The Upper Forcing Geodetic Number of a Graph

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Abstract

For vertices u and v in a nontrivial connected graph G, the closed interval $I[u,v]$ consists of u, v, and all vertices lying in some $u-v$ geodesic of G. For $S \subseteq V(G)$, the set $I[S]$ is the union of all sets $I[u,v]$ for $u, v \in S$. A set S of vertices of a graph G is a geodetic set in G if $I[S] = V(G)$. The minimum cardinality of a geodetic set in G is its geodetic number $g(G)$. A subset T of a minimum geodetic set S in a graph G is a forcing subset for S if S is the unique minimum geodetic set containing T. The forcing geodetic number $f(S)$ of S in G is the minimum cardinality of a forcing subset for S, and the upper forcing geodetic number $f^+(G)$ of the graph G is the maximum forcing geodetic number among all minimum geodetic sets of G. Thus $0 \le f^+(G) \le g(G)$ for every graph G. The upper forcing geodetic numbers of several classes of graphs are determined. It is shown that for every pair a, b of integers with $0 \le a \le b$ and $b \ge 1$, there exists a connected graph G with $f^+(G) = a$ and $g(G) = b$ if and only if $(a, b) \notin \{(1,1), (2,2)\}$.

Key Words: geodetic set, geodetic number, forcing geodetic number.
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1 Introduction

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is also referred to as a $u-v$ *geodesic*. We refer to the book [6] for terminology and notation in graph theory. A vertex w is said to lie in a $u-v$ geodesic P if w is an internal vertex of P, that is, if w is a vertex of P distinct from u and v. The *closed interval* $I[u,v]$ consists of u, v, and all vertices lying in

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some $u-v$ geodesic of G, while for $S \subseteq V(G)$, the set $I[S]$ is the union of all sets $I[u,v]$ for $u,v \in S$. A set S of vertices of a graph G is called a *geodetic set* in G if $I[S] = V(G)$. The minimum cardinality of a geodetic set in G is called the *geodetic number* $g(G)$. A geodetic set of G with cardinality $g(G)$ is called a *g-set*. Certainly, if G is a nontrivial connected graph of order n, then $2 \leq g(G) \leq n$.

The closed intervals $I[u,v]$ in a connected graph G were studied and characterized by Nebeský [12, 13] and were also investigated extensively in the book by Mulder [11], where it was shown that these sets provide an important tool for studying metric properties of connected graphs. The geodetic number of a graph was introduced in [1, 10] and studied further in [2]. Closed intervals and geodetic numbers of digraphs were introduced and studied in [4]. It was showed in [10] that determining the geodetic number of a graph is an NP-hard problem.

A subset T of a g-set S in a nontrivial connected graph G is a *forcing subset* for S if S is the unique g-set containing T. The *forcing geodetic number* $f(S,g)$ *of* S is the minimum cardinality of a forcing subset for S, and the *upper forcing geodetic number* $f^+(G,g)$ of the graph G is the maximum forcing geodetic number among all g-sets of G. Hence for every g-set S in G, there is a subset T of S of cardinality at most $f^+(G,g)$ that uniquely determines S. Since the parameter g is understood in this context, we write $f(S)$ for $f(S,g)$ and $f^+(G)$ for $f^+(G,g)$. Hence if G is a nontrivial connected graph with $f^+(G) = a$ and $g(G) = b$, then $0 \leq a \leq b$. Forcing concepts have been studied for such diverse parameters in graphs as the chromatic number [7], the graph reconstruction number [9], the domination number [5], and the geodetic number [3]. A survey of graphical forcing parameters is discussed in [8].

To illustrate these concepts, consider the graph G of Figure 1. There are four g-sets in G, namely $S_1 = \{u,x,z\}$, $S_2 = \{v,y,w\}$, $S_3 = \{x,y,w\}$, and $S_4 = \{v,y,z\}$. Thus $g(G) = 3$. Since S_1 is the only g-set containing u, it follows that $f(S_1) = 1$. No other vertex of G belongs to only one g-set, so $f(S_i) \geq 2$ for $i = 2,3,4$. Since S_2 is the unique g-set containing $\{v,w\}$, the set S_3 is the unique g-set containing $\{x,y\}$, and S_4 is the unique g-set containing $\{y,z\}$, it follows that $f(S_i) = 2$ for $i = 2,3,4$. Therefore, $f^+(G) = 2$.

