Neighborhood Union Condition with Distance for Vertex-pancyclicity[1]

Lin Wensong Song Zengmin

(Department of Mathematics, Southeast University, Nanjing, 210018, P. R. China)

ABSTRACT: Let G be a 2-connected simple graph with order $n(n \geqslant 5)$ and minimum degree δ. This paper proves that if for any two vertices u,v of G at distance two there holds $|N(u) \cup N(v)| \geqslant n - \delta$, then G is vertex-pancyclic with a few exceptions.

Key words: cycle vertex-pancyclic vertex-pancyclic graph

1. INTRODUCTION

We use the notation and terminology of [1] and [2]. Only simple undirected graphs are considered. A graph G with order n is called *pancyclic* if it contains cycles of length from 3 to n.

For $x_1, x_2, \cdots, x_k \in V(G)$, we use $N(x_1, x_2, \cdots, x_k)$ to denote the set of vertices $\bigcup_{i=1}^{k} N(x_i)$, $n(x_1, x_2, \cdots, x_k)$ to denote the order of $N(x_1, x_2, \cdots, x_k)$ and $\overline{N}(x_1, x_2, \cdots, x_k)$ to denote $N(x_1, x_2, \cdots, x_k) \cup \{x_1, x_2, \cdots, x_k\}$. A cycle of length p is called a p-cycle. Let $C = v_1 v_2 \cdots v_p v_1$ be a p-cycle. We denote by $v_i \vec{C} v_j$ or $\vec{C}[v_i, v_j]$ the path $v_i v_{i+1} \cdots v_j$ on C, while $v_i \overleftarrow{C} v_j$ or $\overleftarrow{C}[v_i, v_j]$ denotes the path $v_i v_{i-1} \cdots v_{j+1} v_j$ on C (the indices of vertices are to be taken modulo p). For $u \in V(C)$, we use u^+, u^- to denote its successor and predecessor vertex on C, respectively. Let $T \subseteq V(C)$. By T^+, T^- we denote the sets $\{u^+ \mid u \in T\}$ and $\{u^- \mid u \in T\}$. We use T^{2+} to denote $(T^+)^+$.

Pancyclic graphs were first considered by Bondy in [3]. Recently people began to study vertex-pancyclic graphs and have obtained many sufficient conditions for a graph to be vertex-pancyclic. For example, in [4] and [5], the authors gave sufficient conditions for vertex-pancyclic graphs which involve degree sum or neighborhood intersections. In [6], Faudree, Gould, Jacobson and Lesniak conjectured that if G has order n,

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connectivity t and minimum degree δ, for any two nonadjacent vertices u,v of G there holds $|N(u) \cup N(v)| \geqslant n - t$ with $\delta \geqslant t + 1$, then G is vertex-pancyclic. In [7], Song reproposed this conjecture in the form that if each pair of nonadjacent vertices u and v in a 2-connected graph of order n and minimum degree δ satisfies $|N(u) \cup N(v)| \geqslant n - \delta + 1$, then G is vertex-pancyclic. Obviously, Song's conjecture can imply the conjecture by Faudree et al. In [8], the authors solved Song's conjecture. In this paper, we give an improvement of Song's conjecture.

Before giving the description of our main result, we define some special graphs.

G_1 is the graph with $V(G) = \{x,u,v\} \cup A \cup B$, $N(x) = \{u,v\}$, $N(u) = A \cup \{x\}$, $N(v) = B \cup \{x\}$ and $G[A \cup B]$ is complete.

G_2 is the graph with $V(G) = \{x,u,v\} \cup A$, $N(x) = \{u,v\}$, $uv \in E$. Furthermore, $N_A(u), N_A(v) \neq \emptyset$ and $N_A(u) \cap N_A(v) = \emptyset$, $G[A]$ is complete.

G_3 is the graph with $V(G) = \{x,u,v,y_1,y_2,y_3\} \cup D$, $(|D| \geq 2)$ and $E(G) = \{xu,xv,uv,xy_3,uy_1,vy_2\} \cup \{y_iw \mid i = 1,2,3, w \in D\} \cup \{wt \mid w, t \in D\}$.

