Graceful Labelings of Graphs Associated with Vertex-saturated Graphs

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ABSTRACT. A graceful graph with n edges and $n+1$ vertices is called a vertex-saturated graph. Each graceful graph corresponds to a vertex-saturated graph. Four classes of graceful graphs associated with vertex-saturated graphs are presented. Three of which generalize the results of [1], [2] and [5].

1 Introduction

In this paper, all graphs considered will be finite, without loops or multiple edges. Let $H = (V(H), E(H))$ be a graph with m edges. A *graceful labeling* of H is a one-to-one mapping θ of the set $V(H)$ into the set $\{0, 1, \dots, m\}$ which has the property that the valves of the edges form the set $\{1, 2, \dots, m\}$ if the value $\theta^*(e)$ of the edge e with the end vertices x, y is given as $\theta^*(e) = |\theta(x) - \theta(y)|$. A graceful labeling is an *α-labeling* if there is an integer λ $(0 \le \lambda \le m-1)$ such that for each edge $e = (x, y)$,

$$\min(\theta(x), \theta(y)) \le \lambda < \max(\theta(x), \theta(y)).$$

The integer λ will be called the boundary value of the α-labeling. Clearly, a graph permitting an α-labeling is always bipartite. Further, if θ is a graceful labeling of H then the labeling $\tilde{\theta}$ defined by $\tilde{\theta} = m - \theta(x)$ for all $x \in V(H)$ is again a graceful labeling of H. The graceful labeling $\tilde{\theta}$ is sometimes called the complementary labeling of H (see [4]). A graph H is called graceful (or λ-graceful) if it has a graceful labeling (or an α-labeling).

Let G be a graph with n edges and $n+1$ vertices $(n \ge 1)$. The graph G is referred to as a *vertex-saturated graph* $G(n)$ (or a *λ-vertex-saturated graph* $G(n, \lambda)$) if it is graceful (or λ-graceful). It is evident that each graceful tree itself is a vertex-saturated graph. Moreover, each graceful graph corresponds to a vertex-saturated graph. As an example, the graceful graph K_3

and its corresponding vertex-saturated graph $G(3)$ are depicted in Figure 1-(2) and (3).

Let $Z = \{0, 1, \ldots, n\}$ $(n \geq 1)$ and a_i and b_i, for $1 \leq i \leq n$, be the distinct elements of Z. A sequence $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ is called a *graceful sequence* $S(n)$ if $|a_i - b_i| = i$ for $1 \leq i \leq n$; each (a_i, b_i) is a *term* of a graceful sequence. Obviously, $(a_n, b_n) = (n, 0)$. The set of distinct elements of all terms in a graceful sequence $S(n)$ is called the *element set* of $S(n)$, denoted by $E(S(n))$. Two graceful sequences $S(n)$ and $S^*(n)$ are said to be *equivalent* (written $S(n) \sim S^*(n)$) if there is a bijection ψ: $E(S(n)) \rightarrow E(S^*(n))$ such that $(a_i, b_i) \in S(n)$ if and only if $(\psi(a_i), \psi(b_i)) \in S^*(n)$.

A graph H corresponding to a graceful sequence $S(m)$ is called an *induced graceful graph* of $S(m)$, denoted by $H(S(m))$; a graceful sequence corresponding to a graph H with m edges and a graceful labeling is said to be the *induced graceful sequence* of H. A graceful sequence $S(3) = \{(1, 0), (3, 1), (3, 0)\}$ and its induced graceful graph $G(S(3))$ (i.e., the complete graph K_3) are shown in Figure 1-(2). From the definitions, the following results are readily given.

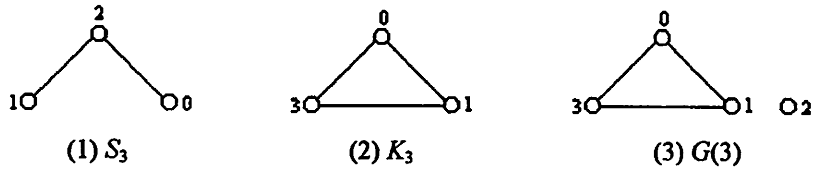


Figure 1

Corollary 1.1. *There is a one-to-one correspondence between a graph with n edges having a graceful labeling θ and a graceful sequence $\{(a_1, b_1), \ldots, (a_n, b_n)\}$ with n terms. The correspondence is given by*

$$a_i = \max\{\theta(x), \theta(y)\},$$
$$b_i = \min\{\theta(x), \theta(y)\}, \quad 1 \leq i \leq n,$$

where x, y are the end vertices of the edge labeled i.

Corollary 1.2. $S(n) \sim S^*(n)$ *if and only if* $G(S(n)) \cong G(S^*(n))$.

Note that the result in Corollary 1.2 will be repeatedly employed in the next section.

