Extremal Matrices of Generalized Exponents of Primitive Nearly Reducible Matrices

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ABSTRACT. The $n \times n$ primitive nearly reducible Boolean matrices whose k-exponents ($1 \le k \le n$) achieve the maximum value are characterized.

1 Introduction

We consider the binary Boolean matrices. They are matrices whose entries are 0 or 1; the arithmetic underlying the matrix multiplication and addition is Boolean, that is, it is the usual integer arithmetic except that $1 + 1 = 1$.

The study of the Boolean matrices is closely related to the combinatorial properties of nonnegative matrices. By combinatorial properties we mean those properties of a matrix A which depend only upon the zero-nonzero pattern of A.

Let B_n be the set of all $n \times n$ Boolean matrices. A matrix $A \in B_n$ is *primitive* if one of its powers, A^k, is the all 1's matrix for some positive integer k. The minimum such k is called the *exponent* of A, which we denote by $\exp(A)$.

Let $A \in B_n$, and let k be an integer with $1 \le k \le n$. If A is primitive, then there is a positive integer m such that there exists a set of k rows in A^m whose entries are all 1's. The minimum such m is called the kth *generalized exponent* or in short the k-exponent of A, which we denote by $\exp(A, k)$. For any primitive matrix $A \in B_n$, clearly we have

$$\exp(A, 1) \le \exp(A, 2) \le \cdots \le \exp(A, n) = \exp(A).$$

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Hence, the concept of k-exponent is a generalization of the traditional concept of exponent.

It has been proved in [1] that

$$\exp(A,k) \le n^2 - 3n + k + 2$$

for any primitive matrix $A \in B_n$, In [2] the matrices whose k-exponents achieve this bound have been characterized.

A matrix $A \in B_n$ is reducible if there is a permutation matrix P such that

$$PAP^t = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where the diagonal blocks A_{11} and A_{22} are square and A_{21} is a (nonvacuous) block of zeros. A square Boolean matrix which is not reducible is said to be *irreducible*. We say A and PAP^t are *permutationally similar*.

Note that primitive matrices are included in the class of irreducible matrices.

A Boolean matrix A is *nearly reducible* if A is irreducible but each matrix obtained from A by replacing a 1 by 0 is reducible.

Let

$$e(n,k) = \begin{cases} n^2 - 5n + 7 + k & 1 \le k \le n-2, \\ n^2 - 4n + 5 & k - n - 1, \\ n^2 - 4n + 6 & k = n. \end{cases}$$

Recently Liu ([3]) showed that $\exp(A,k) \le e(n,k)$ for any $n \times n$ primitive nearly reducible matrix A, and he also gave an example to show this bound can be attained. Thus $e(n,k)$ is the maximum value of the k-exponent over all $n \times n$ primitive nearly reducible matrices. In this paper, we first give a new proof for the expression of the maximum value of the k-exponents of matrices in the class of primitive nearly reducible matrices for each k and n with $1 \le k \le n$. Then we provide a complete characterization of the matrices whose k-exponent achieve $e(n,k)$.

2 Preliminaries

For $A = (a_{ij}) \in B_n$, its *associated digraph* $D(A)$ is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $E = \{(i,j): a_{ij} = 1\}$. Thus A (viewed as a $(0,1)$ matrix) is just the adjacency matrix of $D(A)$. Clearly $D(A)$ allows loops but no multiple arcs. Recall that $A \in B_n$ is permutationally similar to B if and only if $D(A)$ is isomorphic to $D(B)$.

It is well known that a matrix A is irreducible if and only if $D(A)$ is strongly connected; A is nearly reducible if and only if $D(A)$ is minimally strong.

A digraph D is primitive if its adjacency matrix $A(D)$ is primitive. For convenience, we define the exponent of a primitive digraph D to be the exponent of $A(D)$. Thus we write $\exp(D) = \exp(A(D))$.