$G^* = \{G \mid V(G) = \{x,u_1,u_2,\cdots,u_\delta\} \cup A, N(x) = \{u_1,\cdots,u_\delta\}$ and $N(u_i) \cup N(u_j) = A \cup \{x\} (i \neq j)$, $N(x)$ is independent. $G[A]$ can be any graph of order $n - \delta - 1\}$.

Clearly $G_1, K_{\frac{n}{2},\frac{n}{2}} \in G^*$.

The following theorem is our main result.

Theorem Let G be a 2-connected simple graph of order n and minimum degree δ. If for any $u,v \in V(G)$ with $d(u,v) = 2$ there holds $|N(u) \cup N(v)| \geq n - \delta$, then G is vertex-pancyclic unless $G \in G^* \cup \{G_2, G_3\}$.

2. PROOFS

Lemma 1 Let G be a graph of order n satisfying the conditions of Theorem. Then for any vertex x of G, x lies on a 3-cycle of G unless $G \in G^*$, and x lies on a 4-cycle unless $G \in G^* \cup \{G_2, G_3\}$. Furthermore, x lies on a 5-cycle unless $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Proof: We consider two cases.

Case 1 x doesn't lie on any 3-cycle of G

Let $N(x) = \{u_1, u_2, \cdots, u_m\}$. $N(x)$ is independent. If $m > \delta$ then $n(u_i, u_j) < n - \delta$, a contradiction. Thus $m = \delta$ and $n(u_i, u_j) = n - \delta (i, j = 1, 2, \cdots, \delta, i \neq j)$. Let $A = V(G) \setminus \bar{N}(x)$, it is evident that $G \in G^*$.

If for any i and $j (i \neq j)$, $N(u_i) \cap N(u_j) = \{x\}$, then clearly $G \cong G_1$ and x must lie on some 5-cycle. Thus we assume that there exist i and $j \in \{1, 2, \cdots, \delta\}$ such that $|N(u_i) \cap N(u_j)| \geq 2$. Obviously x lies on a 4-cycle. Now we prove that x also lies on a 5-cycle unless $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

Let $y (\neq x) \in N(u_i) \cap N(u_j)$. If there exists $y' \in N(u_j) \setminus N(u_i)$ (or $N(u_i) \setminus N(u_j)$), then $y'y \in E$ or $d(y, y') = 2$. If $yy' \in E$, then x lies on a 5-cycle clearly. Thus we may assume $d(y, y') = 2$. If $N(u_i) \cap N(y, y') \neq \emptyset$, then x lies on a 5-cycle. Thus $N(u_i) \cap N(y, y') = \emptyset$, implying $n(y, y') < n - \delta$, a contradiction.

If there exists no such y', then $N(u_i) \cap N(u_j) = N(u_i, u_j)$, for any $i, j \in \{1, 2, \cdots, \delta\}$. Since $n(u_i, u_j) \geq n - \delta$, $n = 2\delta$. It is easy to see that $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

Case 2 x lies on a 3-cycle xuvx.

Since G is 2-connected, we can assume $N(u) \setminus \{x, v\} \neq \emptyset$ without loss of generality.

Case 2.1 $d(x) = 2$.

If there exists $y \in N(u) \setminus \{x, v\}$ such that $yv \in E$, then x lies on a 4-cycle. Since G is 2-connected, there exists $y' \in N(u, v) \setminus \{x, u, v, y\}$. If $yy' \notin E$, then $d(y, y') = 2$ and $n(y, y') \leq n - 3$. A contradiction. Thus $yy' \in E$ and x lies on a 5-cycle.

If for any $y \in N(u) \setminus \{x, v\}$, $yv \notin E$, then $d(y, v) = 2$. By the hypothesis of Lemma 1, $n(y, v) \geq n - 2$. This implies that for any $s \in V \setminus \bar{N}(v)$, $ys \in E$. Similarly, for any $z \in N(v) \setminus \{x, u\}$ and $q \in V \setminus \bar{N}(u)$, $zq \in E$. It is not difficult to prove that $G \cong G_2$.

