On Quadrangulations of Closed Surfaces Covered by Vertices of Degree 3

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Abstract

A graph is said to be *k-covered* if for each edge xy, $\deg(x) = k$ or $\deg(y) = k$. In this paper, we characterize the 3-covered quadrangulations of closed surfaces.

1 Introduction

In this paper, we assume that a graph has already been 2-cell embedded in some closed surface F^2. We denote the Euler characteristic of F^2 by $\chi(F^2)$. We denote the vertex-set, the edge-set and the face-set of a graph G by $V(G)$, $E(G)$ and $F(G)$, respectively. For $S \subset V(G)$, we denote by $\langle S \rangle$ the subgraph of G induced by the vertices in S. A *k-cycle* C_k is a cycle of length exactly k.

A *quadrangulation* G of a closed surface F^2 is a simple graph on F^2 such that each face is bounded by a 4-cycle. A graph G is called *k-covered* if for any edge xy of G, $\deg(x) = k$ or $\deg(y) = k$.

In this paper, we characterize the 3-covered quadrangulations of closed surfaces. The 4-,5- and 6-covered triangulations have been characterized in [7].

Let G be a graph on a closed surface F^2 with the vertices colored black. Put a white vertex v in a face f of G bounded by a closed walk $v_1 \cdots v_n$, and join v with v_i for $i = 1, \ldots, n$. Apply this procedures to all faces and delete all edges of G. The resulting graph on F^2 is said to be the *radial graph* of G and denoted by $R(G)$. (This was first defined in [2].) Observe that $R(G)$ is bipartite and each face of $R(G)$ is bounded by a 4-cycle.

Embed a cycle $v_1u_1v_2u_2 \ldots v_nu_n$ ($n \geq 2$) of even length into the sphere along the equator, put vertices x and y on the south pole and the north pole respectively, and add edges xv_i and yu_i for $i = 1, \ldots, n$. The resulting quadrangulation of the sphere with $2n + 2$ vertices is said to be the *pseudo double wheel* with rim C_{2n} and denoted by W_{2n}. (A "double wheel" usually means a maximal planar graph

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isomorphic to $C_k + \bar{K}_2$, and hence we attached "pseudo" for representing these quadrangulations.) See the left-hand side in Figure 1.

A *pseudo-triangulation* of a closed surface F^2 is a loopless graph on F^2 possibly with multiple edges such that each face is bounded by a 3-cycle.

Recently, Ando, Komuro and Nakamoto have shown the following theorem.

THEOREM 1 (Ando, Komuro and Nakamoto [1]) *A plane quadrangulation G is 3-covered if and only if either*

(i) G is a pseudo double wheel, or

(ii) G is the radial graph of some plane pseudo-triangulation.

In this paper, we extend this theorem to show the following theorem, introducing another infinite family of quadrangulations on the projective plane which are 3-covered but not radial.

Embed a cycle $C = v_1v_2 \dots v_{2n-1}$ ($n \geq 2$) of odd length into the projective plane so that the tubular neighborhood of C forms a Möbius band. Next, put a vertex x on the center of the unique face of the embedding and join x with v_i for all i so that the resulting graph is a quadrangulation. The resulting quadrangulation of the projective plane with $2n$ vertices is said to be the *Möbius wheel* with rim C_{2n-1} and denoted by $\tilde{W}_{2n-1}$. See the right-hand side in Figure 1, where each antipodal pair of vertices is identified.

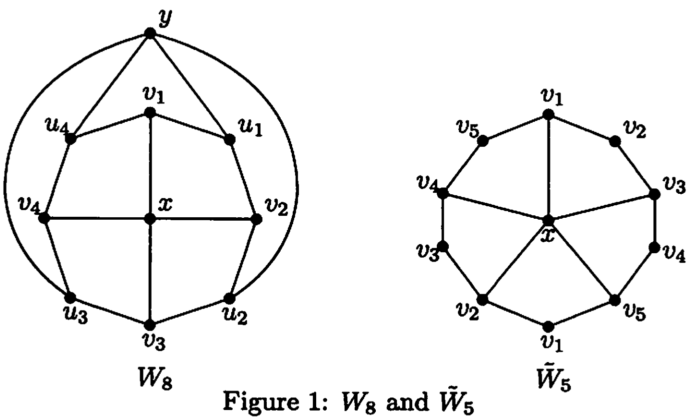


Figure 1: W_8 and $\tilde{W}_5$

THEOREM 2 *A quadrangulation G of a closed surface F^2 is 3-covered if and only if either*

(i) G is the radial graph of some pseudo-triangulation on F^2,

(ii) G is a pseudo double wheel on the sphere, or

(iii) G is a Möbius wheel on the projective plane.