We now assume that D is a primitive digraph of order n. The exponent of a vertex x of D is the minimum integer $\exp_D(x)$ such that for every integer $t \geq \exp_D(x)$ there is a walk of length t from x to each vertex of D (including x). The primitivity of D implies that $\exp_D(x)$ exists for every vertex x.

Assume the vertices of D have been labelled as

$$x_1, x_2, \ldots, x_n$$

in order that

$$1 \leq \exp_D(x_1) \leq \exp_D(x_2) \leq \cdots \leq \exp_D(x_n).$$

Then we call $\exp_D(x_k)$ the k-exponent of D, and denote it by $\exp(D, k)$. Obviously we have $\exp(A, k) = \exp(D(A), k)$ if $A \in B_n$ is primitive.

To prove the main results, we establish the following lemmas.

Lemma 1. ([4]) *A digraph D is primitive if and only if D satisfies the following two conditions:*

(1) *D is strongly connected;*

(2) *the greatest common divisor of the cycle lengths of D is* 1.

Lemma 2. ([1]) *Let D be a primitive digraph of order n, $2 \leq k \leq n$. Then*

$$\exp(D, k) \leq \exp(D, k-1) + 1.$$

The next lemma was first obtained in [5] . For completeness a proof is included here.

Lemma 3. *Let s be the length of the shortest cycle C of the primitive digraph D of order n, and let r be the maximum outdegree of vertices in C. Then*

$$\exp(D, 1) \leq s(n - r) + 1.$$

Proof: Suppose x is a vertex of C with the maximum outdegree r. Let $V_1 = \{y \in V(D)\colon (x, y) \in E(D)\}$. Then $|V(C) \cap V_1| = 1$ and let $x_1 \in V(C) \cap V_1$. Obviously, there is a walk of length s from x_1 to every vertex in V_1. Let $A = A(D)$. Then (x_1, y) is an arc of $D(A^s)$ for every $y \in V_1$ and $D(A^s)$ is strongly connected by the primitivity of D. Hence there is at most one vertex which is not reachable by any walk of length $n - r$ starting from x_1 in $D(A^s)$.

For any vertex which is reachable by some walk of length $n-r$ starting from x_1 in $D(A^s)$, it is reachable by a walk of length $s(n-r)$ starting from x_1 in D, and hence reachable by a walk of length $s(n-r)+1$ starting from x (via x_1) in D.

If there exists a vertex w which is not reachable by any walk of length $n-r$ starting from x_1 in $D(A^s)$, then there is a path Q of length $n-r+1$ from x_1 to w in $D(A^s)$. Note that Q must meet some vertex, say z, in V_1 and there is a path of length $n-r$ from z to w in $D(A^s)$. Thus w is reachable by a walk of length $s(n-r)$ starting from z in D, and hence reachable by a walk of length $s(n-r)+1$ starting from x (via z) in D.

We conclude that each vertex is reachable by a walk of length $s(n-r)+1$ starting from x in D. This completes the proof of this lemma. □

Lemma 4. *Let D be a primitive digraph of order n, $1 \le k \le n$, and let s be the length of the shortest cycle of D. Then*

$$\exp(D,k) \le s(n-2)+k.$$

Proof: Assume C is a cycle of length s in D and r is the maximum outdegree of vertices in C. Using Lemma 1, the primitivity of D implies $r \ge 2$. By Lemma 3, we have $\exp(D,1) \le s(n-2)+1$. For $2 \le k \le n$, by Lemma 2, $\exp(D,k) \le \exp(D,1)+k-1 \le s(n-2)+k$. The proof is complete. □

Let D_{n-2} ($n \ge 4$) be a digraph with $V(D_{n-2}) = \{1,2,\dots,n\}$ and $E(D_{n-2}) = \{(i,i+1)\colon i=1,2,\dots,n-2\} \cup \{(n-1,1),(n-2,2),(n,2)\}$; let D^1_{n-3} ($n \ge 6$) be a digraph with $V(D^1_{n-3}) = \{1,2,\dots,n\}$ and $E(D^1_{n-3}) = \{(i,i+1)\colon i=1,2,\dots,n-3\} \cup \{(n-2,1),(n-4,n-1),(n-1,n),(n,2)\}$; and let D^2_{n-3} ($n \ge 6$, n is even) be a digraph with $V(D^2_{n-3}) = \{1,2,\dots,n\}$ and $E(D^2_{n-3}) = \{(i,i+1)\colon i=1,2,\dots,n-2\} \cup \{(n-1,1),(n-3,n),(n,2)\}$.