Case 2.2 $d(x) \geq 3$.

Case 2.2.1 x doesn't lie on any 4-cycle of G

Let $w \in N(x) \setminus \{u, v\}$, $d(v, w) = 2$ and $n(v, w) \leq n - 2 - (d(u) - 2) = n - d(u)$. Thus $d(u) = \delta$. Similarly $d(v) = \delta$. Since G is 2-connected, $\delta \geq 3$. If $d(x) \geq 4$, let $w_1, w_2 \in N(x) \setminus \{u, v\}$, then $d(v, w_1) = 2$ and $n(v, w_1) \leq n - 2 - (d(u) - 2) - (d(w_2) - 2) < n - \delta$. A contradiction. Thus $d(x) = 3$ and $d(u) = d(v) = d(x) = \delta = 3$. Let $y_1, y_2 \in V \setminus \{x, u, v, w\}$ such that $uy_1, vy_2 \in E$. Since $d(y_1, v) = 2$ and $N(y_1, v) \cap \{y_1, v, w\} = \emptyset$, for any $r \in B = V \setminus \{x, u, v, y_1, y_2, w\}$, $y_1 r \in$

E. Similarly, y_2r and $wr \in E$. For $t_1, t_2 \in B$, since $n(t_1, t_2) < n - 3$ if $d(t_1, t_2) = 2$, we have $t_1t_2 \in E$. Hence $G \cong G_3$ and clearly x lies on a 5-cycle.

Case 2.2.2 x lies on some 4-cycle $x_0x_1x_2x_3x_0 (x = x_0)$.

If x_0x_2 and $x_1x_3 \notin E$. Suppose G has no 5-cycle containing x. Let y be any vertex of $N(x_0) \backslash \{x_1, x_3\}$, clearly $d(y, x_1) = 2$ and $N(y, x_1) \subseteq V \backslash (N(x_2) \cup \{y\})$. If $y \notin N(x_2)$, then $n(y, x_1) < n - \delta$, a contradiction. Thus $y \in N(x_2)$, that is $N(x_0) \subseteq N(x_2)$. Since G has no 5-cycle containing x, $N(x_0) \cap N(x_2)$ is independent. But this contradicts the supposition of Case 2.

If $x_0x_2 \notin E$ and $x_1x_3 \in E$. Suppose G has no 5-cycle containing x. Let y be any vertex of $N(x_0) \backslash \{x_1, x_3\}$, clearly $d(y, x_1) = 2$ and $N(y, x_1) \subseteq V \backslash ((N(x_2) \backslash \{x_3\}) \cup \{y\})$. If $N(x_3) \backslash \bar{N}(x_1) \neq \emptyset$, then $n(y, x_1) < n - \delta$, a contradiction. Thus $N(x_3) \backslash \bar{N}(x_1) = \emptyset$. Similarly $N(x_1) \backslash \bar{N}(x_3) = \emptyset$, and so $\bar{N}(x_1) = \bar{N}(x_3)$. If $d(x_1) > 3$, let $z \in N(x_1) \backslash \{x_0, x_2, x_3\}$, then $d(z, x_2) = 2$ and $n(z, x_2) \leqslant n - |\bar{N}(x_0) \backslash \{x_1, x_3\}| - |\{x_2, z\}| < n - \delta$. This is a contradiction. Thus $d(x_1) = d(x_3) = 3$. Since G is 2-connected, there exists $y' \in V \backslash \bar{N}(x_0, x_1, x_3)$ such that $y'x_2 \in E$. Since $d(y, x_1) = 2$, we have $n(y, x_1) \geqslant n - \delta$. On the other hand, we have $N(y, x_1) \subseteq V \backslash ((N(x_2) \backslash \{x_3\}) \cup \{y\})$, which implies $n(y, x_1) \leqslant n - \delta$. Thus $n(y, x_1) = n - \delta$ and y is adjacent to any vertex of $V \backslash \bar{N}(x_0, x_1, x_2, x_3)$. By the same argument, y' is adjacent to any vertex of $V \backslash \bar{N}(x_0, x_1, x_2, x_3)$. Hence $d(y, y') = 2$. But $n(y, y') \leqslant n - 4 < n - \delta$. A contradiction.

