On Dudeney's Round Table Problem for $p+2$

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Abstract

Dudeney's round table problem was proposed about one hundred years ago. It is already solved when the number of people is even, but it is still unsettled except only few cases when the number of people is odd.

In this paper, a solution of Dudeney's round table problem is given when $n = p + 2$, where p is an odd prime number such that 2 is the square of a primitive root of $GF(p)$, and $p \equiv 3 \pmod 4$.

1 Introduction

In 1905, Dudeney [1, problem 273] proposed the Round Table Problem as follows:

"Seat the same n persons at a round table on $(n-1)(n-2)/2$ occasions so that no person shall ever have the same two neighbours twice. This is, of course, equivalent to saying that every person must sit once, and only once, between every possible pair."

The problem proposed by Dudeney is equivalent to asking for a set of Hamilton cycles in the complete graph K_n with the property that every path of length two (2-path) lies on exactly one of the cycles. We call such

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a set of cycles in K_n a Dudeney set in K_n. The cardinality of a Dudeney set in K_n is $(n-1)(n-2)/2$.

A Dudeney set in K_n has been constructed for all even n [3]. However, a Dudeney set in K_n for an odd n has been constructed only when $n = 2^k + 1$ (k is a natural number) [6], and $n = p+2$, where p is an odd prime number and 2 is a primitive root of $GF(p)$ [2], and some sporadic cases [2].

In this paper, we show:

Theorem 1.1 *There exists a Dudeney set in K_n when $n = p+2$, where p is an odd prime number such that 2 is the square of a primitive root of $GF(p)$, and $p \equiv 3 \pmod 4$.*

Theorem 1.1 combined with the known result makes Theorem 1.2.

Theorem 1.2 *There exists a Dudeney set in K_n when $n = p+2$, where p is an odd prime number and -2 is a primitive root of $GF(p)$.*

Theorem 1.2 is trivial from Lemma 1.3.

Lemma 1.3 *Let p be an odd prime number.*
(1) When $p \equiv 1 \pmod 4$, -2 is a primitive root of $GF(p)$ if and only if 2 is a primitive root of $GF(p)$.
(2) When $p \equiv 3 \pmod 4$, -2 is a primitive root of $GF(p)$ if and only if 2 is the square of a primitive root of $GF(p)$.

Proof. Put $r = (p-1)/2$.
(1) When $p \equiv 1 \pmod 4$, r is even. If -2 is a primitive root, then $2 = (-1)(-2) = (-2)^{r+1}$. Since $(r+1, p-1) = 1$, 2 is a primitive root, where $(\ ,\ )$ means the greatest common divisor. Conversely, if 2 is a primitive root, then $-2 = 2^{r+1}$ and -2 is a primitive root.
(2) When $p \equiv 3 \pmod 4$, r is odd. If -2 is a primitive root, then $2 = (-2)^{r+1} = ((-2)^{(r+1)/2})^2$.

When $(r+1)/2$ is odd, we have $((r+1)/2, p-1) = 1$ and $(-2)^{(r+1)/2}$ is a primitive root. When $(r+1)/2$ is even, then $2 = ((-2)^{(3r+1)/2})^2$. Since $((3r+1)/2, p-1) = 1$, $(-2)^{(3r+1)/2}$ is a primitive root.

Conversely, if $2 = \omega^2$, where ω is a primitive root, then $-2 = \omega^{r+2}$. Since $(r+2, p-1) = 1$, -2 is a primitive root.

2 Preliminaries

Let p be an odd prime integer. Put $n_1 = p+1$ and $r = (p-1)/2$.

We denote by $K_{n_1} = (V_{n_1}, E_{n_1})$ the complete graph on n_1 vertices, where $V_{n_1} = \{0, 1, 2, \cdots, p-1\} \cup \{\infty\} = Z_p \cup \{\infty\}$ is the vertex set (Z_p is the set of integers modulo p), and E_{n_1} is the edge set.

For any integer i, $0 \leq i \leq p-1$, define the 1-factor

$$F_i = \{\{\infty, i\}\} \cup \{\{a, b\} \in E_{n_1} \mid a, b \neq \infty, a + b \equiv 2i \pmod{p}\}.$$

Let σ be the vertex-permutation $(\infty)(0\ 1\ 2\ \ldots\ p-1)$, and put $\Sigma = \{\sigma^j \mid 0 \leq j \leq p-1\}$. Clearly σ induces a permutation of the edges of K_{n_1}; we will also denote this permutation by σ. When C is a set of cycles or circuits in K_{n_1}, define $\Sigma\mathcal{C} = \{C^\tau \mid C \in \mathcal{C}, \tau \in \Sigma\}$.