Surprisingly, Theorem 2 asserts that only the projective plane admits non-bipartite 3-covered quadrangulations. Theorem 2 has interesting applications as well as Theorem 1. Many theorems for triangulations [6, 8, 9] can be translated into those for 3-covered quadrangulations [3, 4], as described in [1].

This research is motivated by the following problem. Let $k \geq 2$ be a positive integer. A *k-quadrangulation* is a quadrangulation with minimum degree at least k. (Note that ordinary quadrangulations correspond to the 2-quadrangulations.) Now we define two local transformations for k-quadrangulations. A *k-diagonal slide* is to replace a diagonal fc with ad in a hexagonal region $abcdef$ formed by two quadrilateral faces $abcf$ and $cdef$ sharing fc. See the upper side of Figure 2. Now suppose that G has a vertex v of degree k with neighbors $\{x_1, \ldots, x_k\}$, where k faces $x_1y_1x_2v, x_2y_2x_3v, \ldots, x_ky_kx_1v$ meet at v. A *k-diagonal rotation* is to delete vx_i and add vy_i for $i = 1, \ldots, k$. See the lower side of Figure 2, which stands for a 2-diagonal rotation. We don't perform either of these operations if the resulting graph is not simple or not a k-quadrangulation. Two k-quadrangulations G_1 and G_2 are said to be *k-equivalent* if G_1 and G_2 can be transformed into each other by a sequence of k-diagonal slides and k-diagonal rotations, through k-quadrangulations.

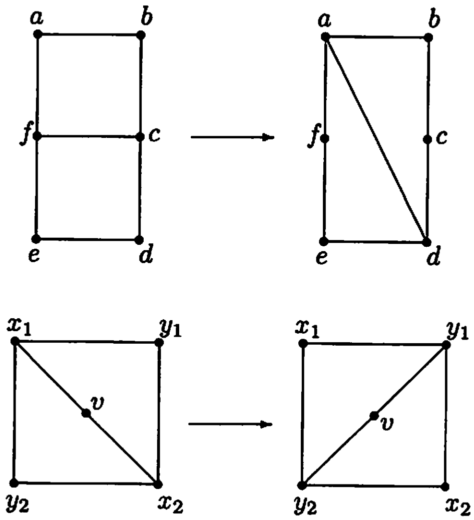


Figure 2: k-diagonal slide and k-diagonal rotation

The second author has shown the following theorem.

THEOREM 3 **(Nakamoto [3])** *Any two quadrangulations G_1 and G_2 of the same closed surface F^2 with the same and sufficiently large number of vertices are 2-equivalent, up to homeomorphism, if for each closed curve l on F^2, a cycle* *in G_1 and a cycle in G_2 each of which is homotopic to l have the same length modulo 2.*

Note that any two cycles in a quadrangulation have the same parity of length if they are homotopic on the surface. Moreover, for any fixed homotopy class $[l]$ on the surface, the length of the cycles belonging to $[l]$ have the same parity of length before and after applying each of a k-diagonal slide and a k-diagonal rotation for any k. (This implies that the both transformations preserve the bipartiteness of given graphs.)

We had conjectured that the 2-equivalence in Theorem 3 was replaceable with the 3-equivalence. However, this conjecture is solved negatively, as follows. By Theorem 2, for any closed surface F^2, there exist infinitely many 3-covered quadrangulations of F^2. Observe that a 3-covered quadrangulation has no edge e which can be moved by a 3-diagonal slide since the operation will decrease the degree of one of endpoints of e to 2. Moreover, we can also construct infinitely many 3-covered quadrangulations G such that each vertex of degree 3 is adjacent to at least one vertex of degree 3. It is easy to see that we can apply no 3-diagonal rotation to each vertex of degree 3 in such G. Thus, for any closed surface F^2 and any arbitrarily large integer n, there exists a bipartite quadrangulation G with at least n vertices such that neither a 3-diagonal slide nor a 3-diagonal rotation can be applied to G.

However, the conjecture remains in non-bipartite case. Except the projective plane, there exist no non-bipartite 3-covered quadrangulations, by Theorem 2. Hence we cannot construct such counterexamples in non-bipartite case. We conjecture the following.