Lemma 5. ([3], [6]) *For D_{n-2}, D^1_{n-3} and D^2_{n-3}, we have*

$$\exp(D_{n-2},k) = e(n,k), 1 \le k \le n,$$

$$\exp(D^1_{n-3}) \le n^2-6n+12,$$

$$\exp(D^2_{n-3},1) \le n^2-5n+5.$$

3 Main Results

Let PNR_n be the set of all $n \times n$ primitive nearly reducible Boolean matrices. The following theorem was obtained in [3]. Here we provide a new proof.

Theorem 1. *For $n \geq 4$, $1 \leq k \leq n$, we have*

$$\max\{\exp(A,k) \colon A \in PNR_n\} = e(n,k).$$

Proof: Suppose $A \in PNR_n$. Then $D(A)$ is primitive and minimally strong. Let s be the length of the shortest cycle of $D(A)$. It follows easily from Lemma 1 that $s \leq n-2$ and $D(A)$ can not have a cycle of length n.

Case 1: $s = n-2$. By Lemma 1, the set of distinct cycle lengths of $D(A)$ is $\{n-2, n-1\}$. Take a cycle C of length $n-1$. There is exactly one vertex, say x, lying outside C. Then x must lie on a cycle of length $n-2$. Otherwise x lies on a cycle C' of length $n-1$, and there is at least one arc besides those in C and C', contradicting the minimally strong connectedness of $D(A)$. We conclude that $D(A)$ is isomorphic to D_{n-2}. Hence by Lemma 5, $\exp(D(A),k) = \exp(D_{n-2},k) = e(n,k)$.

Case 2: $s \leq n-3$. By Lemma 4, we have

$$\begin{aligned}\exp(D(A),k) &\leq s(n-2)+k \\ &\leq (n-3)(n-2)+k \\ &= \begin{cases} n^2-5n+6+k & 1 \leq k \leq n-2 \\ n^2-4n+5 & k = n-1 \\ n^2-4n+6 & k = n \end{cases} \\ &\leq e(n,k).\end{aligned}$$

Hence for any $A \in PNR_n$, $\exp(A,k) = \exp(D(A),k) \leq e(n,k)$, equality holds for $A(D_{n-2})$. The proof is complete. □

Let $EM(n,k) = \{A \in PNR_n \colon \exp(A,k) = e(n,k)\}$. $EM(n,k)$ is called the *extremal matrix* set for k-exponent of the matrix class PNR_n. Any matrix in $EM(n,k)$ is called an *extremal matrix.* The following theorem gives a characterization of $EM(n,k)$.

Theorem 2. *Suppose $n > 4$ and $1 \leq k \leq n$. Then $A \in EM(n,k)$ if and only if A is permutationally similar to the matrix*

$$R_n = \begin{pmatrix} C_{n-1} & \alpha^t \\ \beta & 0 \end{pmatrix}$$

where $\alpha = (0,\ldots,0,1,0,0)$, $\beta = (1,0,\ldots,0)$ are 1 by $n-1$ row vectors and

$$C_{n-1} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}_{(n-1)\times(n-1)}$$

Proof: If A is permutationally similar to R_n, then $D(A)$ is isomorphic to D_{n-2}. By Lemma 4, we have $\exp(A,k) = \exp(D(A),k) = e(n,k)$. Hence $A \in EM(n,k)$.