By the similar argument, if $x_0x_2 \in E$ and $x_1x_3 \notin E$, then we can deduce a contradiction.

If $x_0x_2 \in E$ and $x_1x_3 \in E$. Let $A_i = N(x_i) \backslash \{x_0, x_1, x_2, x_3\} (i = 0, 1, 2, 3)$. Suppose G has no 5-cycle containing x. Clearly $A_i \cap A_j = \emptyset$ and any vertex of A_i is nonadjacent to any vertex of $A_j (i, j \in \{0, 1, 2, 3\})$. Since G is 2-connected, at least two of A_0, A_1, A_2 and A_3 are nonempty. Suppose $A_i, A_j \neq \emptyset (i \neq j)$. Let $y_i \in A_i$, $y_j \in A_j$. For $k \neq i, j$, $n(x_k, y_i) \leqslant n - 2 - (\delta - 3) - (d(x_j) - 3) = n - \delta + 4 - d(x_j) \leqslant n - \delta$, we conclude that y_i is adjacent to any vertex of $V \backslash \bar{N}(x_0, x_1, x_2, x_3)$. Similarly, y_j is adjacent to any vertex of $V \backslash \bar{N}(x_0, x_1, x_2, x_3)$. Thus $d(y_i, y_j) = 2$. But $n(y_i, y_j) \leqslant n - 4 - (\delta - 3) - (d(x_k) - 3) = n - \delta + 2 - d(x_k) < n - \delta$. A contradiction. This complete the proof of Lemma 1.

Lemma 2 Let G be a graph of order n and satisfy the condition of Theorem, then for any vertex x of G, x lies on cycles of length from 6 to n unless $G \subsetneqq K_{\frac{n}{2},\frac{n}{2}}$.

Proof: Let x be a vertex of G. We will prove that if G has a p-cycle containing x, then G also has a $(p+1)$-cycle containing x unless $G \subsetneqq K_{\frac{n}{2},\frac{n}{2}}$. By contradiction. Suppose, to the contrary, that there exists some integer $p(5 \leqslant p < n)$ such that G has a p-cycle containing x, but has no such $(p+1)$-cycle. We shall obtain contradictions. Let $C = v_1v_2\cdots v_pv_1$ be a p-cycle containing x. There exists $u \in R = V\backslash V(C)$ such that $N_C(u) \neq \varnothing$. Set $T = N_C(u) = \{t_1, t_2, \cdots, t_m\}$. Let $C_i = \vec{C}[t_i^+, t_{i+1}^-]$ $(i = 1, 2, \cdots, p)$.

We consider two cases.

Case 1 $m \geqslant 2$.

If x^+ and $x^- \in T$, then $C' = x^+\vec{C}x^-ux^+$ is a p-cycle with $d_{C'}(x) \geqslant 2$. Notice that any $(p+1)$-cycle involved in the following proof of Case 1 contains x and u, so we can assume x^+ or $x^- \notin T$. (otherwise use C' and x to take the places of C and u). Without loss of generality, we assume $x \notin T^+$.

Claim 1 $T^+ \cup \{u\}$ is independent.

Claim 2 $m < \delta$.

If $m \geqslant \delta$, then $|T^+ \cup \{u\}| \geqslant \delta + 1$. Clearly $d(u, t_1^+) = 2$ and $n(u, t_1^+) < n - \delta$. A contradiction. Thus $m < \delta$.

Set $B = N_R(u)$, then

Claim 3 $N(B) \cap T^{2+} = \varnothing$.

Claim 4 T^{2+} is independent.