The following lemma characterizes those graphs G having $f^+(G) = 0$, $f^+(G) = 1$, or $f^+(G) = g(G)$. Since the proof of this lemma is straightforward, we omit it.

Lemma 1.1 *For a graph G, the upper forcing geodetic number $f^+(G) = 0$ if and only if G has a unique g-set, $f^+(G) = 1$ if and only if G has at least two distinct g-sets and every g-set S is the unique g-set containing*

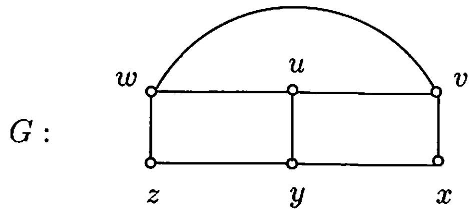

Figure 1: A graph G with $g(G) = 3$ and $f^+(G) = 2$

some element of S, and $f^+(G) = g(G)$ if and only if there is a g-set of G that is not the unique g-set containing any of its proper subsets.

Obviously, the geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. Similarly, the upper forcing geodetic number of a disconnected graph is the sum of the upper forcing geodetic numbers of its components. Consequently, it suffices to consider connected graphs only.

2 Upper Forcing Geodetic Numbers of Certain Classes of Graphs

In this section, we determine the upper forcing geodetic numbers of some well-known graphs. Some additional definitions will be useful. For a vertex v in a graph G, the *link* $L(v)$ of v is the subgraph induced by the neighbors of v. A vertex v in a graph G is called an *extreme vertex* if $L(v)$ is complete. In particular, every end-vertex in G is extreme. Note that every extreme vertex belongs to every geodetic set. In fact, an extreme vertex x lies on an $u - v$ geodesic only if $x = v$ or $x = u$. These observations yield the following lemma.

Lemma 2.1 *Every geodetic set of a connected graph G contains every extreme vertex of G. In particular, if the set W of extreme vertices is a geodetic set of G, then W is the unique g-set of G and $g(G) = |W|$.*

Next we present a lemma, which gives the location of a minimum forcing set in a g-set of a graph G.

Lemma 2.2 *Let W be the set of extreme vertices in a connected graph G. If S is a g-set of G and T is a minimum forcing set of S, then $T \subseteq S - W$.*

Proof. Assume, to the contrary, that $T \cap W \neq \emptyset$. Let $T' = T - (T \cap W)$. Then $|T'| < |T|$. We show that T' is a forcing subset of S. If T' is not a forcing subset of S, then there exists a g-set S' distinct from S such that $T' \subseteq S'$. Since S' is a g-set, it follows from Lemma 2.1 that $T \cap W \subseteq W \subseteq S'$. Thus S' contains $T = T' \cup (T \cap W)$ and so T is not a forcing set of S, which is a contradiction. ∎

The following two corollaries are immediate consequences of Lemmas 1.1 and 2.2.

Corollary 2.3 *If G is a connected graph with k extreme vertices, then*

$$0 \leq f^+(G) \leq g(G) - k.$$

In particular, if the set of extreme vertices of G is a g-set, then $f^+(G) = 0$.

Corollary 2.4 *If G is a complete graph or tree, then $f^+(G) = 0$.*

Now we determine the upper forcing geodetic numbers of cycles C_n of order $n \geq 4$. It is known that $g(C_n) = 2$ if n is even and $g(C_n) = 3$ if n is odd (see [2]). The *diameter* $\operatorname{diam} G$ of a graph G is the largest distance between two vertices of G. Two vertices u and v in G are *antipodal* if $d(u, v) = \operatorname{diam} G$.

