Subgraphs orthogonal to a 1-factorization of complete bipartite graphs*

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Abstract

We take a special 1-factorization of $K_{n,n}$ and investigate the subgraphs suborthogonal to the 1-factorization. Some interesting results are obtained, including an identity involving n^n and $n!$ and a property of permutations.

1 Introduction

In this paper, C_n represents a cycle of length n and P_n a path of length $n-1$. Let K and H be two graphs and k a positive integer. $K \uplus H$ is the vertex disjoint union of K and H, and kH is k vertex disjoint union of $H's$. For other definitions which are not mentioned here, we refer to [3, 8].

A *factorization* $\mathcal{F} = \{F_1, \ldots, F_t\}$ of a graph G is a partition of $E(G)$ into edge-disjoint spanning subgraphs $F_1, \ldots, F_t$, which are called *factors*. A factorization may contain isolated vertices. Let a and b be nonnegative integers with $a \leq b$. An (a, b)-factorization $\mathcal{F} = \{F_1, \ldots, F_t\}$ of a graph G is a factorization such that the degree

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of each vertex of F_i is between a and b for $i = 1, \ldots, t$. If $a = b = 1$, $\mathcal{F}$ is called a 1-factorization of G. A subgraph H of G is *suborthogonal* to $\mathcal{F}$ if $|E(H) \cap E(F_i)| \leq 1$ for $1 \leq i \leq t$, and *orthogonal* to $\mathcal{F}$ if $|E(H) \cap E(F_i)| = 1$ for $1 \leq i \leq t$. We also call H an *$\mathcal{F}$-suborthogonal* or *$\mathcal{F}$-orthogonal* subgraph of G.

We can view $\mathcal{F}$ as an edge-coloring of G using $t = |\mathcal{F}|$ colors, where F_i consists of all edges with color i for $1 \leq i \leq t$.

The majority of work done to date on suborthogonal factorizations has focused on the case when G is a complete graph or a complete bipartite graph and $\mathcal{F}$ is a 1-factorization.

It is well known [5, 8] that the complete bipartite graph $K_{n,n}$ has a 1-factorization; and that a 1-factorization of $K_{n,n}$ is equivalent to a Latin square of order n. Note that given a 1-factorization $\mathcal{F}$ of $K_{n,n}$, an $\mathcal{F}$-suborthogonal graph which is a matching corresponds to a partial transversal in the Latin square corresponding to $\mathcal{F}$. There is a long standing conjecture posed by Ryser that *"every Latin square of order n has a transversal if n is odd, and a partial transversal of length $n-1$ otherwise"*. In [1], Alspach, Heinrich and G. Liu posed a more general problem.

Problem 1 *Given a 1-factorization $\mathcal{F}$ of $K_{n,n}$, describe all subgraphs of $K_{n,n}$ that are suborthogonal to $\mathcal{F}$.*

The converse to Problem 1 is the following:

Problem 2 *Given a subgraph of $K_{n,n}$ with at most n edges, is there some 1-factorization $\mathcal{F}$ of $K_{n,n}$ to which it is suborthogonal?*

In other words, this problem asked which partial Latin square of order n with n filled cells could be extended to an $n \times n$ Latin square. Evans [6] conjectured that *if at most $n-1$ cells of an $n \times n$ array are filled with entries from an n-set, then it can be completed to a Latin square of order n.* This conjecture was verified by Smetaniuk [7].

Problem 2 was solved by Andersen and Hilton [2], and Damerell [4] independently.

Theorem 1 *[2, 4] Every n-edge subgraph of $K_{n,n}$ except for the vertex-disjoint union $K_{1,t} \uplus K_{n-t,1}$ is orthogonal to a 1-factorization of $K_{n,n}$, where $K_{1,t}$ and $K_{n-t,1}$ are two stars in $K_{n,n} = [X, Y]$ with one center in X and the other in Y.*

Compared with Ryser's conjecture, it is no wonder that Problem 1 is difficult. In this paper, we investigate the problem by focusing on a special 1-factorization of $K_{n,n}$. It does not seem to be interesting by looking at a particular 1-factorization at first, but we will see later that some interesting applications and problems are arising, and even in this special case, it is hard to solve it completely.

