Extended Mendelsohn Triple Systems Having a Prescribed Number of Blocks in Common

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Abstract

An extended Mendelsohn triple system of order v (EMTS(v)) is a collection of cyclically ordered triples of the type $[x, y, z]$, $[x, x, y]$ or $[x, x, x]$ chosen from a v-set, such that each ordered pair (not necessarily distinct) belongs to exactly one triple. If such a design with parameters v and a exist, then they will have $b_{v,a}$ blocks, where $b_{v,a} = (v^2 + 2a)/3$. In this paper, we show that there are two (not necessarily distinct) EMTS(v)'s with common triples in the following sets:
$\{0, 1, 2, \ldots, b_v - 4, b_v - 2, b_v\}$, if $v \neq 6$; and
$\{0, 1, 2, \ldots, b_v - 4, b_v - 2\}$, if $v = 6$,
where b_v is $b_{v,v-1}$ if $v \equiv 2 \pmod 3$; $b_{v,v}$ if $v \not\equiv 2 \pmod 3$.

1 Introduction

A Mendelsohn triple system of order v, MTS(v), is a pair (V, B), where V is a v-set and B is a collection of cyclically ordered triples of distinct elements of V, such that every ordered pair of distinct elements of V is contained in only one member of B. This concept was introduced in [9] by N.S. Mendelsohn, who proved that a MTS(v) exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$. In [1] F.E. Bennett introduced the concept of a system, similar to an MTS, in which a triple may have repeated elements.

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An extended Mendelsohn triple system of order v (EMTS(v)) is a pair (V, B), where V is a v-set and B is a collection of cyclically ordered triples of elements of V, where each triple may have repeated elements and will be called a block, such that every ordered pair of elements of V, not necessarily distinct, is contained in only one block of B. It has been well established that an extended Mendelsohn triple system is co-extensive with the variety of quasigroup satisfying the identity $x(yx) = y$. (It is called a semi-symmetric quasigroup). There are three types of blocks: $[x, x, x]$, $[x, x, y]$, $[x, y, z]$ and we call them idempotent, lollipop and directed triangle, respectively. Observe that $[x, x, x]$ contains only the pair (x, x); $[x, x, y] = [x, y, x] = [y, x, x]$ contains the pairs (x, x), (x, y), (y, x); and $[x, y, z] = [y, z, x] = [z, x, y]$ contains (x, y), (y, z), (z, x). An extended Mendelsohn triple system of order v which has a idempotents will be denoted by EMTS(v, a). We define $\mathcal{EMTS}(v, a)$ as the class of all extended Mendelsohn triple systems on v element and having a idempotents. If (V, B) is an extended Mendelsohn triple system with parameters v and a, we say B is an EMTS(v, a) and write $B \in \mathcal{EMTS}(v, a)$. For $B \in \mathcal{EMTS}(v, a)$, $|B| = b_{v,a} = (v^2 + 2a)/3$.

Recently, some papers have investigated the possible number of common blocks with two generalized triple systems having the same parameters, based on the same v-set. G. Lo Faro [8] considered this problem for extended triple systems without idempotent; W. C. Huang [5, 6] for extended triple systems and C. M. Fu, Y. H. Gwo and F. C. Wu [2] for semi-symmetric latin squares. K. B. Huang, W. C. Huang, C. C. Hung and G. H. Wang [7] considered the intersection problem for some classes of extended Mendelsohn triple systems with the following results:

(1) Constructed two EMTS(v,0)'s such that the number of common triples is in the set $\{0, 1, 2, \ldots, b_{v,0} - 3, b_{v,0}\}$, for $v \equiv 0 \pmod 3$.

(2) Constructed two EMTS(v,1)'s such that the number of common triples is in the set $\{0, 1, 2, \ldots, b_{v,1} - 2, b_{v,1}\}$, for $v \not\equiv 0 \pmod 3$.

In this paper, we have considered the intersection problems for the class EMTS(v). Let $J[v]$ be the set of non-negative integers k such that there is a pair of EMTS(v) with k common blocks.The number of blocks of an EMTS is a maximum when the number of idempotents are at a maximum. And in [1] it was shown that the necessary and sufficient conditions for the existence of an EMTS(v, a), with $0 \le a \le v$, are:

(i) if $v \equiv 0 \pmod 3$, then $a \equiv 0 \pmod 3$;

(ii) if $v \not\equiv 0 \pmod 3$, then $a \equiv 1 \pmod 3$;

(iii) if $v = 6$, then $a \le 3$.