Proposition 2.5 *For $n \geq 4$, the upper forcing geodetic number of C_n is*

$$f^+(C_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n = 5, \\ 3 & \text{if } n \geq 7 \text{ is odd.} \end{cases}$$

Proof. If n is even, then $g(C_n) = 2$ and every g-set of C_n consists of a pair of antipodal vertices. Consequently, C_n has $n/2$ g-sets and every vertex v of C_n belongs to the g-set consisting of v and the unique vertex antipodal to v. Thus $f^+(C_n) = 1$. Next assume that n is odd. So $g(C_n) = 3$. Again, C_n has more than one g-set. Moreover, every vertex of C_n belongs to at least two distinct g-sets. By Lemma 1.1, $f^+(C_n) \geq 2$. If $n = 5$, then every g-set S has the form $\{x, y, z\}$, where x and y are adjacent and z is adjacent to neither x nor y. Hence S is the unique g-set containing $\{x, y\}$ and $f(S) = 2$. Thus $f^+(C_5) = 2$. For an odd integer $n = 2k + 1 \geq 7$, let $C_n : v_0, v_1, v_2, \cdots, v_{2k}, v_0$ and $S = \{v_0, v_k, v_{k+2}\}$. Since $I[S] = V(C_n)$, it follows that S is a g-set. Note that $\{v_1, v_k, v_{k+2}\}$, $\{v_0, v_{k-1}, v_{k+2}\}$, and $\{v_0, v_k, v_{k+1}\}$ are g-sets of C_n. Thus, S is not the unique g-set containing any of its proper subsets, and so $f(S) = 3$. Therefore, $f^+(C_n) = 3$ for all odd integers $n \geq 7$. ∎

We have determined the upper forcing geodetic numbers of trees and cycles. A closely related class of graphs is the unicyclic graphs. A graph is *unicyclic* if it is connected and contains exactly one cycle.

Proposition 2.6 *Let G be a unicyclic graph that is not a cycle. If the cycle C of G has length k, and ℓ is the greatest order of a path on C every vertex of which has degree 2 in G, then*

$$f^+(G) = \begin{cases} 0 & \text{if } \ell \leq (k-2)/2, \text{ or if } \ell = k-1 \text{ is either 2 or odd} \\ 1 & \text{if } (k-1)/2 \leq \ell \leq k-2 \\ 2 & \text{if } \ell = k-1 \text{ is even and } \ell \geq 4 \end{cases}$$

Proof. Let $C : v_1, v_2, \cdots, v_k, v_1$ be the unique cycle in G and let W be the set of all end-vertices of G. Assume, without loss of generality, that $P : v_1, v_2, \cdots, v_\ell$, where $\deg v_i = 2$ for $i = 1, 2, \cdots, \ell$. So $\deg v_k \geq 3$ and $\deg v_{\ell+1} \geq 3$. If $\ell \leq (k-2)/2$, then W is the unique g-set of G and so $f^+(G) = 0$. Therefore, we assume that $\ell \geq (k-1)/2$. Since $I(W) = V(G) - V(P)$, it follows that W is not a geodetic set of G; so $g(G) \geq |W| + 1$. Assume first that $\ell = k - 1$. If ℓ is odd, then $W \cup \left\{v_{\frac{\ell+1}{2}}\right\}$ is the unique g-set of G and so $f^+(G) = 0$. If ℓ is even, then there are pairs v_i, v_j of vertices with $1 \leq i < j \leq \ell$ such that $W \cup \{v_i, v_j\}$ is a g-set. Thus $g(G) = |W| + 2$ in this case. It then follows from Corollary 2.3 that $f^+(G) \leq 2$. If $\ell = 2$, then the g-set $W \cup \{v_1, v_2\}$ consists of all extreme vertices of G. Thus $f^+(G) = 0$ by Corollary 2.3. If $\ell \geq 4$ is even, then let $S = W \cup \{v_{\frac{\ell}{2}}, v_{\frac{\ell}{2}+1}\}$. Since S is not the unique g-set containing any of its elements and S is the unique g-set containing $\{v_{\frac{\ell}{2}}, v_{\frac{\ell}{2}+1}\}$, it follows that $f(S) = 2$ and so $f^+(G) = 2$.

Finally, we assume that $(k-1)/2 \leq \ell \leq k-2$. If ℓ is odd, then both $W \cup \left\{v_{\frac{\ell+1}{2}}\right\}$ and $W \cup \left\{v_{\frac{\ell-1}{2}}\right\}$ are g-sets of G; while if ℓ is even, then $W \cup \left\{v_{\frac{\ell}{2}}\right\}$ and $W \cup \left\{v_{\frac{\ell+2}{2}}\right\}$ are g-sets of G. In either case, $f^+(G) = 1$ by Lemmas 1.1 and 2.2. ∎

Next we determine the geodetic and upper forcing geodetic numbers of the hypercubes Q_n, $n \geq 2$. The hypercube Q_n can be defined as the graph whose vertices are labeled by the binary n-tuples $(a_1, a_2, \cdots, a_n)$ and such that two vertices are adjacent if and only if their corresponding n-tuples differ in precisely one position.

