Correction to "All c-Bhaskar Rao designs with block size 3 and $c \geq -1$ exist"

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Abstract: We correct an earlier theorem and reprove its consequences regarding c-BRD's with $v \equiv 5, 8 \pmod{12}$. The original conclusions remain valid.

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It is necessary to correct the statements of Theorem 3 and its Corollary in the authors' paper "All c-Bhaskar Rao designs with block size 3 and $c \geq -1$ exist," Ars Combinatoria 59(2001), pp. 21-32 [3]. The statements of all the other theorems in the paper are correct, but two theorems, Theorems 10 and 13, require further argument which is based on a new result which we provide here in the form of a Lemma.

In the statement of Theorem 3, we hypothesized that v(v-1) is not 0 mod 12. What we proved, and mostly applied, required that v(v-1) is not 0 mod 4. With this correction, the proofs of Theorem 3 and the Corollary immediately following are correct as given. All the other theorems, Theorems 4 to 13 are correctly stated, and we indicate here where "mod 4" must replace "mod 12" in discussions or proofs. Theorem 10 requires one new easy case. The significant consequence is that Theorem 13, while correct, requires consideration of $v \equiv 5, 8$ mod 12.

The reader is referred to [3] for an introduction, notation, terminology, and Tables, all of which need no change. A c-BRD(v, k, λ) is a {0, 1, -1}-matrix whose rows have inner product c and such that the matrix of absolute values is the incidence matrix of a BIBD(v, k, λ). The full statements of the revised theorems are repeated here, and we take the opportunity to improve (smooth) the proof of Theorem 8 and to state Theorem 11 completely (one line of which was missing). We re-organize the proof of Theorem 10 to consider one new case,

Case 3 of the proof of Theorem 13 is new (and necessary), and the Lemma A below is a new result.

Theorem 3. *Suppose v(v-1) is not congruent to 0 mod 4. Then, for any c-BRD(v, 3, λ):*

(A) If c = 2s and λ = 2t, then s $\equiv$ t (mod 2).

(B) Suppose x = 1 or 5. If c = 2s+1 > 0 and λ = 6t+x, then s + t is even. If c = -2s - 1 < 0, then s + t is odd.

(C) Suppose λ = 6t + 3. If c = 2s + 1 > 0, then s + t is odd. If c = -2s - 1 < 0, then s + t is even. In particular, if c = -1, i.e., s = 0, then t must be even.

Corollary. *Suppose v(v-1) is not congruent to 0 mod 4. Then for any c-BRD(v, 3, λ), c $\equiv$ λ (mod 4).*

The proof of Theorem 3 is now correct as given in **[3]**. Lemma 4, Theorem 5, and Theorem 6 require no discussion here. The statements of Theorems 7-13 are correct, but in the proofs or discussions of Theorems 7 to 12, replace "mod 12" with "mod 4" in applying Theorem 3 or its Corollary.

Before re-proving Theorem 10 or Theorem 13, we establish the following Lemma which contains new results.

Lemma A *Whenever v $\equiv$ 0, 1 (mod 4), a 4-BRD(v, 3, 6) exists.*

Proof: First, for v = 4, 9, or 12, the incidence matrix for a 4-BRD(v, 3, 6) may be formed by juxtaposition (placing side-by-side) of the incidence matrices of a 0-BRD(v, 3, 2) and a 4-BRD(v, 3, 4), i.e., a BIBD(v, 3, 4). A 4-BRD(5, 3, 6) is obtained from a 1-BRD(5, 3, 3) and a 3-BRD(5, 3, 3). A 4-BRD(8, 3, 6) is obtained from the following signed difference family:

(∞, -0, 1), (∞, 0, 2), (∞, 0, 3), (1, 2, -4),
and 4 copies of (1, 2, 4).