If claim 4 is not true. Suppose $t_i^{2+}, t_j^{2+} \in T^{2+}$ and $t_i^{2+}t_j^{2+} \in E (i < j)$. It is not difficult to verify that $N(t_i^+, t_j^+) \cap B = \varnothing$. If $N(t_i^+) \cap N(t_j^+) \neq \varnothing$, then $d(t_i^+, t_j^+) = 2$. But $n(t_i^+, t_j^+) \leqslant n - |T^+| - |B \cup \{u\}| < n - \delta$. Thus $N(t_i^+) \cap N(t_j^+) = \varnothing$. Now we prove that $n(u, t_i^+) < n - \delta$. Define a bijection f_1 on $N(t_j^+)$ as follows: for $w \in N(t_j^+)$

$$f_1(w) = \begin{cases} w & w \in N_R(t_j^+) \\ w^- & w \in \vec{C}[t_j^{2+}, t_i] \\ w^+ & w \in \vec{C}[t_i^{2+}, t_j^{2-}]\backslash T^- \\ w^{2+} & w \in \vec{C}[t_i^{2+}, t_j^{2-}] \cap T^- \\ t_i^+ & w = t_j \\ u & w = t_j^- \end{cases}$$

It is not difficult to verify that for $w \in \vec{C}[t_j^{2+}, t_i], w^- \notin N(u, t_i^+)$ and for $w \in \vec{C}[t_i^{2+}, t_j^-]\backslash T^-, w^+ \notin N(u, t_i^+)$. For $w \in \vec{C}[t_j^{2+}, t_j^-] \cap T^-, w^{2+} \notin$

$N(u,t_i^+)$. If $t_j^+w^+ \in E$, then $f(w) = f(w^+) = w^{2+}$. But we can prove that $t_j^+w^+ \notin E$. Otherwise $d(t_j^+, w^{2+}) = 2$. Since $n(t_j^+, w^{2+}) \geqslant n - \delta$ and $N(t_j^+) \cap (\{u\} \cup B \cup T^+) = \emptyset$, $N(w^{2+}) \cap B \neq \emptyset$. Let $y \in N(w^{2+}) \cap B$, then the cycle $C': t_i\ uyw^{2+}\vec{C}t_j^+w^+\ \overleftarrow{C}t_i^{2+}t_j^{2+}\vec{C}t_i$ is a $(p+1)$-cycle containing x. So for any two vertices $w, w' \in N(t_j^+), f(w) \neq f(w')$. Thus $n(u,t_i^+) \leqslant n - \delta$. From the hypothesis of Lemma 2, $n(u,t_i^+) = n - \delta$.

From f_1, if $t_j^+t_j^- \notin E$, then $n(u,t_i^+) < n - \delta$. Thus $t_j^+t_j^- \in E$. Clearly $ut_j^{2+} \notin E$, $t_i^+t_j^{2+} \notin E$. And it is not difficult to see that $(T^+ \setminus \{t_j^+\}) \cup \{t_j^{2+}\}$ is independent. On the other hand, $N(t_i^+, t_j^{2+}) \cap (B \cup \{u\}) = \emptyset$. Thus $n(t_i^+, t_j^{2+}) < n - \delta$. This is a contradiction.

Claim 5 $N(t_i^+) \cap B$ or $N(t_{i+1}^+) \cap B \neq \emptyset (i = 1,2,\cdots,m)$

Suppose $N(t_i^+) \cap B = N(t_{i+1}^+) \cap B = \emptyset$. If $N(t_i^+) \cap N(t_{i+1}^+) \neq \emptyset$, then $d(t_i^+, t_{i+1}^+) = 2$ and $n(t_i^+, t_{i+1}^+) \leqslant n - |T^+| - |B \cup \{u\}| < n - \delta$, thus $N(t_i^+) \cap N(t_{i+1}^+) = \emptyset$. Now we define a bijection $f_2: N(t_{i+1}^+) \rightarrow V \setminus N(u,t_i^+)$ by:

$$f_2(w) = \begin{cases} w & w \in N_R(t_{i+1}^+) \\ w^- & w \in \vec{C}[t_{i+1}^{2+}, t_i] \\ w^+ & w \in \vec{C}[t_i^{2+}, t_{i+1}^{2-}] \\ t_i^+ & w = t_{i+1} \\ u & w = t_{i+1}^- \end{cases}$$

From f_2, $n(u,t_i^+) \leqslant n - \delta$. And if $t_{i+1}^+t_{i+1}^- \notin E$, then $n(u,t_i^+) < n - \delta$. Thus $t_{i+1}^+t_{i+1}^- \in E$.