2 The main theorems

In this section we consider the gracefulness of four classes of graphs associated with vertex-saturated graphs. Let S_m and P_m denote a star and a path with m (≥ 2) vertices, respectively. For any vertices x and y in H, let $d(x, y)$ be the *length* of the shortest path joining x and y in H; in particular, $d(x, x) = 0$.

Construction I: *The join $G \vee H$ of disjoint graphs G and H is the graph obtained by joining each vertex of G to each vertex of H.*

Corollary 2.1. *The graph $G(n) \vee \overline{K_m}$ is graceful for $m \geq 1$.*

Proof: Suppose that $S(n) = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ is the induced graceful sequence of $G(n)$. Set $S_i(n+1) = \{(n + i(n+1), n), (n + i(n+1), n-1), \ldots, (n + i(n+1), 0)\}$, for $1 \leq i \leq m$. Clearly, $|E(G(n) \vee \overline{K_m})| = (n+1)m + n = e$. Define a sequence $S^*(e)$ as

$$\begin{aligned} S^*(e) &= S(n) \cup S_1(n+1) \cup \cdots \cup S_m(n+1) \\ &= \{(c_1, d_1), \ldots, (c_i, d_i), \ldots, (c_e, d_e)\}. \end{aligned}$$

(1) If $1 \leq i \leq n$, then $(c_i, d_i) = (a_i, b_i)$.

(2) If $i = j(n+1) + n - k$, $1 \leq j \leq m$ and $0 \leq k \leq n$, then $(c_i, d_i) = (n + j(n+1), k)$.

It is easily verified that $|c_i - d_i| = i$, for $1 \leq i \leq e$, and hence $S^*(e)$ is a graceful sequence. Therefore the graph $G(n) \vee \overline{K_m}$ induced by $S^*(e)$ is graceful. □

Remark: The graph $T + \overline{K_m}$ [5] is a special case of Corollary 2.1, where T is any graceful tree.

Corollary 2.2. *The graph $G(n) \vee S_m$ is graceful.*

Proof: Suppose that $S_1(n) = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ and $S_2(m-1) = \{(m-1, m-2), (m-1, m-3), \ldots, (m-1, 0)\}$ are the induced graceful sequences of $G(n)$ and S_m, respectively. Clearly, $|E(G(n) \vee S_m)| = (m+1)n + 2m - 1 = e$. Set $A_i(n+1) = \{((m-i+1)(n+2) - 2, n), ((m-i+1)(n+2) - 2, n-1), \ldots, ((m-i+1)(n+2) - 2, 0)\}$ $(1 \leq i \leq m-1)$, $A_m(n+1) = \{(e, n), (e, n-1), \ldots, (e, 0)\}$, and set $B(m-1) = \{(e, m(n+2) - 2), (e, (m-1)(n+2) - 2), \ldots, (e, 2n+2)\}$. Let $S^*(e)$ be a sequence given as

$$\begin{aligned} S^*(e) &= S_1(n) \cup A_1(n+1) \cup \cdots \cup A_m(n+1) \cup B(m-1) \\ &= \{(c_1, d_1), \ldots, (c_i, d_i), \ldots, (c_e, d_e)\}. \end{aligned}$$

(1) If $1 \leq i \leq n$, then $(c_i, d_i) = (a_i, b_i)$.

(2) If $i = (j+1)n + 2j + 1$, $0 \leq j \leq m-2$, then $(c_i, d_i) = (e, (m-j)(n+2) - 2)$.

(3) If $i = (j+1)n + 2j - k$, $1 \le j \le m-1$ and $0 \le k \le n$, then $(c_i, d_i) = ((j+1)(n+2) - 2, k)$.

(4) If $i = e - k$, $0 \le k \le n$, then $(c_i, d_i) = (e, k)$.

An easy computation shows that $|c_i - d_i| = i$, for $1 \le i \le e$, and so $S^*(e)$ is a graceful sequence. Since $G(S_1(n)) \cong G(n)$ and $G(B(m-1)) \cong S_m$, we have therefore the induced graceful graph of $S^*(e)$ is the graph $G(n) \vee S_m$. □

Remark: Corollary 2.2 generalizes Proposition 2.5 of [1].

Combining Corollary 2.1 and Corollary 2.2, we have

Theorem 2.3. *The graph $G(n) \vee (S_m \cup \overline{k_m})$ is graceful.*

The graph $G(3) \vee (S_3 \cup K_1)$, shown in Figure 2, is an example, where S_3 and $G(3)$ are the graphs of Figure 1-(1) and (3), respectively.