Proposition 2.7 *For $n \geq 2$, $g(Q_n) = 2$ and $f^+(Q_n) = 1$.*

Proof. Since every g-set of Q_n is of the form

$$S = \{(a_1, a_2, \cdots, a_n), (1 - a_1, 1 - a_2, \cdots, 1 - a_n)\}$$

where $a_1, a_2, \cdots, a_n \in \{0, 1\}$, it follows that $g(Q_n) = 2$. Certainly, Q_n has more than one g-set and every g-set S is the unique g-set containing $(a_1, a_2, \cdots, a_n)$. Therefore, $f^+(Q_n) = 1$. ∎

Next, we study the upper forcing geodetic numbers of complete bipartite graphs. Let $1 \le r \le s$ be two integers. It was shown in [2] that $g(K_{r,s}) = s$ if $r = 1$; while $g(K_{r,s}) = \min\{r, 4\}$ if $r \ge 2$.

Proposition 2.8 *Let $K_{r,s}$ be a complete bipartite graph with $1 \le r \le s$. Then*

$$f^+(K_{r,s}) = \begin{cases} 0 & r = 1 \quad or \quad r = 2,3 \text{ and } r < s \\ 1 & r = 2,3 \quad and \quad r = s \\ 4 & r \ge 4 \end{cases}$$

Proof. Let $V_1 = \{u_1, u_2, \cdots, u_r\}$ and $V_2 = \{v_1, v_2, \cdots, v_s\}$ be the partite sets of $G = K_{r,s}$. If $r = 1$, then G is a tree and so its upper forcing geodetic number is 0.

We first assume that $r \in \{2, 3\}$. If $r < s$, then G has a unique g-set, namely V_1, and so the upper forcing geodetic number is 0 by Lemma 1.1. If $r = s$, then G has two distinct g-sets, namely V_1 and V_2. So the upper forcing geodetic number is at least 1 by Lemma 1.1. Certainly, for each $v \in V_i$ with $i = 1, 2$, the set V_i is the unique g-set containing v. So the upper forcing geodetic number is 1.

Next we assume that $r \ge 4$ and so $g(G) = 4$. First let $r = 4$. If $r = s$, then V_1, V_2, and $S = \{u_1, u_2, v_1, v_2\}$ are g-sets of G. All other g-sets are similar to S. Since V_1 is the unique g-set containing $\{u_1, u_2, u_3\}$ and V_1 is not the unique g-set containing any 2-element or 1-element subset of V_1, it follows that $f(V_1) = 3$. Similarly, $f(V_2) = 3$. Moreover, since S is not the unique g-set containing any of its proper subsets, $f(S) = 4$. Therefore, $f^+(G) = 4$. If $r < s$, then V_1 and $S = \{u_1, u_2, v_1, v_2\}$ are g-sets of G. All other g-sets are similar to S. Since $f(V_1) = 3$ and $f(S) = 4$, it follows that $f^+(G) = 4$. Now let $r \ge 5$. Then every g-set S of G is of the form $S = \{u_{i_1}, u_{i_2}, v_{j_1}, v_{j_2}\}$, where $1 \le i_1 < i_2 \le r$ and $1 \le j_1 < j_2 \le s$. Since S is not the unique g-set containing any of its proper subsets, $f(S) = 4$ and so $f^+(G) = 4$. ∎

We have seen that if G is a complete graph or a tree, then the set S of extreme vertices of G is the unique g-set of G. Thus the link of each vertex in S is complete. On the other hand, the complete bipartite graph $K_{r,s}$, where $r = 2, 3$ and $r < s$, has a unique g-set S and the link of each vertex in S is an empty graph. In fact, a graph with a unique g-set S can have any prescribed geodetic number and each vertex in S can have any prescribed link. In order to show this, we need some additional definitions. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. An *end-block* of G is a block containing exactly one cut-vertex of G. We present a lemma whose routine proof is omitted.

Lemma 2.9 *Let S be a g-set in a nontrivial connected graph G. Then*

(a) no cut-vertex in G belongs to S, and

(b) for each cut-vertex v of G and every branch B of G at v, $V(B) \cap S \neq \emptyset$.