For any edge $\{a, b\}$ in K_{n_1}, we define the length $d(a, b)$:

$$d(a,b) = \begin{cases} \min\{p - |b-a|, |b-a|\} & (a, b \neq \infty) \\ \infty & (\text{otherwise}), \end{cases}$$

and we define the colour $c(a, b)$:

$$c(a,b) = \begin{cases} a+b & (a, b \neq \infty) \\ 2a & (b = \infty) \\ 2b & (a = \infty), \end{cases}$$

where the calculation is carried out modulo p. We can also write

$$c(a,b) = i, \quad \text{if } \{a,b\} \in F_{2^{-1}i} \quad (0 \leq i \leq p-1),$$

and define the absolute colour $ac(a, b)$:

$$ac(a,b) = \min\{p - c(a,b), c(a,b)\}.$$

Let H be a subset of $Z_m^* = Z_m \setminus \{0\}$, where m is a positive integer. We call H a half-set modulo m if $H \cap (-H) = \phi$ and $H \cup (-H) = Z_m^*$.

Proposition 2.1 *Let p be an odd prime integer. For any half-set H modulo p, $\mathcal{D} = \Sigma\{F_0 \cup F_i \mid i \in H\}$ is a Dudeney set in K_{n_1}.*

Proof. It is well known that $F_0 \cup F_i$ $(1 \leq i \leq p-1)$ is a Hamilton cycle in K_{n_1}. Since $(F_0 \cup F_{-i})^{\sigma^i} = F_0 \cup F_i$, we have $\Sigma\{F_0 \cup F_{-i}\} = \Sigma\{F_0 \cup F_i\}$ $(1 \leq i \leq p-1)$. Hence we have $\mathcal{D} = \Sigma\{F_0 \cup F_i \mid 1 \leq i \leq p-1\}$. It is trivial that any 2-path belongs to $\Sigma\{F_0 \cup F_i \mid 1 \leq i \leq p-1\}$, so any 2-path belongs to $\mathcal{D}$. The number of all 2-paths belonging to $\mathcal{D}$ is at most $(p+1)p(p-1)/2$, which is the number of all 2-paths in K_{n_1}. So every 2-path lies in $\mathcal{D}$ exactly once.

Therefore $\mathcal{D}$ is a Dudeney set in K_{n_1}. □

We now explain what we mean by exchanging edges between two 1-factors. Let H_1 and H_2 be 1-factors of K_{n_1}. Assume that $H_1 \cup H_2$ is not hamiltonian and that we have a cycle C in $H_1 \cup H_2$. Then we exchange edges of H_1 and H_2 via C to obtain new two 1-factors H_1' and H_2':

$$H_1' = (H_1 \setminus C) \cup (H_2 \cap C) \text{ and}$$

$$H_2' = (H_2 \setminus C) \cup (H_1 \cap C).$$

We next construct the complete graph K_n by adding a new vertex λ to K_{n_1}; that is, put $n = n_1 + 1 = p + 2$, $K_n = (V_n, E_n)$ and $V_n = V_{n_1} \cup \{\lambda\}$. Extend σ to a permutation of V_n and denote it also by σ: $\sigma = (\infty)(\lambda)(0\ 1\ 2\ 3\ \cdots\ p-1)$. Further we put $\Sigma = \{\sigma^j \mid 0 \le j \le p-1\}$.

Let A be a 1-factor in K_{n_1} which satisfies (1) and (2):

(1) *$F_0 \cup A$ is a Hamilton cycle in K_{n_1}.*

(2) *If S is the multiset $\{d(a,b) \mid \{a,b\} \in A\}$, then we have $S = \{\infty, 1, 2, \cdots, r\}$, i.e. A has all lengths.*

If we insert the vertex λ into all the edges in A, we get a set of 2-paths in K_n. Denote this set by A^λ; that is,

$$A^\lambda = \{(a, \lambda, b) \mid \{a,b\} \in A\}.$$

We note that paths are undirected, i.e., $(a,\lambda,b) = (b,\lambda,a)$. $F_0 \cup A^\lambda$ is considered to be a circuit in K_n.