Let $K_{n,n} = [X, Y]$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the bipartition of $K_{n,n}$. Let $F_t = \{x_i y_{i+t} | i = 1, 2, \ldots, n\}$, where $t = 0, 1, \ldots, n-1$ and the subscripts are reduced modulo n. The edges in F_t is said to have **slope** t. Let $\mathcal{F} = \{F_0, F_1, \ldots, F_{n-1}\}$. Then $\mathcal{F}$ is a 1-factorization of $K_{n,n}$ which appears frequently in literature. In the following discussion, we will focus on this particular 1-factorization $\mathcal{F}$.

2 A construction

Proposition 2 *If H is a subgraph orthogonal to $\mathcal{F}$, then any subgraph of H is suborthogonal to $\mathcal{F}$. On the other hand, if M is a subgraph of $K_{n,n}$ which is suborthogonal to $\mathcal{F}$, then M can be extended to a subgraph of $K_{n,n}$ which is orthogonal to $\mathcal{F}$.*

Proof: The first statement is obviously true by the definition of sub orthogonality. To prove the second part, let $|M| = k$. W.l.o.g., let $|M \cap F_i| = 1$ for $i = 0, 1, \ldots, k-1$. Choose $e_i \in F_i$ for $i = k, \ldots, n-1$. Let $H = M \cup \{e_k, \ldots e_{n-1}\}$. Then $M \subseteq H$ and H is orthogonal to $\mathcal{F}$. ∎

By the above proposition, we need to only consider $\mathcal{F}$-orthogonal subgraphs. If H is an $\mathcal{F}$-orthogonal subgraph, then $|E(H)| = n$ and $|H \cap F_t| = 1$ for $t = 0, \ldots n-1$. Based on this information, we can construct all $\mathcal{F}$-orthogonal subgraphs of $K_{n,n}$.

A construction: Let $s = x_{i_1}, x_{i_2}, \ldots x_{i_m}$ be a sequence of vertices in X such that $1 \le i_1 < i_2 < \cdots < i_m \le n$. Let $P = (S_1, S_2, \ldots, S_m)$ be an ordered partition of $\{0, 1, \ldots, n-1\}$ where $S_i \neq \emptyset$ for $i = 1, 2, \ldots, m$. We call (s, P) **a compatible pair**. Let $Y_{i_j + S_j} = \{y_{i_j + k} | k \in S_j\}$ for $j = 1, \ldots, m$. We construct a subgraph of $K_{n,n}$, denoted by $H(s, P)$, as following:

$$V(H(s,P)) = \{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} \cup (\cup_{j=1}^{m} Y_{i_j + S_j}),$$

and

$$E(H(s,P)) = \{x_{i_j} y_k | k - i_j \in S_j, j = 1, \ldots, m\},$$

where all subscripts are reduced to modulo n.

It is easy to see that $H(s,P)$ is an edge disjoint union of m stars.

Proposition 3 *A subgraph H of $K_{n,n}$ is orthogonal to $\mathcal{F}$ if and only if $H = H(s,P)$ for some compatible pair (s,P).*

Proof: Sufficiency. Let H be an $H(s,P)$ type subgraph with $s = x_{i_1}, \ldots, x_{i_m}$, and $P = (S_1, \ldots, S_m)$. Then $E(H) = \{x_{i_j} y_k | k - i_j \in S_j, j = 1, \ldots, m\}$. For each t, $0 \leq t \leq n-1$, there exists a unique S_{l_t} such that $t \in S_{l_t}$. Then $e_t = x_{i_{l_t}} y_{i_{l_t}+t} \in E(H)$ is the only edge in H with slope t. Therefore, $H \cap F_t = \{e_t\}$ and hence H is orthogonal to $\mathcal{F}$.

Necessity. Let H be an $\mathcal{F}$-orthogonal subgraph of $K_{n,n}$ and $V(H) \cap X = \{x_{i_1}, \ldots, x_{i_m}\}$ with $i_1 < i_2 < \cdots < i_m$. Then let $s = x_{i_1}, \ldots, x_{i_m}$, and $S_j = \{k | x_{i_j} y_{i_j+k} \in E(H)\}$. Since H is orthogonal to $\mathcal{F}$, it follows that $S_j \neq \emptyset$ and $S_j \cap S_i = \emptyset$ if $i \neq j$. Hence $P = (S_1, \ldots, S_m)$ is an ordered partition of $\{0, 1, \ldots, n-1\}$ and $H = H(s,P)$. ∎

A natural question is that what a subgraph $H(s,P)$ is? We will describe some special $H(s,P)$ in the next section.