We note that, if v $\equiv$ 0, 1 (mod 4), a PB(v, {4, 5, 8, 9, 12}, 1) exists **[1]**. The Lemma now follows from Theorem 1 of **[2]** which states: : *Let PB(v; K; λ) be any pairwise balanced design. If there exists a c-BRD(k', k, μ) for each k' in K, then there exists a cλ-BRD(v, k, $\lambda\mu$).*

Theorem 8. *If λ $\equiv$ 1, 5 (mod 6), then the necessary conditions are sufficient for (-1)-BRD(v, 3, λ) to exist.*

Proof: By Table 1, we note v $\equiv$ 1, 3, 7, 9 (mod 12). First suppose v $\equiv$ 1, 9 (mod 12). We construct the incidence matrix of a (-1)-BRD(v, 3, 6t+1) by juxtaposing, or placing side-by-side, the incidence matrices for 3t-1 copies of a 0-BRD(v, 3, 2) and one copy of a (-1)-BRD(v, 3, 3). If λ = 6t+5, use two more 0-BRD(v, 3, 2). On the other hand, suppose v $\equiv$ 3, 7 (mod 12), i.e., v(v-1) $\neq$ 4m for any m. For this case, the proof is given in **[3]**.

Theorem 10 The necessary conditions are sufficient for the existence of 5-BRD(v, 3, 6t+3) and 3-BRD(v, 3, 6t+3)

Proof: For 3-BRD(v, 3, 6t+3), use the first paragraph of Section 4 replacing mod 12 with mod 4. For 5-BRD(v, 3, 6t+3), there are 3 cases. Case 1: $v \equiv 1, 9 \pmod{12}$. Use the argument given for $v(v-1) \equiv 0 \pmod{12}$. Case 2: $v \equiv 3, 7, 11 \pmod{12}$. Use the remaining argument given, noting $v(v-1) \neq 4t$. Case 3 (*new*): $v \equiv 5 \pmod{12}$. Here create a 5-BRD(v, 3, 6t+3) by juxtaposition of a 4-BRD(v, 3, 6) from Lemma A, (t-1)-copies of a 0-BRD(v, 3, 6), and one 1-BRD(v, 3, 3).

We re-state Theorem 11 to correct a misprint.

Theorem 11. *The necessary conditions are sufficient for the existence of a 3-BRD(v, 3, λ).*

Theorem 13. *Suppose $c \geq -1$. Then, the necessary conditions are sufficient for the existence of all c-BRD(v, 3, λ).*

Proof: The cases c = -1 (Theorem 9) and c = 0 (i.e., for BRD's) are done. We need only consider positive c. We divide the argument into 3 cases.

Case 1: Assume $v(v-1) \equiv 0 \pmod{12}$. Use the proof given.

Case 2: Assume v(v-1) is not congruent to 0 mod 4. Use the proof given.

Case 3: Assume $v \equiv 5, 8 \pmod{12}$. (This is the new case.) It is only necessary here to show how to construct the incidence matrix for the needed c-BRD for those examples not given in Tables 5 and 6 in [3]. Consequently, here $\lambda - c$ is even but not a multiple of 4 as those cases are in the referenced tables.

First suppose $v \equiv 5 \pmod{12}$. It follows that $\lambda \equiv 0, 3, 6, 9 \pmod{12}$. Constructions for all these are in Table 1. In the table, *m means use m copies. We note that, in general, existence of the c-BRD's (and the underlying BIBD's) in Table 1 follow from Theorems 5, 6 in [3], Table 1 in [3], and the new Lemma A. A 1-BRD(v, 3, 3) exists whenever $v \equiv 1 \pmod{4}$ [2]. Now suppose $v \equiv 8 \pmod{12}$. In this case, $\lambda \equiv 0, 6 \pmod{12}$. The ingredients for the c-BRD are in the first two sections of Table 1.

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References:

1. H. Hanani, The existence and construction of balanced incomplete block designs, *Ann. Math. Statis.* **32**(1961), p.361-386.