Let $I[v]$ be the set $\{0, 1, 2, \ldots, b_v - 4, b_v - 2, b_v\}$ if $v \ne 6$; and $\{0, 1, 2, \ldots, b_v - 4, b_v - 2\}$ if $v = 6$, where b_v is $b_{v,v-1}$ if $v \equiv 2 \pmod 3$; $b_{v,v}$ if $v \not\equiv 2$ (mod

3). Analysing the structure of EMTS, we can obtain $J[v] \subseteq I[v]$ and so we have proved the following theorem.

Main Theorem $J[v] = I[v]$, for $v \geq 4$.

2 Basic Lemmas.

From the results of K. B. Huang, W. C. Huang, C. C. Hung and G. H. Wang [7], the set $J[v]$ contains the set $\{0, 1, 2, \ldots, b_{v,0} - 3, b_{v,0}\}$, for $v \equiv 0 \pmod 3$ and $\{0, 1, 2, \ldots, b_{v,1} - 2, b_{v,1}\}$, for $v \not\equiv 0 \pmod 3$. From the spectrum of EMTS(v, a) and the identity $b_{v,a} + 2 = b_{v,a+3}$, we have

$$\begin{array}{ll}
J[v] \supseteq I[v] \setminus \{3k^2 - 2\} \cup \{3k^2 - 1 + 2t \mid t = 0, 1, \ldots, & \\
\quad k - 2\}, & \text{for } v = 3k, v \geq 9; \\
J[v] \supseteq I[v] \setminus \{10, 11\}, & \text{for } v = 6; \\
J[v] \supseteq I[v] \setminus \{3k^2 + 2k + 2t \mid t = 0, 1, \ldots, k - 2\}, & \text{for } v = 3k + 1, v \geq 7; \\
J[v] \supseteq I[v] \setminus \{3k^2 + 4k + 1 + 2t \mid t = 0, 1, \ldots, k - 2\}, & \text{for } v = 3k + 2, v \geq 8; \\
J[v] \supseteq I[v] \text{ (hence } J[v] = I[v]), & \text{for } v = 4, 5.
\end{array}$$

Lemma 2.1 *$J[3] = \{0, 3, 5\}$ and $J[v] = I[v]$, for $v = 6$ and 8.*

Proof. $v = 3$. There are precisely three EMTS(3); we call them designs M_1, M_2 and M_3: $M_1 = \{[1,2,3], [1,3,2], [1,1,1], [2,2,2], [3,3,3]\}$, $M_2 = \{[1,1,2], [2,2,3], [3,3,1]\}$ and $M_3 = \{[1,1,3], [3,3,2], [2,2,1]\}$. So we have $J[3] = \{0, 3, 5\}$.

$v = 6$. Let $M_4 = \{[1,3,2], [1,4,5], [2,5,4], [5,5,3], [3,3,4], [4,4,4], [1,1,1], [1,2,3], [1,5,6], [1,6,4], [2,2,2], [2,4,6], [2,6,5], [6,6,3]\}$. Now, M_5 comes from M_4 with the set $\{[1,3,2], [1,4,5], [2,5,4], [5,5,3]\}$ replaced by $\{[5,5,4], [3,2,5], [3,5,1], [4,2,1]\}$ and M_6 comes from M_4 with the set $\{[5,5,3], [3,3,4], [4,4,4]\}$ replaced by $\{[4,4,3], [3,3,5], [5,5,5]\}$. Then, $|M_4 \cap M_5| = 10$ and $|M_4 \cap M_6| = 11$. Thus, we have $J[6] = I[6]$.

$v = 8$. The missing data is 21. Let $M_7 = \{[1,4,5], [1,5,7], [1,6,8], [1,7,4], [1,8,6], [2,4,7], [2,5,8], [2,6,4], [2,7,6], [2,8,5], [3,4,8], [3,5,6], [3,6,7], [3,7,5], [3,8,4], [4,6,5], [7,7,8], [4,4,4], [5,5,5], [6,6,6], [8,8,8]\} \cup A$, where $A = \{[1,2,3], [1,3,2], [1,1,1], [2,2,2], [3,3,3]\}$. Now, M_8 comes from M_7 with the set A replaced by $\{[1,1,2], [2,2,3], [3,3,1]\}$. Then, $|M_7 \cap M_8| = 21$. Thus, we have $J[8] = I[8]$. ∎