CONJECTURE 4 *Except Möbius wheels, any two non-bipartite 3-quadrangulations G_1 and G_2 of the same closed surface F^2 with the same and sufficiently large number of vertices are 3-equivalent, up to homeomorphism, if for each closed curve l on F^2, a cycle in G_1 and a cycle in G_2 each of which have the same length modulo 2.*

2 Proof of theorems

The following theorem can be easily proved.

THEOREM 5 *A quadrangulation G is 2-covered if and only if G is isomorphic to $K_{2,m}$ on the sphere for some integer $m \geq 2$.*

Proof of Theorem 5. Let G be a 2-covered quadrangulation of a closed surface. Let z_1 be a vertex of degree 2 which is contained in a 4-cycle xz_1yz_2 bounding a face. If x has degree 2, then the path yz_1xz_2 are shared by two adjacent faces of G. Since G has no multiple edges, G is isomorphic to $K_{2,2}$. Thus, we may suppose that $\deg(x) \geq 3$. By symmetry, we also have that $\deg(y) \geq 3$. Since G is 2-covered, we must have that $\deg(z_2) = 2$. Consider the face, say xz_iyz_{i+1}, sharing the path xz_iy with the face $xz_{i-1}yz_i$ for $i = 2, \dots$ since $\deg(z_{i+1}) = 2$, by the 2-covering of G. Continue this until z_{n+1} coincides with z_1. ■

The following lemma is a modified form of the lemma in [1, 5]. Throughout this section, let G be a quadrangulation, and $S = \{v \in V(G) | \deg(v) = 3\}$.

LEMMA 6 (akn,gene) *If H is a component of $\langle S \rangle$, then H is isomorphic to one of the following graphs :*

(i) $H = K_1$ *: an isolated vertex,*

(ii) $H = K_{1,3}$ *: a claw,*

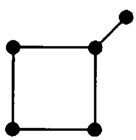
(iii) $H = P_n$ *: a path of length* $n \geq 1$*,*

(iv) $H = C_4$ *: a 4-cycle bounding a face,*

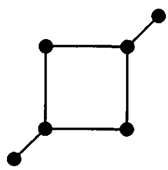
(v) $H = C_{2n}$ *(or* $H = G = W_6$*) : a cycle of even length* $2n$ *with* $n \geq 2$ *and* $n \neq 3$ *(or a pseudo double wheel with rim* C_6*),*

(vi) $H = C_{2n-1}$ *(or* $H = G = \tilde{W}_3$*) : a cycle of odd length* $2n - 1 \geq 5$ *(or a Möbius wheel with rim* C_3*),*

(vii)



(viii)



The 4-cycles in (vii) and (viii) bound faces in G.

In particular, the case (v) happens only on the sphere, and the case (vi) happens only on the projective plane. Moreover, in these cases, G is isomorphic to a pseudo double wheel or a Möbius wheel, respectively.

We denote the graphs in the cases (vii) and (viii) by C_4^+ and C_4^{++}, respectively.

LEMMA 7 *If G is 3-covered but it is not isomorphic to a pseudo double wheel or a Möbius wheel, then each component of $\langle S \rangle$ is isomorphic to either K_1, $K_{1,3}$ or C_4^{++}.*

Proof. Since G is not isomorphic to a pseudo double wheel or a Möbius wheel, it suffices to show that G cannot have $P_n (n \geq 1)$, C_4 and C_4^+ as components of $\langle S \rangle$, by Lemma 6.

Suppose that G has $P_n = v_0 v_1 \cdots v_n$ as a component of $\langle S \rangle$. Since $\deg_{\langle S \rangle}(v_0) = 1$ and $1 \leq \deg_{\langle S \rangle}(v_1) \leq 2$, there are two distinct vertices $x_1, x_2 \notin S$ incident with v_0, and there is a vertex $y \notin S$ incident with v_1. Since $\deg_G(v_1) = 3$, the path $y v_1 v_0$ lies on the boundary of some face. Thus, either $x_1 v_0 v_1 y$ or $x_2 v_0 v_1 y$ forms the boundary 4-cycle of some face in G, and hence either $x_1 y \in E(G)$ or $x_2 y \in E(G)$, contrary to G being 3-covered.