3 Some classes of $\mathcal{F}$-orthogonal subgraphs

If $|s| = 1$, we have the following easy result.

Proposition 4 *If $|s| = 1$, then $H(s,P) \cong K_{1,n}$.*

When $|s| = 2$, we know that $H(s,P)$ is an edge-disjoint union of two stars. Are these two stars necessarily vertex-disjoint?

Proposition 5 *There exists an $\mathcal{F}$-orthogonal subgraph H of $K_{n,n}$ consisting of two vertex-disjoint union of stars $K_{1,m}$ and $K_{1,n-m}$ with two centers in the same partition of $K_{n,n}$ if and only if there exists an integer k such that $1 < k \leq n$ and $(k-1)m \equiv 0 (\mod n)$.*

Proof: Necessity. Suppose, without loss of generality, that $K_{1,m}$ is centered at x_1, and $K_{1,n-m}$ is centered at x_k. Then $s = x_1, x_k$, where $1 < k \leq n$. Let $S_1 = \{i_1, \ldots, i_m\}$ and $S_2 = \{i_{m+1}, \ldots, i_n\}$. Then

$\{i_1+1, i_2+1, \ldots i_m+1, i_{m+1}+k, \ldots, i_n+k\} \equiv \{0, 1, \ldots, n-1\} (\text{mod } n)$ since H is vertex disjoint. Therefore,

$$[\sum_{j=1}^{n} i_j + m + (n-m)k] \equiv \frac{n(n-1)}{2} (\text{mod} \quad n).$$

But $\{i_1, \ldots, i_n\} = \{0, 1, \ldots, n-1\}$, thus $\sum_{j=1}^{n} i_j = \frac{n(n-1)}{2}$. This implies that $(k-1)m \equiv 0(\text{mod } n)$.

Sufficiency. Let m and k be two integers such that $m < n, k > 1$ and $(k-1)m \equiv 0(\text{mod } n)$. Choose $s = x_1, x_k$ as the sequence and $P = (S_1, S_2)$ where $S_1 = \{0, k-1, 2(k-1), \ldots, (m-1)(k-1)\}$ and $S_2 = \{1, 2, \ldots, n-1\} - S_1$. We are going to show that $H(s, P)$ is a vertex-disjoint union of two stars $K_{1,m}$ and $K_{1,n-m}$.

It is easy to see that $H(s, P)$ is an edge-disjoint union of two stars $K_{1,m}$ and $K_{1,n-m}$. If they are not vertex-disjoint, then there exist integers i and j such that $j \in S_2$ and

$$1 + i(k-1) \equiv k + j(\text{mod} \quad n).$$

Then $j \equiv (i-1)(k-1)(\text{mod } n)$. It follows that $j \in S_1$, which is a contradiction. ■

Proposition 6 *There is no $\mathcal{F}$-orthogonal subgraph of $K_{n,n}$ consisting of vertex-disjoint union of two stars with centers in different partition of $K_{n,n}$.*

Proof: Suppose $H = K_{1,m} \uplus K_{n-m,1}$ is orthogonal to $\mathcal{F}$, where the stars $K_{1,m}$ and $K_{n-m,1}$ have centers x and y, respectively. Then there are n colors assigned to the edges of H and no color is available for the edge xy, which is a contradiction. ■

Propositions 5 and 6 will imply the following result.

Corollary 7 *(1) If n is a prime, there is no $\mathcal{F}$-orthogonal subgraph consisting of two vertex-disjoint stars.*

(2) There is no $\mathcal{F}$-orthogonal subgraph consisting of two vertex-disjoint union of $K_{1,n-1}$ and $K_{1,1}$.

Proposition 8 *For any n, there exists an $\mathcal{F}$-orthogonal path of length n.*

Proof: If n is odd, let $P_{n+1} = x_1 y_1 x_n y_2 x_{n-1} y_3 \cdots x_{\frac{n+3}{2}} y_{\frac{n+2}{2}}$.

If n is even, let $P_{n+1} = x_1 y_1 x_n y_2 x_{n-1} y_3 \cdots y_{\frac{n}{2}} x_{\frac{n}{2}+1}$. ■

Proposition 9 *(1) There is an $\mathcal{F}$-orthogonal* 1*-factor if and only if n is odd.*

(2) There is an $\mathcal{F}$-orthogonal subgraph consisting of an $(n-2)$-matching with a disjoint P_3 if and only if n is even.