Since $t_{i+1}^+t_i \notin E$, if $t_i^+t_i^- \notin E$, then $n(u,t_i^+) < n - \delta$. Thus $t_i^+t_i^- \in E$. Since $t_{i+1}^+t_i^{2+} \notin E$, we must have $t_i^+t_i^{3+} \in E$ (otherwise $n(u,t_i^+) < n - \delta$). Thus $t_{i+1}^+t_i^{3+} \notin E$. Similarly, $t_i^+t_i^{4+} \in E$. Continue in this way, we have $t_i^+t_{i+1}^- \in E$. Since $t_{i+1}^+t_{i+1}^- \in E$, $d(t_i^+, t_{i+1}^+) = 2$. This is a contradiction.

Hence Claim 5 holds.

Claim 6 If $N(t_1^+) \cap B \neq \emptyset$, let $y \in N(t_1^+) \cap B$, then $t_1^{2+}u \in E$, that is $|C_1| = 1$.

If claim 6 is not true, suppose $t_1^{2+}u \notin E$.

Since we assume $x \notin T^+$, we have $N(u) \cap N_R(t_1^{2+}) = \emptyset$. If $N(t_1^+) \cap N_R(t_1^{2+}) \neq \emptyset$, then by the assumption of Case 1, either $u = x$ or $u \neq x$ we can easyly got a $(p+1)$-cycle containing x. This is a contradiction. Thus $N(t_1^+, u) \cap N_R(t_1^{2+}) = \emptyset$. Define a bijection f_3 on $N(t_1^{2+})$ as follows: for any $w \in N(t_1^{2+})$

$$f_3(w)=\begin{cases} w & w\in N_R(t_1^{2+}) \\ w^{2-} & w\in \vec{C}[t_1^{5+},t_1]\backslash\{x^+\} \\ u & w=t_1^{3+} \\ t_1^+ & w=t_1^+ \\ t_1^{3+} & w=x^+ \quad (x^+\neq t_1^{3+}) \end{cases}$$

it is easy to verify that for any $w \in N(t_1^{2+})$, $f_3(w) \notin N(u,t_1^+)$, and for any $w,w'(w\neq w') \in N(t_1^{2+})$, $f_3(w)\neq f_3(w')$. Thus $n(u,t_1^+)\leqslant n-\delta$.

Since $t_m^+ \notin N(t_1^+,u)$, by f_3, $t_1^{2+}t_m^{3+} \in E$, implying $d(t_1^{2+},t_m^{2+})=2$. If $|C_m| \geqslant 2$, then by Claim 3,4, $n(t_1^{2+},t_m^{2+}) < n-\delta$. A contradiction. Hence $|C_m|=1$. But then $t_m^+ \notin f_3(N(t_1^{2+}))$, by f_3, $n(u,t_1^+)<n-\delta$. A contradiction. Thus $t_1^{2+}u \in E$.

Claim 7 $|C_i|=1, i=1,2,\cdots,m$.

If Claim 7 is false, suppose there exists an integer k such that $|C_k| \geqslant 2$. If $N(t_k^+)\cap B\neq\varnothing$, then as claim 6 we can prove that $|C_k|=1$. Thus assume $N(t_k^+)\cap B=\varnothing$. By Claim 5, $N(t_{k-1}^+)\cap B\neq\varnothing$. By claim 6, $|C_{k-1}|=1$. We only prove $|C_2|=1$. Suppose $|C_2|\geqslant 2$ and $N(t_2^+)\cap B=\varnothing$. Clearly $|C_1|=1$. Since $t_i^- \notin N(u,t_1^+)\cup f_3(N(t_1^{2+}))$, by f_3, we have $n(u,t_1^+)\leqslant n-\delta$.

