Non-isomorphic Minimal Bicovers of K_8

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ABSTRACT. The number $g_2^{(4)}(8)$ is the minimal number of blocks that contain all pairs from a set of 8 elements exactly twice under the restriction that the longest block have size 4 (this longest block need not be unique). Thus the blocks have lengths 2, 3, and 4. We show that there are three solutions to this problem.

1 Introduction

Suppose we have a set V of v elements, $V = \{1, 2, 3, \ldots, v\}$. A minimal bicover of this set, with longest block of length k, is given by a family F of $g_2^{(k)}(v)$ blocks whose lengths may be $2, 3, \ldots, k$, where every pair from V occurs exactly twice among the blocks of F and where the cardinatlity of F is minimal among the cardinalities of all such possible bicovers.

It is trivial that

$$g_2^{(2)}(v) = 2\binom{v}{2} = v(v-1).$$

For $g_2^{(3)}(v)$, we immediately find the results in the 3 cases of v congruent to 0, 1, 2, modulo 3. For if $v = 3n$, there is a bicover using only triples; thus

$$g_2^{(3)}(3n) = n(3n-1).$$

Similarly, if $v = 3n + 1$, there is a BIBD $(3n+1, n(3n+1), 3n, 3, 2)$, and so

$$g_2^{(3)}(3n+1) = n(3n+1).$$

If $v = 3n + 2$, the situation is slightly different. If $n = 2t$, then $v = 6t + 2$. We illustrate the situation for $n = 2$ (the construction is perfectly general). It is well known (see, for example, [1]) that $D(2,3,8) = \lfloor \frac{8}{3} \lfloor \frac{7}{2} \rfloor \rfloor = 8$ and that the missing pairs form a 1-factor, say (12), (34), (56), (78). Also,

$D(2,3,7) = 7$ and these 7 blocks miss pairs 18, 28, 38, 48, 58, 68, 78. Put the $4+7$ missing pairs in blocks 128, 348, 568, and we have 18 triples and only 2 missing pairs (78). This illustrates the general construction with

$$D_2(2,3,6t+2) = 12t^2 + 6t \text{ triples}$$

and 2 pairs. Thus

$$g_2^{(3)}(6t+2) = 12t^2 + 6t + 2,$$

or we may write

$$g_2^{(3)}(6t+2) = D_2(2,3,6t+2) + 2.$$

If $n = 2t+1$, then $v = 6t+5$. It is well known (cf. again [1]) that

$$D(2,3,6t+5) = \left\lfloor \frac{6t+5}{3} \left\lfloor \frac{6t+4}{2} \right\rfloor \right\rfloor - 1$$

and that this packing contains all pairs except for a cycle, say (1234). Now write down this packing again on the same v elements with 1 and 2 interchanged. The missing pairs are then in the cycle (2134) and these 2 cycles can be combined into 2 triples (123) and (124). Hence one has $2(6t^2+9t+2)+2$ triples with only 2 missing pairs. Again, this is a set of $D_2(2,3,6t+5) = \left\lfloor \frac{6t+5}{3} \left\lfloor \frac{2(6t+4)}{2} \right\rfloor \right\rfloor$ triples and 2 pairs for a total of $12t^2 + 18t + 8$ blocks.

Thus, in both cases,

$$\begin{aligned} g_2^{(3)}(3n+2) &= D_2(2,3,3n+2) + 2 \\ &= \left\lfloor \frac{3n+2}{3}(3n+1) \right\rfloor + 2 \\ &= 3n^2 + 3n + 2. \end{aligned}$$

In a separate discussion [2], we have given results on $g_2^{(4)}(v)$. The purpose of the present note is to determine all possible solutions in the case $v = 8$, which is one of the early interesting cases.

2 Preamble for $k = 4$, $v = 8$

Since $D_2(2,4,8) = \left\lfloor \frac{8}{4} \left\lfloor \frac{2(7)}{3} \right\rfloor \right\rfloor = 8$, we see that there can be at most 8 quadruples in any bicover. However, there is a BIBD(10,15,6,4,2) and deletion of 2 elements leaves a bicover of an 8-set consisting of 2 pairs, 8 triples, and 5 quadruples. Clearly, one can not do better than this with fewer quadruples and so one must have at least 5 quadruples in a minimal bicover.

If there are b_i blocks of length i in the bicover, then

$$b_2 + b_3 + b_4 = g,$$
$$b_2 + 3b_3 + 6b_4 = 56.$$

Then $2b_3 + 5b_4 = 56 - g$. If $g = 15$, then $2b_3 + 5b_4 = 41$, and we know that this is possible with $b_3 = 8$, $b_4 = 5$. Also, we know that 8 quadruples omit 8 pairs and these might be covered by 2 triples and 2 pairs. So, even if this optimum case should be possible, we have $g_2^{(4)}(8) \geq 12$.