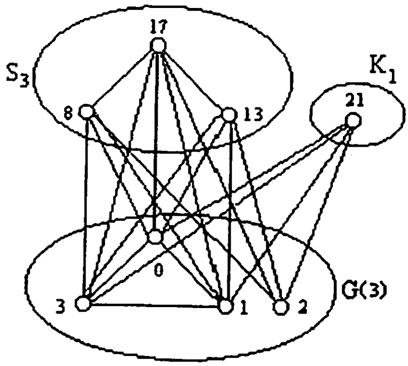


Figure 2

Construction II: The product of graphs G and H is the graph $G \times H$ with vertex set $V(G) \times V(H)$, in which vertex u_1v_1 is adjacent to vertex u_2v_2 if and only if either $u_1 = u_2$ and $(v_1, v_2) \in E(H)$ or $v_1 = v_2$ and $(u_1, u_2) \in E(G)$.

Suppose that (P, Q) is the bipartition of $G(n, \lambda)$, where $\theta(x) \le \lambda$, if $x \in P$ and $\theta(x) > \lambda$, if $x \in Q$. Let $|P| = \lambda + 1 = r$ and $|Q| = n - \lambda = s$.

Theorem 2.4. *If $|r - s| \le 1$, then the graph $G(n, \lambda) \times P_m$ is graceful.*

Proof: We may assume that $r = s$ or $r = s - 1$ since if $r > s$ then we may choose the complementary labeling $\tilde{\theta}$ of $G(n, \lambda)$ such that $r \le s$. Note that $r + s = n + 1$. Obviously, $|E(G(n, \lambda) \times P_m)| = (2m-1)n + m - 1 = e$. Let $S(n) = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ with $a_i > b_i$ $(1 \le i \le n)$ be the induced graceful sequence of $G(n,\lambda)$. Set sequences $A_i(n)$ and $B_j(n+1)$ as the following:

$$\begin{cases} A_{2i+1}(n) = \{(e-n-i(2n+1)+a_t, i(2n+1)+b_t) \mid 1 \le t \le n\}, \\ A_{2i+2}(n) = \{((i+1)(2n+1)-1-a_t, e-n-1-i(2n+1)-b_t) \mid 1 \le t \le n\}, \end{cases}$$

and

$$\begin{cases} B_{2j+1}(n+1) = \{(j(2n+1)+k, e-n-1-j(2n+1)-k), (e-n-j(2n+1)+p, \\ \qquad (j+1)(2n+1)-1-p) \mid 0 \le k \le r-1 \text{ and } r \le p \le n\}, \\ B_{2j+2}(n+1) = \{(e-n-1-j(2n+1)-k, (j+1)(2n+1)+k), ((j+1)(2n+1)-1-p, \\ \qquad e-n-(j+1)(2n+1)+p) \mid 0 \le k \le r-1 \text{ and } r \le p \le n\}. \end{cases}$$

Let $S^*(e)$ be a sequence defined as

$$\begin{aligned} S^*(e) &= A_1(n) \cup B_1(n+1) \cup A_2(n) \cup \cdots \cup B_{m-1}(n+1) \cup A_m(n) \\ &= \{(c_1,d_1),\ldots,(c_i,d_i),\ldots,(c_e,d_e)\}. \end{aligned}$$

Case 1: m is even, say $m = 2u$.

(1) If $i = 2q(2n+1)+t$, $0 \le q \le u-1$ and $1 \le t \le n$, then $(c_i,d_i) = ((u-q)(2n+1)-1-a_t,\ e-n-1-(u-q-1)(2n+1)-b_t)$.

(2) If $i = (2q+1)(2n+1)+t$, $0 \le q \le u-1$ and $1 \le t \le n$, then $(c_i,d_i) = (e-n-(u-q-1)(2n+1)+a_t,\ (u-q-1)(2n+1)+b_t)$.

(3) If $i = 2q(2n+1)+2n-2k$, $0 \le q \le u-1$ and $0 \le k \le r-1$, then $(c_i,d_i) = ((u-q-1)(2n+1)+k,\ e-n-1-(u-q-1)(2n+1)-k)$.

(4) If $i = 2q(2n+1)+2p+1$, $0 \le q \le u-1$ and $r \le p \le n$, then $(c_i,d_i) = (e-n-(u-q-1)(2n+1)+p,\ (u-q)(2n+1)-1-p)$.

(5) If $i = (2q+1)(2n+1)+2n-2k$, $0 \le q \le u-2$ and $0 \le k \le r-1$, then $(c_i,d_i) = (e-n-1-(u-q-2)(2n+1)-k,\ (u-q-1)(2n+1)+k)$.

(6) If $i = (2q+1)(2n+1)+2p+1$, $0 \le q \le u-2$ and $r \le p \le n$, then $(c_i,d_i) = ((u-q-1)(2n+1)-1-p,\ e-n-(u-q-1)(2n+1)+p)$.

Also, it can be verified that $|c_i - d_i| = i$, for $1 \le i \le e$, and thus $S^*(e)$ is a graceful sequence.

To prove that $S(n) \sim A_j(n)$ $(1 \le j \le m)$, define bijections $\hat{\theta}_i$ and $\tilde{\theta}_i$ $(0 \le i \le u-1)$ as the following.