Proof: (1) Sufficiency. Let n be odd. We need to construct a 1-factor which has a form $H(s,P)$. To do that, we select $s = x_1, x_2, \ldots, x_n$ and $P = (\{0\},\{1\},\ldots,\{n-1\})$ as a compatible pair. Then $E(H(s,P)) = \{x_1y_1, x_2y_3, x_3y_5, \ldots, x_iy_{2i-1}, \ldots\}$. If for some i and j, $y_{2i-1} = y_{2j-1}$, then $2i-1 \equiv 2j-1 (\text{mod } n)$. $2(i-j) \equiv 0(\text{mod } n)$. Since n is odd, then $i-j \equiv 0(\text{mod } n)$. But $1 \leq i,j \leq n$, this implies that $i-j=0$ and hence $i=j$. Therefore, all edges in $H(s,P)$ are disjoint and hence $H(s,P)$ is a 1-factor.

Necessity. Suppose that there is an $\mathcal{F}$-orthogonal 1-factor $\{x_1y_{j_1}, x_2y_{j_2}, \ldots, x_ny_{j_n}\}$. Then $\{j_1-1, j_2-2, \ldots, j_n-n\} \equiv \{1,2,\ldots,n-1,n\}(\text{mod } n)$. Since $\{j_1,\ldots,j_n\} = \{1,2,\ldots,n\}$, then $\sum_{i=1}^{n}(j_i - i) = \sum_{i=1}^{n} j_i - \sum_{i=1}^{n} i = 0$. Hence $0 \equiv \frac{n(n+1)}{2}(\text{mod } n)$. It follows that n is odd.

(2) Sufficiency. Let n be even. By (1), there is no $\mathcal{F}$-orthogonal 1-factor. Choose $s = x_1, x_2, \ldots, x_n$, and $P = (S_1, S_2, \ldots, S_n)$, where $S_1 = \{0\}, S_2 = \{1\}, \ldots, S_{\frac{n}{2}} = \{\frac{n}{2}-1\}, S_{\frac{n}{2}+1} = \{n-1\}, S_{\frac{n}{2}+2} = \{\frac{n}{2}\}, S_{\frac{n}{2}+3} = \{\frac{n}{2}+1\}, \ldots, S_n = \{n-2\}$. Then $E(H(s,P)) = \{x_1y_1, x_2y_3, \ldots, x_iy_{2i-1}, \ldots, x_{\frac{n}{2}}y_{n-1}, x_{\frac{n}{2}+1}y_{\frac{n}{2}}, x_{\frac{n}{2}+2}y_2, x_{\frac{n}{2}+3}y_4, \ldots, x_{\frac{n}{2}+i}y_{2i-2}, \ldots, x_ny_{n-2}\}$. Note that the edges in $E(H(s,P))$ appearing before the edge $x_{\frac{n}{2}+1}y_{\frac{n}{2}}$ have the form x_iy_{2i-1} while the edges appearing after $x_{\frac{n}{2}+1}y_{\frac{n}{2}}$ have the form $x_{\frac{n}{2}+i}y_{2i-2}$. They are all different. But we know that $H(s,P)$ cannot be a 1-factor, therefore, we have either $y_{\frac{n}{2}} = y_{2i-1}$ for some i or $y_{\frac{n}{2}} = y_{2j}$ for some j. Hence $H \cong (n-2)K_2 \uplus P_3$.

Necessity. Suppose H is orthogonal to $\mathcal{F}$ and $H \cong (n-2)K_2 \uplus P_3$. Without loss of generality, let $H \cap X = X$. Then we may assume that $E(H) = \{x_iy_{j_i} | i = 1,2,\ldots n\}$. We have $\{j_i - i | i = 1,2,\ldots,n\} \equiv \{1,2,\ldots,n\}(\text{mod } n)$. Therefore, $\sum_{i=1}^{n}(j_i - i) \equiv \frac{n(n+1)}{2}(\text{mod } n)$. Since we have one P_3 with the middle vertex in Y, we assume that k appears twice and j does not appear in $(j_1,\ldots,j_n)$. Then $\sum_{i=1}^{n}(j_i - i) = \sum_{i=1}^{n} j_i - \sum_{i=1}^{n} i = (\frac{n(n+1)}{2} - j + k) - \frac{n(n+1)}{2}$, where $(j \neq k \leq n)$. Hence $k - j \equiv \frac{n(n+1)}{2}(\text{mod } n)$. If n is odd, then we must have $k=j$, which is a contradiction. Hence n is even. ∎

We note that if A is a Latin square which corresponds to $\mathcal{F}$, then it has the following form.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{1n} & a_{11} & \cdots & a_{1n-1} \\ & & \ddots & \\ a_{12} & a_{13} & \cdots & a_{11} \end{pmatrix}.$$

We call such Latin square a cyclic Latin square.