Proposition 2.2 *Let H be a half-set modulo p. Then*

$$\Sigma(\{F_0 \cup A^\lambda\} \cup \{F_0 \cup F_i \mid i \in H\})$$

has every 2-path in K_n exactly once.

Proof. Divide the set of all 2-paths in K_n into 8 classes:
(i) (a,b,c), (ii) (a,∞,b), (iii) (∞,a,b), (iv) (a,λ,b), (v) (λ,a,b), (vi) (λ,∞,a), (vii) (λ,a,∞), (viii) (∞,λ,a), where $a,b,c \ne \infty, \lambda$.

(i), (ii), (iii) are also 2-paths in K_{n_1}, so they belong to $\Sigma\{F_0 \cup F_i \mid i \in H\}$ by Prop. 2.1.

(iv), (viii) Since A has all lengths, we have $\Sigma A = E_{n_1}$. Hence 2-paths (a,λ,b) and (∞,λ,a) belong to ΣA^λ because $\{a,b\},\{\infty,a\} \in E_{n_1}$. So (a,λ,b) and (∞,λ,a) clearly belong to $\Sigma\{F_0 \cup A^\lambda\}$.

(v) We have $\{a,b\}^{\sigma^t} \in F_0$ for some t $(0 \le t \le p-1)$, as F_0 has all lengths. So we can assume $\{a,b\} \in F_0$ without loss of generality. Since A is a 1-factor, we have $\{a,c\} \in A$ for some $c \in V_{n_1}$. Then we have $(a,\lambda,c) \in A^\lambda$. So (λ,a,b) belongs to $F_0 \cup A^\lambda$

(vi), (vii) Similarly, we can assume $a = 0$. (If necessary, apply σ^t to these 2-paths for some t.) Then the 2-paths $(\lambda, \infty, 0)$ and $(\lambda, 0, \infty)$ belong to $F_0 \cup A^\lambda$.

By counting the number of 2-paths, we prove that every 2-path in K_n lies only once. □

3 Gray 1-factor

From now to the end of this paper, we assume that p is an odd prime number with $p \equiv 3 \pmod 4), p > 7$, and that 2 is the square of a primitive root of $GF(p)$. Put $n_1 = p + 1$, $r = (p-1)/2$ and $s = (r-1)/2$.

We define a special 1-factor in K_{n_1}, which is called the gray 1-factor;

$$G = \{\{\infty, 1\}, \{-1, -2\}, \{2, 2^2\}, \{-2^2, -2^3\}, \{2^3, 2^4\}, \\ \{-2^4, -2^5\}, \cdots, \{2^{r-2}, 2^{r-1}\}, \{-2^{r-1}, 0\}\}.$$

We call these edges in G gray edges.

Proposition 3.1 *Let G be the gray 1-factor in K_{n_1}.*
(1) $F_0 \cup G$ is a Hamilton cycle in K_{n_1}.
(2) If S is the multiset $\{d(a,b) \mid \{a,b\} \in G\}$, then we have $S = \{\infty, 1, 2, \cdots, r\}$, i.e. G has all lengths.
(3) If M is the multiset $\{ac(a,b) \mid \{a,b\} \in G\}$, then we have $M = \{1, 2, 2, 3, 4, \cdots, r-2, r, r\}$, i.e. G has the absolute colours 2 and r twice each, G doesn't have the absolute colour $r-1$, and G has the other absolute colours once each.

Proof. From our assumption, it holds $2 = \omega^2$ for some primitive root ω of $GF(p)$. So we have $2^r = 1$. Since $p \equiv 3 \pmod 4$, -1 is not a quadratic residue; that is $-1 \notin \langle 2 \rangle$, where $\langle 2 \rangle = \{1, 2, 2^2, \cdots, 2^{r-1}\}$. Then we have $GF(p)^* = \langle 2 \rangle \cup -\langle 2 \rangle$, and $\langle 2 \rangle$ is a half-set modulo p.

It is clear that G is a 1-factor in K_{n_1} and that $F_0 \cup G$ is a Hamilton cycle.