(1) $\hat{\theta}_i\colon E(S(n)) \to E(A_{2i+1}(n))$ with $\hat{\theta}_i(x) = i(2n+1)+x$, if $0 \le x \le r-1$ and $\hat{\theta}_i(x) = e-n-i(2n+1)+x$, if $r \le x \le n$.

(2) $\tilde{\theta}_i\colon E(S(n)) \to E(A_{2i+2}(n))$ with $\tilde{\theta}_i(x) = e-n-1-i(2n+1)-x$, if $0 \le x \le r-1$ and $\tilde{\theta}_i(x) = (i+1)(2n+1)-1-x$, if $r \le x \le n$.

It is not difficult to see that $S(n) \sim A_{2i+1}(n)$ and $S(n) \sim A_{2i+2}(n)$ (i.e., $S(n) \sim A_i(n), 1 \leq i \leq m$), and it follows that $G(A_i(n)) \cong G(n, \lambda)$.

Next, we shall prove that the graphs $G(A_i(n))$ $(1 \leq i \leq m)$ are all edge-disjoint. To do it, we must show that $E(A_i(n)) \cap E(A_j(n)) = \emptyset$, for $1 \leq i$, $j \leq m$ and $i \neq j$.

(1) $E(A_{2i+1}(n)) \cap E(A_{2j+1}(n)) = \emptyset$, for $0 \leq i$, $j \leq u-1$ and $i \neq j$.

Suppose that $e - n - i(2n+1) + p_1 = e - n - j(2n+1) + p_2$ or $i(2n+1) + k_1 = j(2n+1) + k_2$, for $0 \leq k_1$, $k_2 \leq r-1$ and $r \leq p_1$, $p_2 \leq n$. Then we have $(j-i)(2n+1) = p_2 - p_1$ or $k_2 - k_1$, and it implies that $p_2 - p_1 = 0$ or $k_2 - k_1 = 0$ and hence $i = j$, a contradiction.

If $e - n - i(2n+1) + p_1 = j(2n+1) + k_1$, for $0 \leq k_1 \leq r-1$ and $r \leq p_1 \leq n$, then $p_1 - k_1 = (i+j+1-m)(2n+1) < 0$, a contradiction.

(2) $E(A_{2i+2}(n)) \cap E(A_{2j+2}(n)) = \emptyset$, for $0 \leq i$, $j \leq u-1$ and $i \neq$j.

Similar to (1) and omitted.

(3) $E(A_{2i+1}(n)) \cap E(A_{2j+2}(n)) = \emptyset$, for $0 \leq i$, $j \leq u-1$.

Suppose that $e - n - i(2n+1) + p = e - n - 1 - j(2n+1) - k$, for $0 \leq k \leq r-1$ and $r \leq p \leq n$. Then we have $(i-j)(2n+1) = p+k-1$, a contradiction.

If $i(2n+1) + k = (j+1)(2n+1) - 1 - p$, for $0 \leq k \leq r-1$ and $r \leq p \leq n$, then $(j+1-i)(2n+1) = p+k+1$, a contradiction.

If $e - n - i(2n+1) + p_1 = (j+1)(2n+1) - 1 - p_2$, for $r \leq p_1$, $p_2 \leq n$, then $(i+j+2-m)(2n+1) = p_1 + p_2 + 1$ and it implies that $i + j = m - 1$, contradicting the fact that $i + j \leq m - 2$.

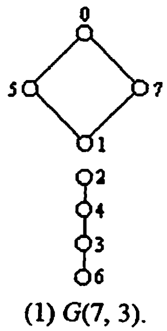
If $i(2n+1) + k_1 = e - n - 1 - j(2n+1) - k_2$, for $0 \leq k_1$, $k_2 \leq r-1$, then $k_1 + k_2 = (m-i-j-1)(2n+1) - 1 \geq 2n$, a contradiction.

Thus we assert that the graphs $G(A_i(n))$ $(1 \leq i \leq m)$ are all edge-disjoint.

