$

Proof. Since $P_1(1, k, m, v) = p(k, m, v) - p(k, m-1, v)$,

therefore, $A_1(1,k,m;q) = \begin{bmatrix} k+m \\ m \end{bmatrix} - \begin{bmatrix} k+m-1 \\ m-1 \end{bmatrix}$,

which leads to (2.11) in view of (1.8).

Similarly, to prove (2.12), we use the fact

$$p(k,m,v) = \sum_{s=0}^{m} P_1(1,k,s,v).$$

3. SECOND ANALOGUE OF THE GAUSSIAN POLYNOMIALS

Following the method of proof of theorem 2.1 one can prove that $P_2(r,\infty,m,v) = \lim_{k\to\infty} P_2(r,k,m,v)$ equals the number of plane partitions of v with $\leq$ r rows and with conjugate trace m.

Stanley's formula [12, Eq. (6), p. 59] leads us to the following result. :

Theorem 3.1. We have

$$\sum_{v=0}^{\infty}\sum_{m=0}^{\infty} P_2(r,k,m,v) z^m q^v = \prod_{v=1}^{k} (1-zq^v)^{-\min(r,v)}. \tag{3.1}$$

Now we define the second analogue $A_2(r, k, m; q)$ of the

Gaussian polynomial by

$$A_2(r,k,m;q)=\sum_{\nu\geq 0}P_2(r,k,m,\nu)q^{\nu}. \qquad (3.2)$$

Remark. Since $P_2(r, k, m, \nu) = 0$ if $\nu >$ mk and $P_2(r, k, m; mk) = 1$, we see that $A_2(r, k, m; q)$ is a polynomial in q of degree mk.

Using Theorem 3.1 and the Taylor's theorem we get the formula

$$A_2(r,k,m;q)=\frac{1}{m!}\frac{d^m}{dz^m}\left[\prod_{j=1}^{k}(1-zq^j)^{-\min(r,j)}\right]_{z=0}. \qquad (3.3)$$

Like Formula (2.3), Formula (3.3) is also implementable on computer. Following is a MACSYMA produced table of $A_2(r,k,3; q)$ for $1\leq r, k\leq 4$.

TABLE-3

	1	2	3	4
1	q^3	$q^3+q^4+q^5+q^6$	$q^3+q^4+2q^5+2q^6+2q^7+q^8+q^9$	$q^3+q^4+2q^5+3q^6+3q^7+3q^8+3q^9+2q^{10}+q^{11}+q^{12}$
2	q^3	$q^3+2q^4+3q^5+4q^6$	$q^3+2q^4+5q^5+8q^6+9q^7+6q^8+4q^9$	$q^3+2q^4+5q^5+10q^6+13q^7+16q^8+15q^9+12q^{10}+6q^{11}+4q^{12}$
3	q^3	$q^3+2q^4+3q^5+4q^6$	$q^3+2q^4+6q^5+10q^6+15q^7+12q^8+10q^9$	$q^3+2q^4+6q^5+13q^6+21q^7+30q^8+34q^9+30q^{10}+18q^{11}+10q^{12}$
4	q^3	$q^3+2q^4+3q^5+4q^6$	$q^3+2q^4+6q^5+10q^6+15q^7+12q^8+10q^9$	$q^3+2q^4+6q^5+14q^6+23q^7+36q^8+44q^9+44q^{10}+30q^{11}+20q^{12}$

We shall now discuss the properties of the polynomials $A_2(r, k, m; q)$ which are analogous to the properties (1.4) – (1.8) of the Gaussian polynomials.

Theorem 3.2. The polynomials $A_2(r, k, m; q)$ satisfy the following relations

$$\text{degree } A_2(r, k, m; q) = mk. \tag{3.4}$$

$$A_2(r, k, 0; q) = 1. \tag{3.5}$$

$$\text{if } r > k \text{ then } A_2(r, k, m; q) = A_2(k, k, m; q), \tag{3.6}$$

for $1 \le r \le m$,

$$q^m A_2(r,k,m;q) = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} (-1)^i q^{j(j+1)/2} A_2(r,k+1,m-j;q), \tag{3.7}$$

$$A_2(r,k,m;q) = q^m \sum_{j=0}^{m} \begin{bmatrix} r+m-j-1 \\ m-j \end{bmatrix} A_2(r,k,j;q). \tag{3.8}$$

Proof. Equation (3.4) follows from the remark given just after Equation (3.2). Equation (3.5) is obvious from the definition of $A_2(r, k, m; q)$ given by Equation (3.2). Equation (3.6) follows from the fact that in an n-color partition $\Pi = (a_1)_{b_1} + (a_2)_{b_2} + \ldots + (a_m)_{b_m}$ no subscript b_i can exceed the part a_i. Tthe proofs of Equations (3.7) and (3.8) are similar to those of Equations (2.7) and (2.8), respectively and are hence omitted.