Now suppose that G has C_4 or C_4^+ as a component of $\langle S \rangle$. Let $v_1 v_2 v_3 v_4$ be the cycle in C_4 or C_4^+ bounding a face. We may suppose that $\deg_{\langle S \rangle}(v_1) = \deg_{\langle S \rangle}(v_2) = 2$. Focus on the face, say $v_1 v_2 x y$, sharing the edge $v_1 v_2$ with the face

$v_1v_2v_3v_4$. The edge xy is one such that $\deg_G(x) \neq 3$ and $\deg_G(y) \neq 3$, contrary to G being 3-covered. ■

Now we show Theorem 2.

Proof of Theorem 2. Suppose that G is 3-covered but not isomorphic to either a pseudo double wheel or a Möbius wheel, and that all vertices of G are colored black. By Lemma 7, G can have only K_1, $K_{1,3}$ and C_4^{++} as components of $\langle S \rangle$.

Paint all vertices in S in white. If each component of $\langle S \rangle$ is K_1, then G must be a bipartite quadrangulation of F^2, by the 3-covering of G. Moreover, since each white vertex of G has degree 3, G must be the radial graph of some graph T with all faces of T bounded by 3-cycles. Since T has no loop, T is a loopless pseudo-triangulation of F^2.

In general, G might have $K_{1,3}$ and C_4^{++} as components of $\langle S \rangle$. However, giving the 2-coloring of $K_{1,3}$ and C_4^{++} shown in Figure 3, we can obtain a 2-coloring of G such that each white vertex has degree 3. Thus, G is the radial graph of some loopless pseudo-triangulation of F^2. ■

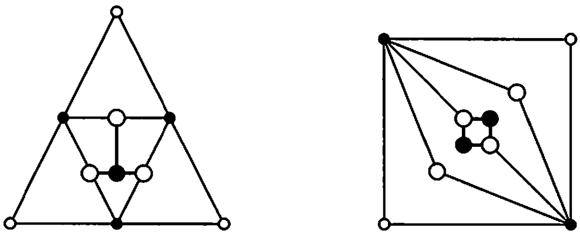


Figure 3: 2-coloring of $K_{1,3}$ and C_4^{++}

By Theorems 2 and 5, we have the conclusion that for $k = 2, 3$, every k-covered quadrangulations with few exceptions is the radial graph of a graph with each face bounded by a k-cycle. However, for $k = 4$, this fact does not hold.

Let G be a bipartite 4-covered quadrangulation. Let $e_1, e_2, \ldots, e_m$ be a sequence of distinct edges of G such that for $i = 1, \ldots, m$, e_i and e_{i+1} lie on the same face of G but they don't share endpoints, where the subscripts are taken modulo m. Put a vertex v_i on the middle of e_i for each i, and join v_i and v_{i+1} for $i = 1, \ldots, m$. (If some v_iv_{i+1} and v_jv_{j+1} cross at the center of some face, then put a vertex on the intersecting point.) Then, the resulting quadrangulation is 4-covered, but it is not necessarily bipartite. This operation does not preserve the bipartiteness of graphs in general.

References

[1] K. Ando, H. Komuro and A. Nakamoto, Quadrangulations with minimum degree three, to appear in *Discrete Math.*

[2] D. Archdeacon and R.B. Richter, The construction and classification of self-dual spherical polyhedra, *J. Combin. Theory Ser. B* **54** (1992), 37–63.

[3] A. Nakamoto, Diagonal transformations of quadrangulations of surfaces, *J. Graph Theory* **21** (1996), 289–299.

[4] A. Nakamoto, Diagonal transformations and cycle parity of quadrangulations on surfaces, *J. Combin. Theory, Ser. B* **67** (1996), 202–211

[5] A. Nakamoto, Generating quadrangulations of surfaces with minimum degree at least 3, *J. Graph Theory* **30** (1999), 223–234.

[6] S. Negami, Diagonal flips in triangulations of surfaces, *Discrete Math.* **135** (1994), 225–232.

[7] S. Negami and A. Nakamoto, Triangulations on closed surfaces covered by vertices of given degree, to appear in *Graphs and Combinatorics.*

[8] S. Negami and T. Watanabe, Diagonal flips in pseudo-triangulations on closed surfaces without loops, *Yokohama Math. J* **47** Special Issue (1999), 213–223.

[9] S. Negami and S. Watanabe, Diagonal transformations of triangulations on surfaces, *Tsukuba J. Math.* **14** (1990), 155–166.