Remark: *Proposition tells us that any cyclic Latin square A has a transversal if n is odd. However, if n is even, A has no transversal but has a partial transversal of length $n-1$.*

Proposition 10 *There exists an $\mathcal{F}$-orthogonal $\frac{n}{2}P_3$ if and only if n is even.*

Proof: Necessity. Obvious.

Sufficiency. Let $s = x_1, x_2, \ldots, x_n$ and $P = (\{0\}, \{1\}, \ldots, \{n-1\})$. Then $E(H(s,P)) = \{x_i y_{2i-1} | i = 1, 2, \ldots, n\}$. If $y_{2i-1} = y_{2j-1}$, then $2i-1 \equiv 2j-1(\text{mod } n)$, or $2(i-j) \equiv 0(\text{mod } n)$. Without loss of generality, let $j > i$. Then there exists an integer $l > 0$ such that $j - i = l\frac{n}{2}$. But $j \leq n$, and $i \geq 1$, this implies that $l = 1$ and $j = i + \frac{n}{2}$. Therefore, $x_i y_{2i-1} x_{\frac{n}{2}+i}$ is a P_3 for each $i = 1, \ldots, \frac{n}{2}$. This shows that $H(s,P) \cong \frac{n}{2}P_3$. ∎

Proposition 11 *There exists a cycle C_n which is orthogonal to $\mathcal{F}$ if and only if $n \equiv 0(\text{mod } 4)$.*

Proof: Necessity. Suppose there exists an $\mathcal{F}$-orthogonal C_n. Then n is even, say $n = 2m$. We may assume that $C_n = x_{i_1} y_{j_1} x_{i_2} y_{j_2} \cdots x_{i_m} y_{j_m} x_{i_1}$. This implies that $\{j_1 - i_1, j_1 - i_2, j_2 - i_2, j_2 - i_3, \ldots, j_{m-1} - i_{m-1}, j_{m-1} - i_m, j_m - i_m, j_m - i_1\} \equiv \{1, 2, \ldots, n\}(\text{mod } n)$, where $m = \frac{n}{2}$. Hence

$$2\sum_{k=1}^{m}(j_k - i_k) \equiv \frac{n(n+1)}{2}(\text{mod} \quad n).$$

In the case of n being even, $\frac{n(n+1)}{2}$ is even if and only if $n \equiv 0(\text{mod } 4)$.

Sufficiency. Suppose that $n \equiv 0 (\text{mod } 4)$. Then

$$C_n = x_1 y_2 x_{n-1} y_4 x_{n-3} \cdots y_{\frac{n}{2}} x_{\frac{n}{2}+1} y_{\frac{n}{2}+3} x_{\frac{n}{2}-1} \cdots x_3 y_1 x_1$$

is an $\mathcal{F}$-orthogonal cycle of length n. ∎

Proposition 12 *(1) If $n \equiv 2(\text{mod } 4)$, then there exists an $H = C_{n-2} \uplus 2K_2$ which is orthogonal to $\mathcal{F}$.*

(2) If n is odd, there exists an $\mathcal{F}$-orthogonal $H = C_{n-1} \uplus K_2$.

Proof: (1) If $n \equiv 2(\text{mod } 4)$, we let $H = \{x_4 y_3, x_{\frac{n}{2}+3} y_{\frac{n}{2}+1}\} \uplus$

$$\{x_1 y_2 x_{n-1} y_4 x_{n-3} \cdots y_{\frac{n}{2}-1} x_{\frac{n}{2}+2} y_{\frac{n}{2}+4} x_{\frac{n}{2}} y_{\frac{n}{2}+6} \cdots x_5 y_1 x_1\},$$

which is orthogonal to $\mathcal{F}$ and is a vertex-disjoint union of a C_{n-2} and a $2K_2$.