Now suppose $A \in EM(n,k)$. Hence $D(A)$ is primitive minimally strong and $\exp(D(A),k) = e(n,k)$. We are going to show that A is permutationally similar to the matrix R_n. So we need only to prove $D(A)$ is isomorphic to D_{n-2}. Let s be the length of the shortest cycle of $D(A)$. Note that $s \leq n-2$ since $D(A)$ is primitive minimally strong.

The case $k = n$ has been proved in [6].

For $1 \leq k \leq n-2$, if $s \leq n-3$, then by the proof in Theorem 1 we have $\exp(D(A),k) \leq n^2 - 5n + 6 + k \leq n^2 - 5n + 6 + k = e(n,k)$, a contradiction; so $s = n-2$. It has been proved in Theorem 1 that $D(A)$ is isomorphic to D_{n-2}, and so we are done.

In the following we assume $k = n-1$. We have shown that $s = n-2$ implies $D(A)$ is isomorphic to D_{n-2}. We will prove $s \leq n-3$ is impossible.

Case 1: $s \leq n-4$. By Lemma 4 we have

$$\begin{aligned}\exp(D(A), n-1) &\leq s(n-2) + n - 1\\ &\leq (n-4)(n-2) + n - 1\\ &= n^2 - 5n + 7\\ &< n^2 - 4n + 5 = e(n, n-1),\end{aligned}$$

contradicting the fact $A \in EM(n, n-1)$.

Case 2: $s = n-3$. Since $D(A)$ has no loops, we have $s \geq 2$ and $n \geq 5$.

If $n = 5$, $s = 2$, then the set of distinct cycle lengths of $D(A)$ is $\{2,3\}$. As may be easily verified, $D(A)$ must be isomorphic to one of the digrpphs D_1, D_2 and D_3, where $V(D_1) = V(D_2) = V(D_3) = \{1,2,3,4,5\}$, $E(D_1) = \{(1,2),(2,3),(3,1),(1,5),(5,1),(3,4),(4,3)\}$, $E(D_2) = \{(1,2),(2,3),(3,1),(1,5),(5,1),(1,4),(4,1)\}$, and $E(D_3) = \{(1,2),(2,3),(3,1),(1,4),(4,1),(4,5),(5,4)\}$. It is easy to see that $\exp(D_i) \leq 6$ using the fact that $(A(D_i)^6$ is a all 1's matrix for $i = 1$, 2 or 3. Hence $\exp(A, 5-1) = \exp(D(A), 5-1) \leq \exp(D(A)) \leq 6 < 10 = e(5, 5-1)$, a contradiction. Assume $n > 5$. By Lemma 1, $D(A)$ must contain a cycle of length $n-2$ or $n-1$.

Case 2.1: $D(A)$ contains no cycle of length $n-1$. Then $D(A)$ must contain a cycle of length $n-2$. Take a cycle C of length $n-2$ in $D(A)$. There are exactly two vertices, say x and y, lying outside C.

Case 2.1.1: $D(A)$ contains one of the arcs (x,y) or (y,x). Suppose that $D(A)$ contains the arc (x,y). Then (y,x) can not be an arc of $D(A)$. Otherwise, we have $n-3 = s = 2$ and so $n = 5$, which is a contradiction. By the strong connectedness of $D(A)$, there are vertices u, v of C such that (u,x) and (y,v) are both arcs of $D(A)$. If $u = v$, then $n-3 = s = 3$, so we have $n = 6$ and $D(A)$ is isomorphic to D^1_{6-3}. If $u \neq v$, then since $D(A)$

has precisely two cycles, of lengths $n-2$ and $n-3$ respectively, it follows that $D(A)$ is isomorphic to D^1_{n-3} $(n \geq 7)$. Applying Lemma 5, we have

$$\begin{aligned} \exp(D(A), n-1) &\leq \exp(D(A), n) \\ &= \exp(D^1_{n-3}, n) \\ &\leq n^2 - 6n + 12 \\ &< n^2 - 4n + 5 = e(n, n-1), \end{aligned}$$

a contradiction.