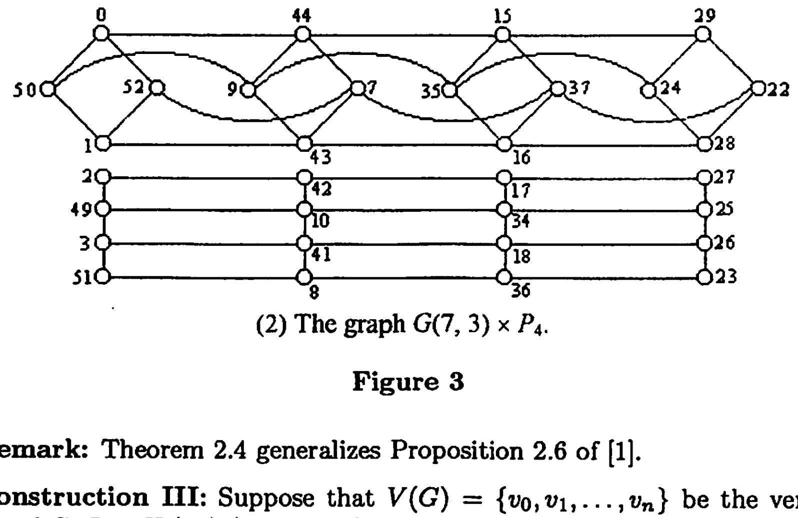
It remains to show that the $n+1$ edges between graphs $G(A_j(n))$ and $G(A_{j+1}(n))$ are contained in $B_j(n+1)$, $1 \leq j \leq m-1$. This can be done as follows. According to the definitions of bijections $\hat{\theta}_i$ and $\tilde{\theta}_i$, the terms $(i(2n+1)+k, e-n-1-i(2n+1)-k)$ and $(e-n-i(2n+1)+p, (i+1)(2n+1)-1-p)$ are contained in $B_{2i+1}(n+1)$, where $0 \leq k \leq r-1$ and $r \leq p \leq n$. Moreover, the terms $(e-n-1-i(2n+1)-k, (i+1)(2n+1)+k)$ and $((i+1)(2n+1)-1-p, e-n-(i+1)(2n+1)+p)$ are contained in $B_{2i+2}(n+1)$, where $0 \leq k \leq r-1$ and $r \leq p \leq n$.

We therefore conclude that the graph induced by $S^*(e)$ is indeed the graph $G(n, \lambda) \times P_m$ (see the graph $G(7,3) \times P_4$ drawn in Figure 3-(2), where $G(7,3)$ is the graph of Figure 3-(1)).

Case 2: p is odd. Similar to case 1 and omitted. □



(1) $G(7,3)$.



(2) The graph $G(7,3) \times P_4$.

Figure 3

Remark: Theorem 2.4 generalizes Proposition 2.6 of [1].

Construction III: Suppose that $V(G) = \{v_0, v_1, \ldots, v_n\}$ be the vertex set of G. Let $H_i(w_i)$ $(0 \leq i \leq n)$ be a connected graph with an arbitrary vertex w_i. Based upon the graph G, adjoin the graph $H_i(w_i)$ to each vertex v_i of G in such a manner that v_i and w_i are identified for $1 \leq i \leq n$. The resulting graph will be denoted by $G \oplus (H_1(w_1), \ldots, H_n(w_n))$ (see Figure 4).

Theorem 2.5. *Let $H(w,p)$ be a connected and graceful bipartite graph with m edges containing an arbitrary vertex w having label p $(0 \leq p \leq m)$. Then the graph $G(n) \oplus (H_0(w_0,p), \ldots, H_n(w_n,p))$ is graceful, where $H_i(w_i,p)$ is the isomorphic copy of $H(w,p)$, and w_i be the isomorphic image of w in $H(w,p)$ for $0 \leq i \leq n$.*

Proof: Suppose that $S(n) = \{(a_1,b_1), \ldots, (a_n,b_n)\}$ is the induced graceful sequence of $G(n)$. Let $B(m) = \{(c_1,d_1), \ldots, (c_m,d_m)\}$ be the induced graceful sequence of $H(w,p)$ with the property that $d(p,d_i)$ is odd and

$d(p, c_j)$ is even (≥ 0). Set $B_j(m) = \{(j(m+1) + c_1, (n-j)(m+1) + d_1), \ldots, (j(m+1) + c_m, (n-j)(m+1) + d_m)\}$, for $0 \leq j \leq n$, and set $\hat{S}(n) = \{((m+1)a_1 + p, (m+1)b_1 + p), \ldots, ((m+1)a_n + p, (m+1)b_n + p)\}$. Obviously, $|E(G(n) \oplus (H_0(w_0, p), \ldots, H_n(w_n, p)))| = n(m+1) + m = z$. Let us introduce a sequence $S^*(z)$ as follows:

$$S^*(z) = B_0(m) \cup \cdots \cup B_n(m) \cup \hat{S}(n)$$
$$= \{(e_1, f_1), \ldots, (e_i, f_i), \ldots, (e_z, f_z)\}.$$

Case 1: n is odd, say $n = 2k+1$.

(1) If $i = t(m+1)$, $1 \leq t \leq n$, then $(e_i, f_i) = ((m+1)a_t + p, (m+1)b_t + p)$.

(2) If $i = (2t+1)(m+1) \pm j$, $0 \leq t \leq k$ and $1 \leq j \leq m$, then

$$\begin{cases} (e_{(2t+1)(m+1)+j}, f_{(2t+1)(m+1)+j}) = ((k+t+1)(m+1) + c_j, (k-t)(m+1) + d_j) \\ (e_{(2t+1)(m+1)-j}, f_{(2t+1)(m+1)-j}) = ((k-t)(m+1) + c_j, (k+t+1)(m+1) + d_j) \end{cases},$$

if $c_j > d_j$

or

$$\begin{cases} (e_{(2t+1)(m+1)+j}, f_{(2t+1)(m+1)+j}) = ((k-t)(m+1) + c_j, (k+t+1)(m+1) + d_j) \\ (e_{(2t+1)(m+1)-j}, f_{(2t+1)(m+1)-j}) = ((k+t+1)(m+1) + c_j, (k-t)(m+1) + d_j) \end{cases},$$

if $c_j < d_j$.