In 1982, Hoffman and Lindner [4] proved that $\{v, v+1, v+2, \ldots, b_{v,v} - 6, b_{v,v} - 4, b_{v,v}\} \subseteq J[v]$, for $v \not\equiv 2 \pmod 6$ and $v \neq 6$. So, we can improve on the unsolvable results and separate the case of $v \equiv 2 \pmod 3$ as follows:

$J[v] \supseteq I[v] \setminus \{3k^2 + 2k - 5\},$ for $v = 3k, v \geq 9$;
$J[v] \supseteq I[v] \setminus \{3k^2 + 4k - 4\},$ for $v = 3k + 1, v \geq 7$;
$J[v] \supseteq I[v] \setminus \{12k^2 + 20k + 8 + 2t \mid t = 0, 1, \ldots, 2k - 1\},$ for $v = 6k + 5, v \geq 11$;
$J[v] \supseteq I[v] \setminus \{12k^2 + 32k + 21 + 2t \mid t = 0, 1, \ldots, 2k\},$ for $v = 6k + 8, v \geq 14$.

In 1981, Hoffman and Lindner [3] gave the embeddings of Mendelsohn triple systems as follows.

Lemma 2.2 *[3] Any MTS(v) can be embedded in a MTS(w) for all $w \geq 2v + 1$, $w \not\equiv 2$ (mod 3).*

Lemma 2.3 *If $v \not\equiv 2$ (mod 3) and $v \geq 7$, then $J[v] = I[v]$.*

Proof. From Lemma 2.2, there exists a MTS(v) B with subsystem $A = \{[1,2,3], [1,3,2]\}$. Let $M = B \cup \{[a,a,a] \mid a = 1,2,\ldots v\}$. Then $M \in \mathcal{EMTS}(v,v)$ with blocks $3k^2 + 2k$ for $v = 3k$; $3k^2 + 4k + 1$ for $v = 3k + 1$. Let $M_1 = \{M \setminus A_1\} \cup A_2$, where $A_1 = A \cup \{[1,1,1], [2,2,2], [3,3,3]\}$ and $A_2 = \{[1,1,2], [2,2,3], [3,3,1]\}$. Then $|M \cap M_1| = 3k^2 + 2k - 5$ for $v = 3k$ and $3k^2 + 4k - 4$ for $v = 3k + 1$. ∎

Solving the case of $v \equiv 2 (\text{mod } 3)$, we need two special patterns of EMTS(5) as follows.

I $= \{[1,1,1], [2,2,1], [3,3,3], [4,4,4], [5,5,5], [1,3,5], [1,4,3], [1,5,4], [2,3,4], [2,4,5], [2,5,3]\}$, and

II $= \{[1,1,1], [2,2,1], [3,3,1], [4,4,1], [5,5,1], [2,3,4], [2,4,5], [2,5,3], [3,5,4]\}$.

Lemma 2.4 *If $v \equiv 5$ (mod 6) and $v \geq 11$, then $J[v] = I[v]$.*

Proof. Let $v = 6k+5$, $S = \{a, b, x_0, x_1, \ldots, x_{2k}, y_0, y_1, \ldots, y_{2k}, z_0, z_1, \ldots, z_{2k}\}$ and let T be the following collection of blocks:

$[x_i, y_{i-r}, y_{i+r}], [x_i, y_{i+r}, y_{i-r}],$

$[y_i, z_{i-r}, z_{i+r}], [y_i, z_{i+r}, z_{i-r}],$

$[z_i, x_{i-r}, x_{i+r}], [z_i, x_{i+r}, x_{i-r}]$ and D_i,

for $i = 0, 1, \ldots, 2k$; $r = 1, 2, \ldots, k$, where $D_i = \{[a, x_i, y_i], [a, y_i, z_i], [a, z_i, x_i], [b, x_i, z_i], [b, z_i, y_i], [b, y_i, x_i]\}$.