Theorem 2.10 *For every integer $k \geq 2$ and every k graphs $G_1, G_2, \cdots, G_k$, there exists a connected graph with a unique g-set $\{v_1, v_2, \cdots, v_k\}$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.*

Proof. We construct a graph G with the desired property. For each integer i ($1 \leq i \leq k$), let $F_i = \overline{K}_2 + G_i$, where $V(\overline{K}_2) = \{u_i, v_i\}$. Then the graph G is constructed from the graphs F_i by adding a new vertex x and the k edges xu_i ($1 \leq i \leq k$). Thus in G, $L(v_i) = G_i$ for $1 \leq i \leq k$. Let $S = \{v_1, v_2, \cdots, v_k\}$. Then S is a g-set of G and so $g(G) = k$. We show that S is a unique g-set of G. Assume, to the contrary, that S' is a g-set of G distinct from S. By Lemma 2.9, S' must contain exactly one vertex from each subgraph F_i ($1 \leq i \leq k$). Since $S \neq S'$, we may assume that $v_1 \notin S'$. However, v_1 lies only on those geodesics having v_1 as an end-vertex or having both end-vertices in $V(F_1)$. Thus, $v_1 \notin I[S']$, which is impossible. Therefore, S is the unique g-set of G, as desired. ■

The graph G constructed in the proof of Theorem 2.10 has a cut-vertex and so is not 2-connected. However, we can extend Theorem 2.10 by modifying the structure of the graph G in the proof of Theorem 2.10 to construct a 2-connected graph with the properties described in Theorem 2.10.

Corollary 2.11 *For every integer $k \geq 2$ and every k graphs $G_1, G_2, \cdots, G_k$, there exists a 2-connected graph with a unique g-set $\{v_1, v_2, \cdots, v_k\}$ such that $L(v_i) = G_i$ for $1 \leq i \leq k$.*

Proof. For each integer i ($1 \leq i \leq k$), let $F_i = \overline{K}_3 + G_i$, where $V(\overline{K}_3) = \{u_i, v_i, w_i\}$. Then a 2-connected graph G is constructed from the graphs F_i by adding $2k$ edges u_iw_i and w_iu_{i+1} for $1 \leq i \leq k$, where the subscripts are expressed modulo k. For $k = 3$, the graph G is shown in Figure 2. Thus in G, $L(v_i) = G_i$ for $1 \leq i \leq k$. A proof similar to that of Theorem 2.10 shows that G has the desired properties. ■

For a vertex v of in a connected graph G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. A vertex v is called a *peripheral vertex* of G if $e(G) = diam\ G$. In every example we have seen thus far, each vertex in every g-set is a peripheral vertex. This may not seem surprising; in fact, one may suspect that this is true in general. However, this is not the case. Indeed, there are graphs possessing a g-set in which no vertex is a peripheral vertex.

Figure 3 shows the subdivision graph $S(K_3 \times K_2)$ of the Cartesian product $K_3 \times K_2$. The diameter of $S(K_3 \times K_2)$ is 5 and $\{u, v, w\}$ is a g-set,

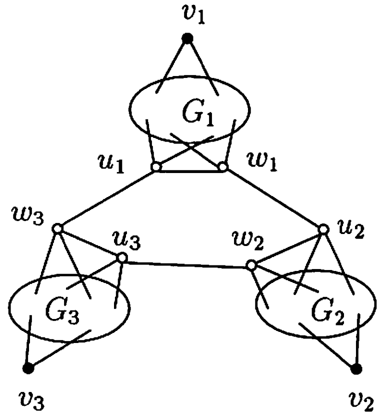

Figure 2: A graph G with the properties described in Theorem 2.11 for $k = 3$

each of whose vertices has eccentricity 4. In fact, this g-set consists exactly of those vertices that are *not* the peripheral vertices of $S(K_3 \times K_2)$. While $S(K_3 \times K_2)$ does contain g-sets consisting entirely of peripheral vertices, the graph H of Figure 3, which is a modification of $S(K_3 \times K_2)$, has a unique g-set, namely $\{x, y, z\}$, and so no peripheral vertex of H belongs to any g-set.