Case 2.1.2: Neither (x, y) nor (y, x) is an arc of $D(A)$. By the strong connectedness of $D(A)$, there must exist vertices u, v, u' and v' of C such that (u, x), (x, v), (u', y) and (y, v') are all arcs of $D(A)$. Then $u \neq v$, $u' \neq v'$. Otherwise we have $n - 3 = s = 2$, and so $n = 5$, a contradiction. Since $D(A)$ contains no cycle of length $n-1$, neither (u, v) nor (u', v') is an arc of $D(A)$. Suppose $uu_1u_2\ldots u_rv$ and $u'v_1v_2\ldots v_tv'$ are two paths in the cycle C. If $r = t = 1$, then the minimally strong connectedness of $D(A)$ implies that $D(A)$ has no cycles of length $n-3$, a contradiction; if $r \geq 3$ or $t \geq 3$, then the length of the shortest cycle of $D(A)$ is at most $n-4$, also a contradiction. So we conclude that $2 \in \{r, t\}$, which implies that $D(A)$ contains a subdigraph which is isomorphic to $D_{(n-1)-2}$. We assume without loss of generality that $D_{n-1)-2}$ is a subdigraph of $D(A)$. Note that $V(D_{(n-1)-2}) = \{1, 2, \ldots, n-1\}$, and that there is a vertex j in $\{1, 2, \ldots, n-1\}$ such that (j, n) is an arc of $D(A)$.

For any two vertices $u, v \in \{1, 2, \ldots, n-1\}$, there is a walk of length $\exp(D_{(n-1)-2}, n-1) = n^2 - 6n + 11$ (and hence also every length greater) from u to v. Therefore there is a walk of length $n^2 - 6n + 12$ from u to n via j. It follows from the definition of the $(n-1)$-exponent that $\exp(D(A), n-1) \leq n^2 - 6n + 12 < n^2 - 4n + 5 = e(n, n-1)$, which is a contradiction.

Case 2.2: $D(A)$ contains a cycle of length $n-1$. Since $D(A)$ is minimally strong, $D(A)$ does not contain any cycle of length $n-2$. Note that $s = n-3$. We know that n must be even and $D(A)$ is isomorphic to D^2_{n-3}. By Lemmas 2 and 5,

$$\begin{aligned} \exp(D(A), n-1) &\leq \exp(D(A), 1) + (n-1) - 1 \\ &= \exp(D^2_{n-3}, 1) + n - 2 \\ &\leq n^2 - 5n + 5 + n - 2 \\ &= n^2 - 4n + 3 \\ &< n^2 - 4n + 5 = e(n, n-1), \end{aligned}$$

a contradiction.

Now we have proved that the case $s = n - 3$ is impossible. Combining Cases 1 and 2, we conclude that $D(A)$ is isomorphic to D_{n-2}. The proof is complete. □

Let P_n be the set of all $n \times n$ permutation matrices. Theorem 2 implies $EM(n,k) = \{PR_nP^t : P \in P_n\}$. Thus we have characterized completely the extremal matrices for k-exponent of the matrix class PNR_n.

Furthermore, let $EX(n,k) = \{\exp(A,k) : A \in PNR_n\}$. $EX(n,k)$ is called the k-exponent set of the matrix class PNR_n. Obviously, $EX(n,k) \subseteq \{1, 2, \dots, e(n,k)\}$. We may prove that there are gaps in $EX(n,k)$, i.e., not every integer between 1 and $e(n,k)$ is in $EX(n,k)$. In fact, it follows from the proof of Theorem 2 that $m \notin EX(n, n-1)$ if $n^2 - 5n + 7 < m < e(n, n-1) = n^2 - 4n + 5$ when n is odd and $n^2 - 4n + 4 \notin EX(n, n-1)$ when n is even. In general, the complete determination of $EX(n,k)$ will be an interesting and difficult problem.

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