Compare the series, T and U:

$$T = (1, 2, 2^2, 2^3, \cdots, 2^{r-1}, 1)$$
$$U = (\infty, 1, 2, 2^2, 2^3, \cdots, 2^{r-1}, 0).$$

The set of differences of the pairs of consecutive elements in T is

$$\{1-2,\ 2-2^2,\ 2^2-2^3,\ \cdots,\ 2^{r-1}-1\} \\ = \{-1,\ -2,\ -2^2,\ \cdots,\ -2^{r-1}\} \\ = -\langle 2 \rangle.$$

which is a half-set modulo p. The set of additions of the pairs of consecutive elements of T is

$$\begin{aligned}&\{1+2,\ 2+2^2,\ 2^2+2^3,\ \cdots,\ 2^{r-1}+1\}\\&=\{3,\ 3\cdot 2,\ 3\cdot 2^2,\ \cdots,\ 3\cdot 2^{r-1}\}\\&=3\langle 2\rangle.\end{aligned}$$

which is also a half-set modulo p.

On the other hand, the set of differences of the pairs of consecutive elements in U is

$$\begin{aligned}&\{\infty-1, 1-2,\ 2-2^2,\ 2^2-2^3,\ \cdots,\ 2^{r-1}-0\}\\&=\{\infty,\ -1,\ -2,\ -2^2,\ \cdots,\ -2^{r-2},\ 2^{r-1}\}\end{aligned}$$

which is $\{\infty\}\cup H$, where H is a half-set. Therefore G has all lengths.

We have $ac(\infty,1)=2$, $ac(2^{r-1},0)=r$, $ac(2^{r-1},1)=r-1$, as $2^{r-1}=2^{-1}=r+1=p-r$. From our assumption that $p>7$, three numbers $2, r, r-1$ are all different. Therefore, comparing the two series T and U, we see that G has the absolute colours 2 and r twice each, G doesn't have the absolute colour $r-1$, and G has the other absolute colours once each. □

Insert the vertex λ into all edges in G and define G^λ same as before; that is,

$$G^\lambda=\{(a,\lambda,b)\mid\{a,b\}\in G\}.$$

Proposition 3.2 *Let H be a half-set modulo p. Then*

$$\Sigma(\{F_0\cup G^\lambda\}\cup\{F_0\cup F_i\mid i\in H\})$$

has every 2-path in K_n exactly once.

Proof. Trivial from Prop. 2.2 and Prop. 3.1(1), (2). □

In Prop. 3.2, there are $r+1$ λs in $F_0\cup G^\lambda$, and there is no λ in $\{F_0\cup F_i\mid i\in H\}$. We would like to leave one λ in $F_0\cup G^\lambda$ and scatter the remaining r λs over $\{F_0\cup F_i\mid i\in H\}$, but that is impossible by Prop. 3.1(3). So, we will exchange edges.

4 Exchanging edges

First we note that $-2^{r-1}=r$. $G\cup F_r$ is not a Hamilton cycle and it has a cycle C_1:

$$C_1=(0,-1,-2,1,\infty,r).$$

If we exchange edges of G and F_r via C_1, we obtain new 1-factors. Denote them by G' and F_r^*, respectively. Then G' has edges $\{0,-1\}$, $\{-2,1\}$, $\{\infty,r\}$ and F_r^* has edges $\{-1,-2\},\{1,\infty\},\{r,0\}$.

Next we consider F_r^* and F_s $(s=(r-1)/2)$. $F_r^* \cup F_s$ is not a Hamilton cycle and it has a cycle C_2,

$$C_2 = (0,\ r-1,\ -r,\ -2,\ -1,\ r).$$

If we exchange edges of F_r^* and F_s via C_2, we obtain new 1-factors. Denote them by F_r' and F_s', respectively. Then F_r' has edges $\{0,r-1\}$, $\{-r,-2\}$, $\{-1,r\}$ and F_s' has edges $\{r-1,-r\}$, $\{-2,-1\}$, $\{r,0\}$.