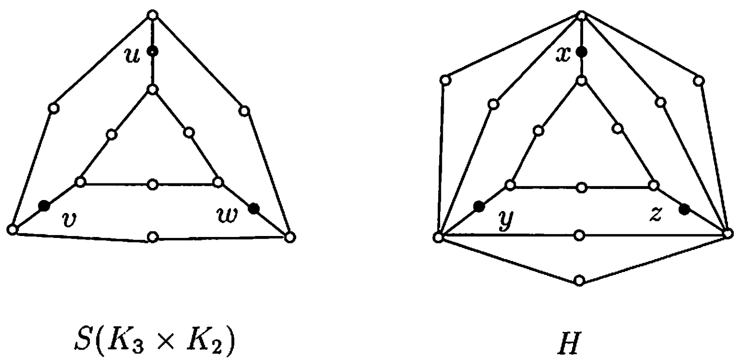

Figure 3: g-sets containing no peripheral vertices

3 Graphs with Prescribed Geodetic and Upper Forcing Geodetic Numbers

We have already noted that if G is a graph with $f^+(G) = a$ and $g(G) = b$, then $0 \leq a \leq b$ and $b \geq 1$. In this section we determine all those pairs (a, b) of integers with $0 \leq a \leq b$ and $b \geq 1$ that are realizable as the geodetic and upper forcing geodetic numbers of some graph. We first determine the only forbidden (f^+, g) pairs. Certainly, if G is a nontrivial connected graph, then $g(G) \geq 2$. Moreover, if $G = K_1$, then $g(G) = 1$, but $f^+(G) = 0$ by Lemma 1.1. Hence $(1,1)$ is a forbidden (f^+, g) pair. Next we show that there is no connected graph G with $f^+(G) = g(G) = 2$ and so $(2,2)$ is a forbidden (f^+, g) pair as well.

Theorem 3.1 *If G is a connected graph with $g(G) = 2$, then $f^+(G) \leq 1$.*

Proof. Assume, to the contrary, that $f^+(G) = 2$. Then there exists a g-set $S = \{u, v\}$ in G such that $f(S) = 2$. Since S is a g-set, u and v are two antipodal vertices of G. Moreover, every vertex w in G that is distinct from u and v lies in some $u-v$ geodesic of G. Since $f(S) = 2$, it follows that S is not the unique g-set containing u. Then there exists $x \neq v$ such that $\{u, x\}$ is also a g-set of G. Therefore, u and x are two antipodal vertices of G and v lies in some $u - x$ geodesic in G. However, the fact that x lies in some $u - v$ geodesic of G implies that $d(u, x) < d(u, v) = diam\ G$, which is a contradiction. ■

We have seen that in Theorem 3.1 that $f^+(G) < g(G)$ if $g(G) = 2$. Next we show that every integer $a \geq 3$ is simultaneously realizable as both the geodetic number and upper forcing geodetic number of some connected graph. Note that if G is a graph with $g(G) = f^+(G)$, then G contains no extreme vertices by Corollary 2.3.

Theorem 3.2 *For every integer $a \geq 3$, there exists a connected graph G such that*

$$f^+(G) = g(G) = a.$$

Proof. For $a = 3$, let $G = C_7$. Then $f^+(G) = g(G) = 3$ by Proposition 2.5. Thus we assume that $a \geq 4$. For each integer i with $0 \leq i \leq a$, let $F_i : u_i, v_i$ be a path of order 2. Then the graph G is obtained from the graphs F_i by adding the $2a$ edges u_0u_j, v_0v_j for all j with $1 \leq j \leq a$. First we show that $g(G) = a$. Let $U = \{u_1, u_2, \cdots, u_a\}$ and $V = \{v_1, v_2, \cdots, v_a\}$. Observe that a set S of vertices of G is a g-set if and only if S has the following two properties: (1) S contains exactly one vertex from each set $\{u_j, v_j\}$ for all $1 \leq j \leq a$ and (2) $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Then (1) implies that $g(G) \geq a$. Since $S' = \{u_1, u_2, v_3, v_4, \cdots, v_a\}$ is a geodetic set of G with $|S'| = a$, it follows that $g(G) = a$.