Again, a routine verification shows that $|e_i - f_i| = i$, for $1 \leq i \leq z$, and so $S^*(z)$ is a graceful sequence.

To prove that $B(m) \sim B_j(m)$ $(0 \leq j \leq n)$ and $S(n) \sim \hat{S}(n)$, first define bijections θ_j $(0 \leq j \leq n)$: $E(B(m)) \to E(B_j(m))$ with $\theta_j(q) = j(m+1) + q$, if $d(p, q)$ (≥ 0) is even and $\theta_j(q) = (n-j)(m+1) + q$, if $d(p, q)$ is odd. For each term (c_i, d_i) in $B(m)$, $(c_i, d_i) \in B(m)$ if and only if $(\theta_j(c_i), \theta_j(d_i)) = (j(m+1) + c_i, (n-j)(m+1) + d_i) \in B_j(m)$. Furthermore, define a bijection $\theta^*: E(S(n)) \to E(\hat{S}(n))$ with $\theta^*(q) = q(m+1) + p$. For each term (a_i, b_i) in $S(n)$, $(a_i, b_i) \in S(n)$ if and only if $(\theta^*(a_i), \theta^*(b_i)) = ((m+1)a_i + p, (m+1)b_i + p) \in \hat{S}(n)$. It follows that $B(m) \sim B_j(m)$ and $S(n) \sim \hat{S}(n)$, and hence $G(B_j(m)) \cong H(w, p)$ (i.e., $H_j(w_j, p) \cong H(w, p), 0 \leq j \leq n$) and $G(\hat{S}(n)) \cong G(n)$.

Now, we shall show that the graphs $G(B_i(m))$ and $G(B_j(m))$ $(0 \leq i, j \leq n$ and $i \neq j)$ are all edge-disjoint, and the graphs $G(B_j(m))$ $(0 \leq j \leq n)$ and $G(\hat{S}(n))$ have only one vertex in common.

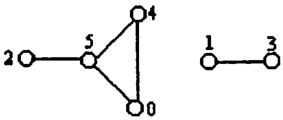
Suppose that $i(m+1) + c_r = j(m+1) + c_s$ or $(n-i)(m+1) + d_r = (n-j)(m+1) + d_s$. We have $(j-i)(m+1) = c_r - c_s$ (or $d_s - d_r$) which implies that $c_r = c_s$ (or $d_r = d_s$) and hence $i = j$, a contradiction.

If $i(m+1) + c_r = (n-j)(m+1) + d_s$ or $(n-i)(m+1) + d_r = j(m+1) + c_s$, then $c_r - d_s = c_s - d_r = (n-i-j)(m+1)$. If $n = i+j$, then $c_r - d_s = c_s - d_r =$

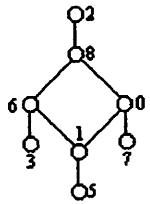
0, contradicting the fact that $c_r - d_s \neq 0$ and $c_s - d_r \neq 0$; if $n \neq i + j$, it means that $c_r - d_s$ (or $c_s - d_r$) $> m+1$ or $c_r - d_s$ (or $c_s - d_r$) $< -(m+1)$, a contradiction. Moreover, since $E(\hat{S}(n)) = \{p, m+1+p, \ldots, n(m+1)+p\}$, it is clear that $E(B_j(m)) \cap E(\hat{S}(n)) = j(m+1)+p$, for $0 \leq j \leq n$.

We thus conclude that the graphs $G(B_i(m))$ and $G(B_j(m))$ are edge-disjoint, and the graphs $G(B_j(m))$ and $G(\hat{S}(n))$ have only one vertex with label $j(m+1)+p$ in common. Therefore, the graph $G(n) \oplus (H_0(w_0, p), \ldots, H_n(w_n, p))$ is the induced graceful graph of $S^*(z)$ (the graph $G(5) \oplus (H_0(w_0, 2), \ldots, H_5(w_5, 2))$ of Figure 4-(3) is an example, where $G(5)$ and $H(w, 2)$ are depicted in Figure 4-(1) and (2), respectively).