Remark. It is interesting to note that from (2.1) and (3.1) it follows that

$$P_1(1, k, m, v) = P_2(1, k, m, v)$$

which implies

$$A_1(1, k, m; q) = A_2(1, k, m; q),$$

and so Theorem 2.3 also holds good for the polynomials A_2 (1. k. m; q).

4. GENERATING FUNCTIONS FOR OTHER n-COLOR RESTRICTED PARTITION FUNCTIONS.

In the classical theory of partitions several restricted partition functions have been studied. $p(r, k, \nu)$ is one such function.

Other restrictions are partitions into distinct parts, partitions into odd parts, partitions satisfying certain modulo conditions etc. In general a restricted partition function is of the type $p(S,n)$ which counts the number of partitions of n that have all their parts in a set of positive integers S.

Analogues to this classical restricted partition function $p(S, n)$ we in this section study various restricted n-color partition functions.

Definition 4.1. Let $P(S; T; \nu)$ denote the number of all n-color partitions of ν of the form $\sum_i (a_i)_{b_i}$ such that the parts $a_i \varepsilon S$ and the subscripts $b_i \varepsilon T$

Definition 4.2. Let $P^m(S; T; \nu)$ denote the number of all n-color partitions of ν enumerated by $P(S, T; \nu)$ with the added restriction that there be exactly m parts.

Definition 4.3. If $f(\nu)$ denotes the number of any kind of partitions of ν then $f_k(\nu)$ will denote the number of partitions of ν enumerated by $f(\nu)$ with the added restriction that each part $\leq$ k.

Definition 4.4. Let $P(D; \nu)$ denote the number of n-color partitions of ν into distinct parts.

Definition 4.5. Let $Q(\nu)$ denote the number of n-color partitions of ν in which even parts appear with even subscripts and odd with odd subscripts.

We will denote the set of all positive integers, the set of all even positive integers and the set of all odd positive integers by *N*, *E* and *O*, respectively.

We obtain the following generating functions by a straight forward application of the standard techniques of partition theory [10, Chap. 1) :

$$1+\sum_{\nu=1}^{\infty} P(O,O;\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-n}$$

$$=1+q+q^2+3q^3+3q^4+6q^5+\ldots, \quad (4.1)$$

$$1+\sum_{\nu=1}^{\infty} P(O,E;\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-(n-1)}$$

$$=1+q^3+2q^5+\ldots, \quad (4.2)$$

$$1+\sum_{\nu=1}^{\infty} P(E,E;\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^{2n})^{-n}$$

$$=1+q^2+3q^4+\ldots, \quad (4.3)$$

$$1+\sum_{\nu=1}^{\infty} P(E,O;\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^{2n})^{-n}$$

$$=1+q^2+3q^4+\ldots, \quad (4.4)$$

$$1+\sum_{\nu=1}^{\infty} P(E,N;\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^{2n})^{-2n}$$

$$=1+2q^2+7q^4+\ldots, \quad (4.5)$$

$$1+\sum_{\nu=1}^{\infty} P(O,N;\nu)q^{\nu} = \prod_{n=1}^{\infty}(1-q^{2n-1})^{-(2n-1)}$$

$$=1+q+q^2+4q^4+9q^5+\ldots, \quad (4.6)$$

$$1+\sum_{v=1}^{\infty}P(D;v)q^{v} \quad =\prod_{n=1}^{\infty}(1+q^{n})^{n}$$

$$=1+q+2q^{2}+5q^{3}+8q^{4}+16q^{5}+\ldots, \qquad (4.7)$$

$$1+\sum_{v=1}^{\infty}Q(v)q^{v} =\prod_{n=1}^{\infty}(1-q^{v})^{-\left[\frac{n+1}{2}\right]}$$

$$=1+q+2q^{2}+4q^{3}+7q^{4}+12q^{5}+\ldots, \qquad (4.8)$$

where [ ] is the greatest integer function.

The two variable generating functions for $P^{m}(S, T; v)$ can be given by using a double series.