Next we show that $f^+(G) = a$. Consider $S' = \{u_1, u_2, v_3, v_4, \cdots, v_a\}$. Since $a \geq 4$, it follows that S' contains at least two vertices from U and at least two vertices from V. Let T be a proper subset of S' and $x \in S' - T$. If $x = u_i$ for $i = 1, 2$, then $S_1 = (S' - \{u_i\}) \cup \{v_i\}$ satisfies properties (1) and (2). Thus S_1 is a g-set in G. Since $S_1 \neq S'$ and $T \subseteq S_1$, it follows that S' is not a unique g-set containing T. If $x = v_i$ for some i with $3 \leq i \leq a$, then $S_2 = (S' - \{v_i\}) \cup \{u_i\}$ is a g-set distinct from S' and containing T. Again, S' is not a unique g-set containing T. Therefore, $f(S') = a$ and so $f^+(G) = a$. ∎

Next we show that every pair a, b of integers with $0 \leq a < b$ and $b \geq 1$ can be realized as the upper forcing geodetic number and geodetic number, respectively, of some graph.

Theorem 3.3 *For every pair a, b of integers with $0 \leq a < b$ and $b \geq 1$, there exists a connected graph G such that $f^+(G) = a$ and $g(G) = b$.*

Proof. We have already seen that if $G = K_b$, then $f^+(G) = 0$ and $g(G) = b$. Thus, we assume that $0 < a < b$. We consider two cases.

Case 1. $a = 1$. If $b = 2$, then every even cycle has upper forcing geodetic number 1 and geodetic number 2. So we assume that $b \geq 3$. Let G be obtained from the cycle $C_4 : v_1, v_2, v_3, v_4$ by first adding the $b-2$ new vertices $u_1, u_2, \cdots, u_{b-2}$ and the $b-2$ edges v_1u_i for $1 \leq i \leq b-2$ and then adding two new vertices x, y and three edges v_1x, xy, yv_3. Let $U = \{u_1, u_2, \cdots, u_{b-2}\}$ be the set of end-vertices of G. Then $S_1 = U \cup \{v_3, x\}$ and $S_2 = U \cup \{v_3, y\}$ are the only two g-sets of G. Thus $g(G) = b$. Moreover, since S_1 is the unique g-set containing x and S_2 is the unique g-set containing y, it follows that $f(S_i) = 1$ for $i = 1, 2$ and so $f^+(G) = 1$.

Case 2. $a = 2$. Then $b \geq 3$. Let G be the graph obtained from the cycle $C_5 : v_1, v_2, \cdots, v_5, v_1$ by adding the $b-2$ new vertices $u_1, u_2, \cdots, u_{b-2}$ and joining each u_i $(1 \leq i \leq b-2)$ to v_1. Let $U = \{u_1, u_2, \cdots, u_{b-2}\}$ be the set of end-vertices of G. Then G contains exactly three g-sets, namely $S_1 = U \cup \{v_2, v_4\}$, $S_2 = U \cup \{v_3, v_5\}$, and $S_3 = U \cup \{v_3, v_4\}$. Thus $g(G) = |U| + 2 = b$. Since S_1 is the unique g-set containing v_2 and S_2 is the unique g-set containing v_5, it follows that $f(S_1) = f(S_2) = 1$. Certainly, S_3 is not the unique g-set containing any of its elements. On the other hand, S_3 is the unique g-set containing $\{v_3, v_4\}$ and so $f(S_3) = 2$ by Lemma 2.2. Therefore, $f^+(G) = 2$.

Case 3. $a = 3$. Then $b \geq 4$. For $i = 1, 2$, let $F_i : u_i, v_i, x_i, y_i, z_i, u_i$ be a copy of the cycle C_5. Let G be the graph obtained from F_1 and F_2 by first identifying the vertices z_1 and z_2 and then adding the $b-3$ new vertices $w_1, w_2, \cdots, w_{b-3}$ and the $b-3$ edges v_1w_ℓ, where $1 \leq \ell \leq b-3$. Let $W = \{w_1, w_2, \cdots, w_{b-3}\}$ be the set of the end-vertices of G. Then every g-set must contain at least one vertex from each of the sets $\{x_1, y_1\}$, $\{u_2, v_2\}$, and $\{x_2, y_2\}$, which implies that $g(G) \geq |W| + 3 = b$. Let $S = W \cup \{x_1, v_2, x_2\}$. Since $I[S] = V(G)$, it follows that S is a geodetic set of G and so $g(G) = b$. Next we show that $f^+(G) = 3$. Since G contains $g(G) - 3$ extreme vertices, $f^+(G) \leq 3$ by Corollary 2.3. Let $S_1 = W \cup \{x_1, v_2, y_2\}$, $S_2 = W \cup \{x_1, u_2, x_2\}$, and $S_3 = W \cup \{y_1, v_2, x_2\}$. Since S_i is a g-set for $i = 1, 2, 3$, it follows that S is not a unique g-set containing any proper subset of $S - W = \{x_1, v_2, x_2\}$. Since S is a unique g-set containing $\{x_1, v_2, x_2\}$, we have $f(S) = 3$ by Lemma 2.2. Therefore, $f^+(G) = 3$.