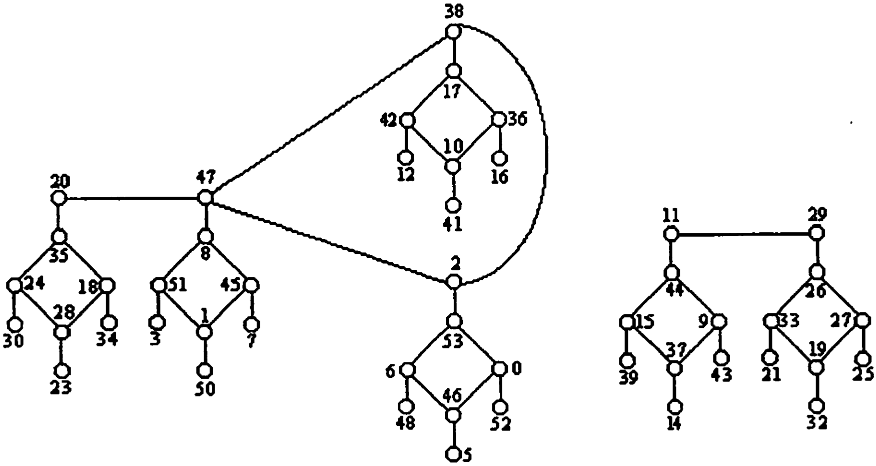
Case 2: n is even. Similar to case 1 and omitted. □



(1) $G(5)$



(2) $H(w, 2)$



(3) $G(5) \oplus (H_0(w_0, 2), \cdots, H_5(w_5, 2))$

Figure 4

Remark: Theorem 2.5 generalizes Theorem 3 of [2]; i.e., if both $G(n)$ and $H_i(w, p)$ are graceful trees, say $G(n) = T(n)$ and $H_i(w, p) = T_i(w, p)$, then the tree $T(n) \oplus (T_0(w_0, p), \ldots, T_n(w_n, p))$ is graceful.

As have seen in Theorem 2. 5, the graph $G(n)\oplus(H_0(w_0,p),\ldots,H_n(w_n,p))$ is graceful, in which $H_i(w_i,p)$ is isomorphic to $H(w,p)$ for $0 \le i \le n$. The following theorem permits graphs $H_i(w_i,p)$ $(0 \le i \le n)$ to be non-isomorphic if the number of edges in each $H_i(w_i,p)$ $(0 \le i \le n)$ is the same and each pair of graphs $H_i(w_i,p)$ and $H_{n-i}(w_{n-i},p)$ $(0 \le i \le [\frac{n-1}{2}])$ has the same boundary value.

Theorem 2.6. *Let $H_i(w_i,p,\lambda_i)$ $(0 \le i \le n)$ be a connected and λ-graceful bipartite graph with m edges containing an arbitrary vertex w_i having label p $(0 \le p \le m)$. If each pair of graphs $H_i(w_i,p,\lambda_i)$ and $H_{n-i}(w_{n-i},p,\lambda_{n-i})$ $(0 \le i \le [\frac{n-1}{2}])$ has the same boundary value (i.e., $\lambda_i = \lambda_{n-i}$), then the graph $G(n) \oplus (H_0(w_0,p,\lambda_0),\ldots,H_n(w_n,p,\lambda_n))$ is graceful.*

Proof: Suppose that $S(n) = \{(a_1,b_1),\ldots,(a_n,b_n)\}$ is the induced graceful sequence of $G(n)$. Let $S_i(m) = \{(c_{i1},d_{i1}),\ldots,(c_{im},d_{im})\}$ $(0 \le i \le n)$ be the induced graceful sequence of $H_i(w_i,p,\lambda_i)$ with the property that $d(p,d_{ij})$ is odd and $d(p,c_{ij})$ is even (≥ 0), for $1 \le j \le m$. Set $\hat{S}_i(m) = \{(i(m+1)+c_{i1},(n-i)(m+1)+d_{i1}),\ldots,(i(m+1)+c_{im},(n-i)(m+1)+d_{im})\}$, for $0 \le i \le n$, and set $\hat{S}(n) = \{((m+1)a_1+p,(m+1)b_1+p),\ldots,((m+1)a_n+p,(m+1)b_n+p)\}$. Clearly, $|E(G(n)\oplus(H_0(w_0,p,\lambda_0),\ldots,H_n(w_n,p,\lambda_n)))| = n(m+1)+m = z$. Define a sequence $S^*(z)$ as

$$\begin{aligned} S^*(z) &= \hat{S}(n) \cup \hat{S}_0(m) \cup \cdots \cup \hat{S}_n(m) \\ &= \{(e_1,f_1),\ldots,(e_i,f_i),\ldots,(e_z,f_z)\}. \end{aligned}$$

Case 1: n is odd, say $n = 2k+1$.

(1) If $i = t(m+1)$, $1 \le t \le n$, then $(e_i,f_i) = ((m+1)a_t+p,(m+1)b_t+p)$.