Thus, for example,

$$\sum_{v=0}^{\infty}\sum_{m=0}^{\infty}P^{m}(O,O;v)z^{m}q^{v}=\prod_{n=1}^{\infty}(1-zq^{2n-1})^{-n}. \qquad (4.9)$$

The generating function for $P_k(S, T; n)$ is obtained by using a finite product instead of an infinite one on the right-hand side. Thus, for example,

$$1+\sum_{v=1}^{\infty}P_{k}(O,O;v)q^{v}=\prod_{n=1}^{k}(1-q^{2n-1})^{-n}, \qquad (4.10)$$

$$1+\sum_{v=1}^{\infty}P_{k}(D;v)q^{v}=\prod_{n=1}^{k}(1+q^{n})^{n}, \qquad (4.11)$$

etc.

5. COMBINATORIAL IDENTITIES

Using the generating functions of the previous section several combinatorial identities can be obtained. For the brevity of the paper we give only two.

Theorem 5.1. Let $B(\nu)$ denote the number of n-color partitions of ν such that the parts are either odd with any subscript or even with even subscripts only. Then $P(D;\nu) = B(\nu)$.

Example. $P(D,4) = 8$, since the relevant partitions are

4_1, 4_2, 4_3, 4_4

$3_1 1_1$, $3_2 1_1$, $3_3 1_1$

$2_1 2_2$

Also, $B(4) = 8$, in this case the relevant partitions are

4_2, 4_4

$3_1 1_1$, $3_2 1_1$, $3_3 1_1$

$2_2 2_2$, $2_2 1_1 1_1$, $1_1 1_1 1_1 1_1$.

Proof. We have

$$\sum_{\nu=0}^{\infty} P(D;\nu) q^{\nu} = \prod_{n=1}^{\infty} (1+q^n)^n$$

(by(4.7))

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^n}{(1-q^n)^n}$$

$$= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^n}{(1-q^{2n})^{2n}(1-q^{2n-1})^{2n-1}}$$

$$= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^n (1-q^{2n-1})^{2n-1}}$$

$$= \sum_{\nu=0}^{\infty} B(\nu) q^{\nu}.$$

(by (4.3) and (4.6))

Since a power series expansion of a function is unique, we see that

$A(\nu) = B(\nu)$, for all ν.

Remark 1. Theorem 5.1 may be considered as an n-color analogue of Euler's identity (*).

Remark 2. the condition "even parts appear with even subscripts only" on the partitions enumerated by $B(\nu)$ can be replaced by the condition "even parts appear with old subscripts only" in view of the generating functions (4.3) and (4.4).

Theorem 5.2. Let $R(\nu)$ denote the number of strict plane partitions of ν. Then

$$Q(\nu) = R(\nu), \text{ for all } \nu.$$

Example. $Q(4) = 7$, since the relevant partitions are 4_2, 4_4, $3_1 1_1$, $3_3 1_1$, $2_2 2_2$, $2_2 1_1 1_1$, $1_1 1_1 1_1 1_1$.

Also, $R(4) = 7$, since the number of strict plane partitions of 4 is also 7. They are

$$\begin{array}{ccccccc} 4\,, & 3\,, & 2\,, & 1\,, & 31\,, & 21\,, & 2 \\ & 1 & 1 & 1 & & 1 & 2 \\ & & 1 & 1 & & & \\ & & & 1 & & & \end{array}$$

Proof. This theorem is an immediate consequence of the fact that the right-hand side of (4.8) is also a generating function for $R(\nu)$ (cf. Gordon and Houten [7]).

6. CONCLUSION

We hope that like Gaussian polynomials our polynomials A_1 *(r, k, m; q)* and A_2 *(r, k, m; q)* will find more applications in number theory, combinatorics, special functions, algebra, statistics, theoretical physics and computer algebra. We conclude by posing three open problems.

Problem 1. Is it possible to find explict expressions (in terms of q only) for the polynomials A_1 *(k, r, m; q)* and A_2 *(k, r, m; q)* as we have for Gaussian polynomials given by (1.1)?

Problem 2. we know that $\lim_{q\to 1}\begin{bmatrix} r \\ k \end{bmatrix}$ equals the binomial coefficient $\begin{pmatrix} r \\ k \end{pmatrix}$, do $A_1(r, k, m; 1)$ and A_2 $(r, k, m; 1)$ have interpretations other than partition theoretic?

Problem 3. We have used generating functions for proving Theorems 5.1 and 5.2, is it possible to prove them combinatorially?

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