If $g_2^{(4)}(8) = 12$, then

$$2b_3 + 5b_4 = 56 - 12 = 44.$$

The only possiblity is $b_4 = 8$, $b_3 = b_2 = 2$. Let the frequency of an element in blocks of length i be α_i; then

$$\alpha_2 + 2\alpha_3 + 3\alpha_4 = 2(8-1) = 14.$$

Also $\alpha_2 \leq 2$, $\alpha_3 \leq 2$, $\alpha_4 \leq 8$. Hence $(\alpha_2, \alpha_3, \alpha_4) = (2,0,4)$ or $(1,2,3)$ or $(0,1,4)$. Let there be a, b, c, of these types of elements. Then

$$a + b + c = 8,$$
$$2a + b = 4,$$
$$2b + c = 6,$$
$$4a + 3b + 4c = 32.$$

The last equation can be replaced by $a + c = 8$. Thus $b = 0$, $a = 2$, $c = 6$.

Let the a-elements be x, y; the c-elements be 1, 2, 3, 4, 5, 6. Then the blocks are

xy	123	$x - - -$	(three times)
xy	456	$y - - -$	(three times).

So, if we delete x and y, there remains a BIBD(6,10,5,3,2) containing 2 disjoint blocks.

In a design (6,10,5,3,2), let x_i be the number of blocks that intersect a specified block in i points. Then

$$x_0 + x_1 + x_2 + x_3 = 9,$$
$$x_1 + 2x_2 + 3x_3 = 3(4) = 12,$$
$$x_2 + 3x_3 = 3.$$

Add these equations with multipliers 1, -1, 1, and we find $x_0 + x_3 = 0$. So it is not possible to have such a design with 2 disjoint blocks. This proves that $g_2^{(4)}(8) > 12$.

3 Discussion of Case $g_2^{(4)}(8) = 13$

In this case $b_2 + b_3 + b_4 = 13$, $b_2 + 3b_3 + 6b_4 = 56$. It follows that $2b_3 + 5b_4 = 43$, and hence $b_4 = 7$, $b_3 = 4$, $b_2 = 2$. Let α_2, α_3, α_4, be as in the last section; then $(\alpha_2, \alpha_3, \alpha_4) = (2,0,4)$ or $(2,3,2)$ or $(1,2,3)$ or $(0,1,4)$ or $(0,4,2)$. Let the number of elements of each of these types be a, b, c, d, e. Then

$$\begin{aligned} a + b + c + d + e &= 8, \\ 2a + 2b + c &= 4, \\ 3b + 2c + d + 4e &= 12, \\ 4a + 2b + 3c + 4d + 2e &= 28. \end{aligned}$$

Then $2b + c + 2e = 32 - 28 = 4$; and so $e = a$. Now $e = 0$ or 1. If $e = 0$, we have

$$\begin{aligned} b + c + d &= 8, \\ 2b + c &= 4, \\ 3b + 2c + d &= 12, \\ 2b + 3c + 4d &= 28. \end{aligned}$$

Then $c + 2d = 12$, and the only solutions are $c = 4$, $d = 4$, $b = 0$; $c = 2$, $b = 1$, $d = 5$; $c = 0$, $b = 2$, $d = 6$.

If $b = 2$, $d = 6$, we have an immediate contradiction, since the b-elements form 2 pairs in both pairs and triples. If $b = 1$, $c = 2$, $d = 5$, then we may write the blocks as

$$\begin{array}{lll} xc_1 & x12 & x \\ xc_2 & xc_1 3 & x \\ & xc_2 4 & c_1 \\ & c_1 c_2 5 & c_1 \\ & & c_1 c_2 \\ & & c_2 \\ & & c_2 \end{array}$$

The elements 1, 2, 3, 4, 5, must now be placed in the quadruples (4 times each). Element 5 must appear once with c_1 and with c_2, twice with x. Element 3 must appear once with x and with c_1, twice with c_2; this results in the following quadruples

$$\begin{array}{l} x541 \\ x532 \\ c_1 542 \\ c_1 431 \\ c_1 c_2 12 \\ c_2 531 \\ c_2 432 \end{array}$$

Now the appearances of 1 and 2 are forced as shown (up to interchange). So we have a solution.

xc_1	$x12$	$x145$
xc_2	xc_13	$x235$
	xc_24	c_1245
	c_1c_25	c_1134
		c_1c_212
		c_2135
		c_2234

If $c = d = 4$, we can write the pairs as

$$c_1c_2,\ c_3c_4.$$

The triples contain 8 elements c_i and the quadruples contain 12 elements c_i. This forces at least 4 pairs c_ic_j in the triples, 5 pairs c_ic_j in the quadruples. Hence, with c_1c_2 and c_3c_4, we have at least 11 pairs c_ic_j forced. Since there are 12 pairs needed, there must be a triple $c_1c_2c_3$ in either the triples or the quadruples.