Proposition 4.1 *$F_0 \cup G'$, $F_0 \cup F_r'$ and $F_0 \cup F_s'$ are Hamilton cycles in K_{n_1}.*

Proof.
(1) We consider $F_0 \cup G'$. There are 6 points in G' which are concerned in exchanging edges, that are $\infty, 0, 1, -1, -2, r$. We have

$$F_0 \cup G = (\cdots, -r, r, 0, \infty, 1, -1, -2, 2, \cdots).$$

On the other hand, we have

$$F_0 \cup G' = (\cdots, -r, r, \infty, 0, -1, 1, -2, 2, \cdots).$$

Thus $F_0 \cup G'$ is hamiltonian since $F_0 \cup G$ is hamiltonian.
(2) We consider $F_0 \cup F_r'$. There are 8 points in F_r' which are concerned in exchanging edges, that are $\infty, 0, 1, -1, -2, r-1, r, -r$. We have

$$F_0 \cup F_r = (\cdots, -(r-1), r-1, -r, r, \infty, 0, -1, 1, -2, 2, \cdots).$$

On the other hand, we have

$$F_0 \cup F_r' = (\cdots, -(r-1), r-1, 0, \infty, 1, -1, r, -r, -2, 2, \cdots).$$

Thus $F_0 \cup F_r'$ is hamiltonian since $F_0 \cup F_r$ is hamiltonian.
(3) We consider $F_0 \cup F_s'$. There are 6 points in F_s' which are concerned in exchanging edges, that are $0, -1, -2, r-1, r, -r$. Two paths

$$(s, \infty, 0, r-1, -(r-1)) \text{ and } (1, -1, r, -r, -2, 2)$$

appear in $F_0 \cup F_s$. We note again that cycles and paths are undirected. On the other hand, two paths

$$(s, \infty, 0, r, -r, r-1, -(r-1)) \text{ and } (1, -1, -2, 2)$$

appear in $F_0 \cup F'_s$. Thus $F_0 \cup F'_s$ is hamiltonian since $F_0 \cup F_s$ is hamiltonian. □

Proposition 4.2 $G \cup F_r \cup F_s = G' \cup F'_r \cup F'_s$.

Proof. It is trivial because edges in $G' \cup F'_r \cup F'_s$ are obtained from edges in $G \cup F_r \cup F_s$ by exchaging edges. □

5 Construction of a Dudeney set in K_n

The 1-factor F'_r has a gray edge $\{\infty, 1\}$. If we insert λ into the edge, we get a set of edges and a 2-path in K_n. Denote this set by F'^{λ}_r; that is,

$$F'^{\lambda}_r = F'_r \setminus \{\{\infty, 1\}\} \cup \{(\infty, \lambda, 1)\}.$$

The 1-factor F'_s has gray edges $\{0, r\}$, $\{-1, -2\}$. Insert λ only into the edge $\{0, r\}$ and define F'^{λ}_s as follows:

$$F'^{\lambda}_s = F'_s \setminus \{\{0, r\}\} \cup \{(0, \lambda, r)\}.$$

The 1-factor G' still has $r-2$ gray edges. Let $\{c, d\}$ be a gray edge in G' whose absolute colour is 1. Insert λ into the edge and define G'^{λ}:

$$G'^{\lambda} = G' \setminus \{\{c, d\}\} \cup \{(c, \lambda, d)\}.$$

(Note that G' has a gray edge whose absolute colour is 1, because G has an edge whose absolute colour is 1 by Prop. 3.1(3), and three exchanging edges $\{\infty, 1\}, \{0, r\}, \{-1, -2\}$ in G have $2, r, 3$ as absolute colours, respectively.)

G'^{λ} still has $r-3$ gray edges. Let $e_1, e_2, \cdots, e_{r-2}$ be the $r-3$ gray edges in G'^{λ} and a gray edge $\{-1, -2\}$ in F'^{λ}_s. (The order doesn't matter.) Let $c_1, c_2, \cdots, c_{r-2}$ be the colours of $e_1, e_2, \cdots, e_{r-2}$, respectively, and $a_1, a_2, \cdots, a_{r-2}$ be the absolute colours of $e_1, e_2, \cdots, e_{r-2}$, respectively. Then we have $\{a_1, a_2, \cdots, a_{r-2}\} = \{2, 3, 4, \cdots, r-2, r\}$ by Prop 3.1(3).