Let $J = \{[x,x,x] \mid x \in S \setminus \{b\}\} \cup \{[b,b,a]\}$ and put $M_0 = T \cup J$. Then $M_0 \in$ EMTS$(6k+5, 6k+4)$ and M_0 contains a collection Δ of $2k+1$ copies of pattern **I** on the set $\{a, b, x_i, y_i, z_i\}$ for $i = 0, 1, \ldots, 2k$. For every integer $t \in \{1, 2, \ldots, 2k+1\}$, we derive from M_0 a system $M_t \in EMTS(6k+5, 6k+$

$4-3t$) by removing the first t copies of Δ and replacing them with t copies of pattern **II**. And M^* is a system which comes from M_0 by removing blocks $\{[x_0, y_{2k}, y_1], [x_0, y_1, y_{2k}], [x_0, x_0, x_0], [y_{2k}, y_{2k}, y_{2k}], [y_1, y_1, y_1]\}$ and replacing them with blocks $\{[x_0, x_0, y_{2k}], [y_{2k}, y_{2k}, y_1], [y_1, y_1, x_0]\}$. Then $|M_0 \cap M^*| = 12k^2 + 24k + 6$ and $|M_i \cap M_{i+1}| = 12k^2 + 24k + 11 - 2i - 7$ for $i = 0, 1, \ldots, 2k-2$. Thus we have $J[v] = I[v]$. ∎

Lemma 2.5 *If $v \equiv 8$ (mod 6) and $v \geq 14$, then $J[v] = I[v]$.*

Proof. Let $v = 6k + 8$, $S = \{a, b, c, d, e, x_0, x_1, \ldots, x_{2k}, y_0, y_1, \ldots, y_{2k}, z_0, z_1, \ldots, z_{2k}\}$ and let T be the following collection of blocks:

$\{[d, e, b], [d, c, a], [d, a, e], [d, b, c], [e, c, b], [e, a, c]\}$,
$[d, x_i, y_{i+1}], [d, y_i, z_{i+1}], [d, z_i, x_{i+1}]$,
$[e, y_{i-1}, x_i], [e, z_{i-1}, y_i], [e, x_{i-1}, z_i]$,
$[c, y_{i+1}, y_{i-1}], [c, z_{i+1}, z_{i-1}], [c, x_{i+1}, x_{i-1}]$,
$[x_i, y_{i-1}, y_{i+1}], [y_i, z_{i-1}, z_{i+1}], [z_i, x_{i-1}, x_{i+1}]$,
$[x_i, y_{i-r}, y_{i+r}], [x_i, y_{i+r}, y_{i-r}]$,
$[y_i, z_{i-r}, z_{i+r}], [y_i, z_{i+r}, z_{i-r}]$,
$[z_i, x_{i-r}, x_{i+r}], [z_i, x_{i+r}, x_{i-r}]$ and D_i,

for $i = 0, 1, \ldots, 2k$; $r = 2, \ldots, k$, where $D_i = \{[a, x_i, y_i], [a, y_i, z_i], [a, z_i, x_i], [b, x_i, z_i], [b, z_i, y_i], [b, y_i, x_i]\}$.

Let $J = \{[x, x, x] \mid x \in S \setminus \{b\}\} \cup \{[b, b, a]\}$ and put $M_0 = T \cup J$. Then $M_0 \in$ EMTS$(6k + 8, 6k + 7)$ and M_0 contains a collection Δ of $2k + 1$ copies of of pattern **I** on the sets $\{a, b, x_i, y_i, z_i\}$ for $i = 0, 1, \ldots, 2k$. For every integer $t \in \{1, 2, \ldots, 2k + 1\}$, we derive from M_0 a system $M_t \in EMTS(6k + 5, 6k + 4 - 3t)$ by removing the first t copies of Δ and replacing them with t copies of patterm **II**. And M^* is a system which comes from M_0 by removing blocks$\{[x_0, y_{2k-1}, y_2], [x_0, y_2, y_{2k-1}], [x_0, x_0, x_0], [y_{2k-1}, y_{2k-1}, y_{2k-1}], [y_2, y_2, y_2]\}$ and replacing them with blocks $\{[x_0, x_0, y_{2k-1}], [y_{2k-1}, y_{2k-1}, y_2], [y_2, y_2, x_0]\}$. Then $|M_0 \cap M^*| = 12k^2 + 36k + 21$ and $|M_i \cap M_{i+1}| = 12k^2 + 36k + 24 - 2i - 7$ for $i = 0, 1, \ldots, 2k-1$. Thus we have $J[v] = I[v]$. ∎

3 Conclusions.

Using Lemmas 2.1, 2.3, 2.4 and 2.5, we obtained the following results:

Main Theorem $J[v] = I[v]$, for $v \geq 4$.

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