By f_3, $t_1^{2+}t_2^{3+} \in E$, otherwise $n(u,t_1^+)<n-\delta$. Since $N(t_2^+)\cap B=\varnothing$, we have $N(u,t_1^+)\cap N_R(t_2^+)=\varnothing$. Suppose $x\neq t_2^{2+}$. Define a bijection f_4 on $N(t_2^+)$ as follows: for any $w \in N(t_2^+)$

$$f_4(w)=\begin{cases} w & w\in N_R(t_2^+) \\ w^- & w\in \vec{C}[t_2^{4+},t_1] \\ t_2^+ & w=t_2^{2+} \\ t_1^+ & w=t_2 \end{cases}$$

Since $u \notin N(u,t_1^+)\cup f_4(N(t_2^+))$, by f_4, $n(u,t_1^+)<n-\delta$. A contradiction.

If $x=t_2^{2+}$, we define a bijection $f_5: N(t_1^{2+}) \to V\backslash N(u,t_1^+)$ by:

$$f_5(w)=\begin{cases} w & w\in N_R(t_1^{2+}) \\ w^{2-} & w\in \vec{C}[t_2^{3+},t_1] \\ t_1^+ & w=t_2^+ \\ t_1^- & w=t_1^+ \end{cases}$$

Since $u \notin N(u,t_1^+)\cup f_5(N(t_1^{2+}))$, we have $n(u,t_1^+)<n-\delta$, a contradiction. This completes the proof of claim 7.

By claim 7. $|C_i|=1$, $i=1,2,\cdots,m$, clearly p is even. Let $A_1=N_R(T^+)$, $A_2=N_R(T)$. Since $d(u,t_i^+)=2$, from the above discussions, it is easy to see that $N_C(t_i^{2+}) = T^+$, which implies $G[T \cup T^+] \cong K_{m,m}$. Clearly A_1 and A_2 are independent sets and $A_1 \cap A_2 = \emptyset$. Let $a \in A_2 \backslash \{u\}$, then $d(u,a) = 2$, and $n(u,a) \leqslant n - |A_2| - m \leqslant n - \delta$. Thus $n(u,a) = n - \delta$. That is $|A_2| + m = \delta$. Since $n(t_1,t_2) \leqslant n - m - |A_1|$ and $n(t_1,t_2) \geqslant n - \delta$, we have $m + |A_1| = \delta$. If $V \backslash (V(C) \cup A_1 \cup A_2) \neq \emptyset$, then clearly $n(t_1,t_2) < n - \delta$. Thus $V = V(C) \cup A_1 \cup A_2$. Since $\delta = m + |A_1| = m + |A_2|$ and $A_1 \cup T$, $A_2 \cup T^+$ are independent, $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Case 2 $m \leqslant 1$. Assume for any $u \in R$, $d_C(u) \leqslant 1$.

Claim 8 For any i, there is no edge between $N_R(v_i)$ and $N_R(v_{i+1})$.

Suppose, to the contrary, assume there is an edge between $N_R(v_1)$ and $N_R(v_2)$. Let $u_1,u_2 \in R$ such that $u_1u_2 \in E$ and $u_1v_1, u_2v_2 \in E$. If $p \geqslant 6$, suppose $x \neq v_2,v_3,v_4$, clearly $d(v_3,v_5) = 2$. But since $N(v_3,v_5) \cap (\overline{N}(u_1) \backslash \{v_1,u_2\}) = \emptyset$, we have $n(v_3,v_5) < n - \delta$. A contradiction. When $p = 5$, if $x \neq v_4$, the proof is the same as $p \geqslant 6$. Thus assume $x = v_4$. Clearly $d_C(x) = 2$. Since $\{v_4,v_2,u_1\}$ is independent, $\delta \geqslant 3$. Let $y \in N_R(x)$, then $d(y,v_3) = 2$. But since $N(y,v_3) \cap N_R(u_1) = \emptyset$, we have $n(y,v_3) < n - \delta$. This is a contradiction.

Now we consider two cases. Assume $x = v_1$.