Put $H = \{c_1, c_2, \cdots, c_{r-2}\} \cup \{-1, r-1\}$. Then H is a half-set modulo p, because $\{a_1, a_2, \cdots, a_{r-2}\} \cup \{1, r-1\} = \{1, 2, 3, 4, \cdots, r-2, r-1, r\}$. Since p is an odd prime integer, $2^{-1}H$ is also a half-set. For i $(1 \le i \le r-2)$, $F_{2^{-1}c_i}$ has an edge e_i. Insert the vertex λ in the edge e_i in $F_{2^{-1}c_i}$ and define $F^{\lambda}_{2^{-1}c_i}$:

$$F^{\lambda}_{2^{-1}c_i} = F_{2^{-1}c_i} \setminus \{e_i\} \cup \{e^{\lambda}_i\},$$

where $e^{\lambda}_i = (a_i, \lambda, b_i)$ if $e_i = \{a_i, b_i\}$.

Put

$$\mathcal{C} = \{F_0 \cup G'^{\lambda}, F_0 \cup F'^{\lambda}_r, F_0 \cup F'^{\lambda}_s\} \cup \{F_0 \cup F^{\lambda}_{2^{-1}c_i} \mid 1 \le i \le r-2\},$$

and

$$C' = \{F_0 \cup G^\lambda\} \cup \{F_0 \cup F_r, F_0 \cup F_s\} \cup \{F_0 \cup F_{2^{-1}c_i} \mid 1 \leq i \leq r-2\}.$$

Proposition 5.1 *The set of all 2-paths in C and the set of all 2-paths in C' are the same.*

Proof. We have

$$G \cup F_r \cup F_s \cup (\cup_{i=1}^{r-2} F_{2^{-1}c_i}) = G' \cup F'_r \cup F'_s \cup (\cup_{i=1}^{r-2} F_{2^{-1}c_i})$$

by Prop. 4.2. All λs in G^λ was scattered over G', F'_r, F'_s and $F'_{2^{-1}c_i}$ $(1 \leq i \leq r-2)$, so we have

$$G^\lambda \cup F_r \cup F_s \cup (\cup_{i=1}^{r-2} F_{2^{-1}c_i}) = G'^\lambda \cup F'^\lambda_r \cup F'^\lambda_s \cup (\cup_{i=1}^{r-2} F^\lambda_{2^{-1}c_i}).$$

Every 2-path in C can be represented as (a, b, c), where $\{a, b\} \in F_0$ and $\{b, c\} \in G'^\lambda \cup F'^\lambda_r \cup F'^\lambda_s \cup (\cup_{i=1}^{r-2} F^\lambda_{2^{-1}c_i})$. Therefore the proposition follows. □

Proposition 5.2 *$\mathcal{H} = \Sigma C$ is a Dudeney set in K_n.*

Proof. Since $F_0 \cup G'$, $F_0 \cup F'_r$ and $F_0 \cup F'_s$ are Hamilton cycles in K_{n_1} (Prop. 4.1), $F_0 \cup G'^\lambda$, $F_0 \cup F'^\lambda_r$ and $F_0 \cup F'^\lambda_s$ are also Hamilton cycles in K_n. Since $F_0 \cup F_{2^{-1}c_i}$ $(1 \leq i \leq r-2)$ is a Hamilton cycle in K_{n_1}, $F_0 \cup F^\lambda_{2^{-1}c_i}$ is also a Hamilton cycle in K_n. So every cycle in C is a Hamilton cycle in K_n.

Since $2^{-1}H = \{r, s\} \cup \{2^{-1}c_i \mid 1 \leq i \leq r-2\}$ is a half-set modulo p, $\Sigma C'$ has every 2-path in K_n exactly once by Prop. 3.2. So ΣC has every 2-path in K_n exactly once by Prop. 5.1.

Therefore $\mathcal{H}$ is a Dudeney set in K_n. □

By Prop. 5.2 and the fact that there is a Dudeney set in K_{p+2} when $p = 7$ [1, p206], we complete the proof of Theorem 1.1.

6 Conjecture for primitive roots

In the last section, we mention the existence of odd prime numbers such that

(1) 2 is the square of a primitive root of $GF(p)$, and

(2) $p \equiv 3 \pmod 4$.

Conjecture 6.1[4, 5] *There exist infinitly many odd prime numbers satisfying* (1) *and* (2).

An affirmative resolution of the above conjecture is obtained under the Extended Riemann Hypothesis.

Proposition 6.2 [4, 5] *If we assume the Extended Riemann Hypothesis, then Conjecture* 6.1 *is true.*

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