If the triples contain $c_1c_2c_3$, then the quadruples are c_1, c_1, c_1c_3, c_2, c_2, c_2c_3, c_3; this forces the triples to be $c_1c_2c_3$, c_1, c_2, c_3. Now c_4 must be placed in the quadruples so as to meet 3 elements of c_1, c_2, c_3. This requires c_4 not to meet any pairs in the quadruples. So we have quadruples

$$c_1c_4, c_1, c_1c_3, c_2c_4, c_2, c_2c_3, c_3c_4$$

and triples

$$c_1c_2c_3, c_1c_4, c_2c_4, c_3.$$

Now let c_312 be a triple. Element 1 appears 4 times in the quadruples with c_1 (twice), c_2 (twice), c_3 (once), c_4 (twice). So 1 must appear with 3 of c_1c_4, c_1c_3, c_2c_4, c_2c_3, c_3c_4. This requires c_1c_41, c_2c_41, c_2c_31. A similar argument applies to element 2 and one has too many pairs (12). So this case can not occur.

On the other hand, if the quadruples contain $c_1c_2c_3$, then we have quadruples and triples

$c_1c_2c_3-$	c_1c_3-
c_1c_4--	c_1c_4-
c_1---	c_2c_3-
c_2c_4--	c_2c_4-
c_2---	
c_3c_4--	
c_3---	

Let 1 occur with $c_1c_2c_3$. Then it occurs in quadruples c_1, c_2, c_3, and this leads to a contradiction. So this case also can not occur.

We now consider the case $e = 1$. Then $a = 1$, and

$$b + c + d = 6$$
$$2b + c = 2.$$

The possibilities are $c = 2$, $b = 0$, $d = 4$, and $c = 0$, $b = 1$, $d = 5$. But $e = 1$ implies $b = 0$. Hence $a = 1$, $c = 2$, $d = 4$, $e = 1$. Then we may write the blocks as

ac_1	xc_1	xa
ac_2	xc_1	xa
	x	ac_1
	x	a
		c_1
		c_1
		$-$

with c_2 occurring twice in the triples, thrice in the quadruples.

Let the d-elements be 1, 2, 3, 4. Call this set D. D occurs with xa and with a in the quadruples and with x in the triples. If ac_1c_2 occurs in a quadruple, then we have triples xc_1, xc_1, xc_2, xc_2, Then the triples are xc_11, xc_12, xc_23, xc_24, and the quadruples are $xa12$, $xa34$, ac_1c_24, $a123$. Then 4 occurs in quadruples c_1 and c_2 and also in a triple with c_1 or c_2. this is a contradiction. So we must have quadruples ac_1 and ac_2. If we have triples xc_1, xc_1c_2, xc_2, x, then the quadruples are $xa12$, $xa34$, ac_1, ac_2, and c_1c_2, c_1, c_2. So we must take ac_113 and ac_234. This forces us to have the quadruples c_1c_2, c_1, c_2, having 2 occurences each of 1, 2, 3, 4. Now consider the elements that do not occur with c_i in the triples. They must appear twice with c_1 and c_2 in the quadruples; so they occur with c_1c_2. This forces $x14$, c_1c_214, c_1234, c_2123. The triples then are xc_12, xc_23, xc_1c_2, $x14$. So we have a solution (Solution 2).

ac_1	xc_12	$xa12$
ac_2	xc_23	$xa34$
	xc_1c_2	ac_113
	$x14$	ac_234
		c_1c_214
		c_1234
		c_2123

Finally, if we have triples xc_1, xc_1, xc_2, xc_2, then the quadruples are

$xa12$, $xa34$, ac_1, ac_2, c_1c_2, c_1c_2, $-$. This produces solution 3:

ac_1	xc_12	$xa12$
ac_2	xc_14	$xa34$
	xc_21	ac_113
	xc_23	ac_224
		c_1c_214
		c_1c_223
		1234

4 Conclusion

$g_2^{(4)}(8) = 13$ and the result is obtained by 3 non-isomorphic solutions. For convenience, we list them using symbols 1, 2, 3, 4, 5, 6, 7, 8.

Solution 1:

68	128	1458
78	368	2358
	478	2456
	567	1346
		1267
		1357
		2347

Solution 2:

68	256	1258
78	357	3458
	567	1368
	145	2478
		1467
		2346
		1237

Solution 3:

68	256	1234
78	456	1258
	157	3458
	357	1368
		2478
		1467
		2367

References

[1] R.G. Stanton and M.J. Rogers, Packings and Coverings by Triples, *Ars Combinatoria* **13** (1982), 61–69.

[2] R.G. Stanton, On the Bipacking Numbers $g_2^{(4)}(v)$, to appear.