Case 2.1 $p \leqslant \delta$

Clearly $N_R(v_i) \neq \emptyset (i = 1,2,\cdots,p)$. Let $y_1 \in N_R(v_1), y_3 \in N_R(v_3)$. Since $N(v_2) \cap \overline{N}_R(y_3) = \emptyset$, if $N(y_1) \cap \overline{N}_R(y_3) = \emptyset$, then $n(y_1,v_2) < n - \delta$. Thus $N(y_1) \cap \overline{N}_R(y_3) \neq \emptyset$, implying $d(y_1,y_3) = 2$. But since $N(y_1,y_3) \cap \overline{N}(v_2) = \emptyset$, we have $n(y_1,y_3) < n - \delta$. A contradiction.

Case 2.2 $p \geqslant \delta + 1$

Let $u_1,u_2 \in R$ such that $d(u_1,u_2) = 2$, clearly $n(u_1,u_2) < n - \delta$. Thus R is the union of complete graphs. Let R_1 be a complete graph of R.

Since G is 2-connected, there exist $v_i,v_j \in V(C)$ and $u_1,u_2 \in R_1$ such that $u_1v_i, u_2v_j \in E$. Clearly $v_i^+ \neq v_j$ and $v_j^+ \neq v_i$. $(i < j)$. Without loss of generality, assume $x \in \vec{C}[v_i,v_j]$. Define a bijection f_6 on $N(v_j^{2+})$ as follows: for $w \in N(v_j^{2+})$

$$f_6(w) = \begin{cases} w & w \in N_R(v_j^{2+}) \\ w^- & w \in \vec{C}[v_j^{3+}, v_i] \\ w^+ & w \in \vec{C}[v_i^{2+}, v_j^-] \\ v_i^+ & w = v_j \\ u_1 & w = v_{j+1} \end{cases}$$

Clearly $f_6(N(v_j^{2+})) \cap N(u_1, v_i^+) = \varnothing$. Thus $n(u_1, v_i^+) = n - \delta$. This implies $v_i^+ v_j^+ \in E$. If $v_j^{2+} v_j^{4+} \notin E$, then clearly $v_i^+ v_j^{3+} \in E$. The cycle C': $v_i u_1 u_2 v_j \overleftarrow{C} v_i^+ v_j^{3+} \overrightarrow{C} v_i$ is a p-cycle containing x with $d_{C'}(v_j^+) \geqslant 2$. By Case 1, we can got a $(p+1)$-cycle containing x or prove that $G \cong K_{\frac{n}{2},\frac{n}{2}}$. Thus $v_j^{2+} v_j^{4+} \in E$. But then $v_i u_1 u_2 v_j \overleftarrow{C} v_i^+ v_j^+ v_j^{2+} v_j^{4+} \overrightarrow{C} v_i$ is a $(p+1)$-cycle containing x.

This completes the proof of Lemma 2.

By Lemmas 1 and 2, Theorem holds immediately.

References

[1] J. A. Bondy and U. S. R. Murty. Graph Theory with Applications. Macmillan Co. , London, 1976.

[2] Z. M. Song, Graph Theory and Network Optimization. Southeast University Press, Nanjing, China, 1990.

[3] J. A. Bondy, Pancyclic graphs, J. Combin. Theory B 11(1971). 80 — 84.

[4] K. M. Zhang and Z. M. Song, On Vertex-pancyclic Graphs with the Distance Two Condition. J. of Nanjing University (semiyearly), vol. 2. (1990), 157 — 162.

[6] Z. M. Song and Y. S. Qin, Neighborhood Intersections and Vertex Pancyclicity. J. of Southeast University. Vol. 20, No. 3(1991),65 — 68.

[6] R. J. Faudree, R. J. Gould, M. S. Jacobson and L. M. Lesniak, Neighborhood Unions and Highly Hamiltonian graphs, Ars Combinatoria 31,1991, 129 — 148.

[7] Z. M. Song, Conjecture 5. 1, Proceedings of the Chinese Symposium on Cycle Problems in Graph Theory, J. of Nanjing University, 27(1991),234.

[8] B. L. Liu, D. J. Lou and K. W. Zhao, A Neighborhood Union Condition for Pancyclicity. To appear in Australasian J. of Combinatorics.