Case 4. $a \geq 4$. Let $F_0 : u_0, z, v_0$ be a copy of the path of order 3. For each integer i with $1 \leq i \leq a$, let $F_i : u_i, v_i$ be the path of order 2. Let G be the graph obtained from the graphs F_i $(0 \leq i \leq a)$ by first adding the $2a$ edges u_0u_j, v_0v_j for $1 \leq j \leq a$ and then the $b-a$ new vertices $w_1, w_2, \cdots, w_{b-a}$ and the $b-a$ edges zw_ℓ for $1 \leq \ell \leq b-a$. This completes the construction of the graph G. We first show that $g(G) = b$. Let $W = \{w_1, w_2, \cdots, w_{b-a}\}$ be the set of end-vertices of G. Also, let $U = \{u_1, u_2, \cdots, u_a\}$, $V = \{v_1, v_2, \cdots, v_a\}$ and $X = \{u_1, u_2, v_3, v_4, \cdots, v_a\}$. Observe that a set S of vertices of G is a g-set if and only if S has the following three properties: (1) S contains W, (2) S contains exactly one vertex from each pair u_j, v_j for all j with $1 \leq j \leq a$, and (3) $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. Then (1) and (2) imply that $g(G) \geq |W| + a = b$. Since $S' = W \cup X$ is a geodetic set of G, it follows that $g(G) = b$.

We now show that $f^+(G) = a$. By Corollary 2.3, $f^+(G) \leq a$. Next consider the g-set $S' = W \cup X$. Certainly, S' is a unique g-set containing X. By Lemma 2.2, it suffices to show that S' is not a unique g-set containing any proper subset of $S' - W = X$. Since $a \geq 4$, it follows that S' contains at least two vertices from U and at least two vertices from V. Let T be a subset of $S' - W$ with $|T| < a$. Then there exists $x \in X$ such that $x \notin T$. Assume first that $x = u_i$ for $i = 1, 2$. Let $S_1 = (S' - \{u_i\}) \cup \{v_i\}$. Since S_1 satisfies properties (1), (2), and (3), it follows that S_1 is a g-set of G. Because $S_1 \neq S'$ and S_1 contains T, it follows that S' is not the unique g-set containing T. Next assume that $x = v_i$ for some i with $3 \leq i \leq a$. Now let $S_2 = (S' - \{v_i\}) \cup \{u_i\}$. Again, S_2 satisfies properties (1), (2), and (3). Thus S_2 is a g-set that is distinct from S' and contains T, and so S' is not the unique g-set containing T. Therefore, $f(S') = a$ and so $f^+(G) = a$.
■

Combining Theorems 3.1, 3.2, and 3.3, we have the following result.

Corollary 3.4 *For every pair a, b of integers with $0 \leq a \leq b$ and $b \geq 1$, there exists a connected graph G with $f^+(G) = a$ and $g(G) = b$ if and only* *if* $(a, b) \notin \{(1,1),(2,2)\}$.

A closely related forcing parameter is the forcing geodetic number $f(G)$ of a graph G, studied in [3] and defined as the minimum forcing geodetic number among all minimum geodetic sets of G. Thus $0 \leq f(G) \leq f^+(G) \leq g(G)$ for every graph G. For example, the graph G of Figure 1 has $f(G) = 1$, $f^+(G) = 2$, and $g(G) = 3$. For all graphs G that we have studied thus far, $f(G)$ and $f^+(G)$ differ by at most 1. We conclude with a question suggested by this remark.

Problem 3.5 *For which pairs a, b of integers with $0 \leq a \leq b$, does there exist a connected graph G with $f(G) = a$ and $f^+(G) = b$?*

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