(2) If $i = (2t+1)(m+1) \pm s$, $0 \le t \le k$ and $1 \le s \le m$, then

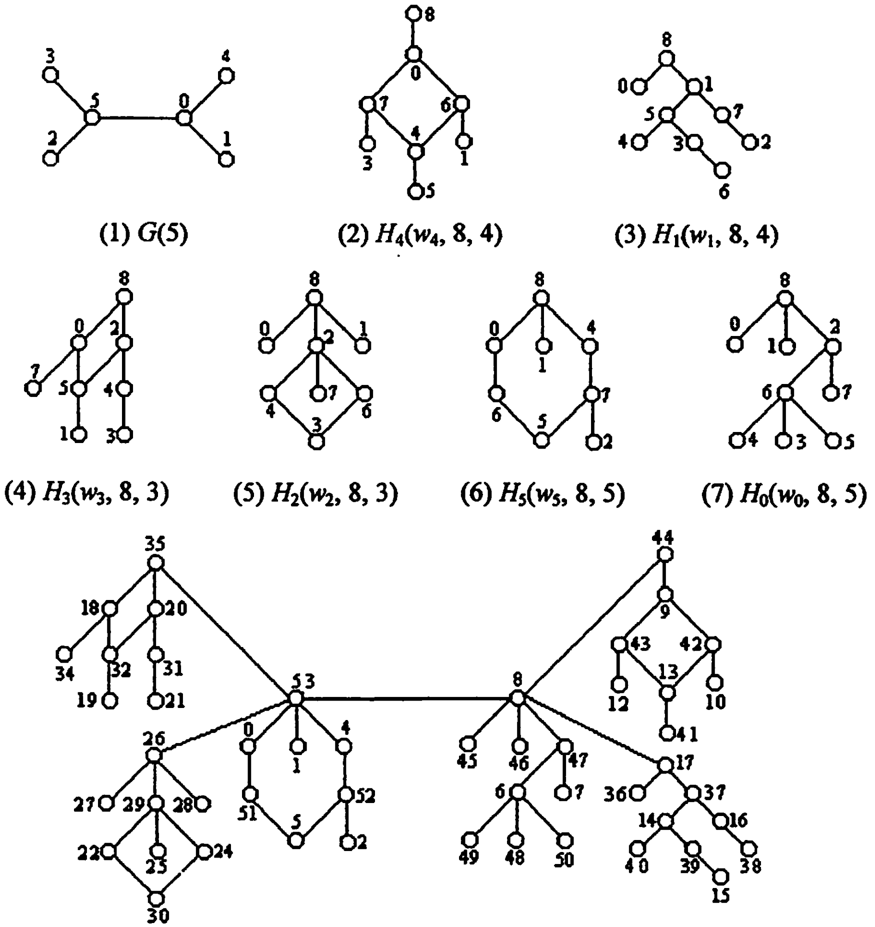
$$\begin{cases} (e_{(2t+1)(m+1)+s}, f_{(2t+1)(m+1)+s}) = ((k+t+1)(m+1)+c_{(k+t+1)s}, (k-t)(m+1)+d_{(k+t+1)s}) \\ (e_{(2t+1)(m+1)-s}, f_{(2t+1)(m+1)-s}) = ((k-t)(m+1)+c_{(k-t)s}, (k+t+1)(m+1)+d_{(k-t)s}) \end{cases},$$
$$\text{if } c_{(k+t+1)s} > d_{(k+t+1)s}$$

or

$$\begin{cases} (e_{(2t+1)(m+1)+s}, f_{(2t+1)(m+1)+s}) = ((k-t)(m+1)+c_{(k+t+1)s}, (k+t+1)(m+1)+d_{(k+t+1)s}) \\ (e_{(2t+1)(m+1)-s}, f_{(2t+1)(m+1)-s}) = ((k+t+1)(m+1)+c_{(k-t)s}, (k-t)(m+1)+d_{(k-t)s}) \end{cases},$$
$$\text{if } c_{(k+t+1)s} < d_{(k+t+1)s}.$$

The remaining verifications are analogous to those of Theorem 2.5 and omitted (see Figure 5-(1) $\sim$ (8)).

Case 2: n is even. Similar to case 1 and omitted. □



(1) $G(5)$ (2) $H_4(w_4, 8, 4)$ (3) $H_1(w_1, 8, 4)$

(4) $H_3(w_3, 8, 3)$ (5) $H_2(w_2, 8, 3)$ (6) $H_5(w_5, 8, 5)$ (7) $H_0(w_0, 8, 5)$

(8) $G(5) \oplus (H_0(w_0, 8, 5), \cdots, H_5(w_5, 8, 5))$

Figure 5

In Theorem 2.6, suppose that both $G(n)$ and $H_i(w_i, p, \lambda_i)$ are graceful trees. Let $G(n) = T(n)$ and let $H_i(w_i, p, \lambda_i) = T_i(w_i, p, \lambda_i)$. We have

Corollary 2.7: *The tree $T(n) \oplus (T_0(w_0, p, \lambda_0), \ldots, T_n(w_n, p, \lambda_n))$ is graceful.*

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