(2) If $n \equiv 1(\text{mod } 4)$, then we let $H = \{x_{\frac{n+5}{2}} y_{\frac{n+3}{2}}\} \uplus$

$$\{x_1 y_2 x_{n-1} y_4 x_{n-3} \cdots y_{\frac{n-1}{2}} x_{\frac{n+3}{2}} y_{\frac{n+7}{2}} x_{\frac{n-1}{2}} y_{\frac{n+11}{2}} \cdots x_4 y_1 x_1\}.$$

If $n \equiv 3(\text{mod } 4)$ and $n > 7$, then $H = \{x_{\frac{n+1}{2}} y_{\frac{n-5}{2}}\} \uplus$

$$\{x_1 y_2 x_{n-1} y_4 x_{n-3} \cdots x_{\frac{n+5}{2}} y_{\frac{n+1}{2}} x_{\frac{n-3}{2}} y_{\frac{n+5}{2}} \cdots y_{n-3} x_2 y_1 x_1\}.$$

If $n = 7$, let $H = \{x_1 y_2 x_6 y_4 x_2 y_1 x_1\} \uplus \{x_3 y_7\}$. ∎

Proposition 13 *For any even integer k with $4 \leq k < n$, there exists a C_k suborthogonal to $\mathcal{F}$.*

Proof: From Propositions 11 and 12, we have cycles of lengths $n-1$ or $n-2$ according to the parity of n. The idea of the proof is that we add in one chord to these cycles to obtain two smaller cycles such that one of them has length k. Therefore we can assume that $k \leq \frac{n}{2}$.

(1) $n \equiv 0(\text{mod } 4)$. Note that we assume that $k \leq \frac{n}{2}$. Choose a $(k-2)$-subpath from the cycle in Proposition 11: $x_1 y_2 x_{n-1} \cdots x_m$, we have that m is odd, then $x_m y_1$ is of even slope. Let $C_k = x_1 y_2 x_{n-1} \cdots x_m y_1 x_1$. Then C_k is a cycle of length k which is orthogonal to $\mathcal{F}$.

(2) $n \equiv 2(\text{mod } 4)$. In this case, we can assume that $k \leq \frac{n-2}{2}$. The proof is exactly the same as (1).

(3) $n \equiv 3 \pmod 4$. In this case, we can assume that $k \le \frac{n-1}{2}$. Choose a $(k-2)$-subpath from the cycle in Proposition 12(2) when $n \equiv 3 \pmod 4$ as follows: $y_1 x_2 y_{n-3} \cdots y_m$. All edges in this path have even slope and also m is even. Therefore, $y_1 x_2 y_{n-3} \cdots y_m x_1 y_1$ is a cycle of length k and is orthogonal to $\mathcal{F}$.

(4) $n \equiv 1 \pmod 4$. In this case, we can assume that $k \le \frac{n-1}{2}$. Choose a $(k-2)$-subpath from the cycle in Proposition 12(2) when $n \equiv 1 \pmod 4$ as follows: $y_1 x_4 \cdots y_m$, we have that m is even and all edges in this path have even slope. Thus $y_1 x_4 \cdots y_m x_1 y_1$ is the desired cycle. ∎

A *double 2k-partial transversal* of a Latin square L is a collection of $2k$ cells in L such that all the cells have different entries and each row and column each contains either two elements or none from the collection.

Proposition 14 *Let A be a cyclic Latin square of order n. Then*

(1) there exists a double n-transversal of A if and only if $n \equiv 0 \pmod 4$.

(2) for any $2k < n$, there exists a double $2k$-partial transversal in A.

Proof: Note that a double partial transversal of A corresponds to a cycle C which is suborthogonal to $\mathcal{F}$. (1) follows from Proposition 11, and (2) follows from Proposition 13. ∎

The following problem might be interesting.

Problem 3 *Which $n \times n$ Latin square has a double $2k$-partial transversal for any $2k \le n$? Equivalently, which 1-factorization of $K_{n,n}$ has a suborthogonal C_{2k} for $2k \le n$?*

4 Applications

In this section, we present two interesting applications which we are not sure are new. But at least we have new proofs here.

Proposition 15 *Let n be a positive integer. Then*

$$n^n = \sum_{i=1}^{n} \sum_{k_1+\cdots+k_i=n, k_j>0} \binom{n}{i} \frac{n!}{k_1! k_2! \cdots k_i!}.$$

Proof: Let $\mathcal{F} = \{F_0, F_1, \ldots F_{n-1}\}$ be the 1-factorization of $K_{n,n}$. Each time we choose arbitrarily an edge from each F_t, for $t = 0, 1, \ldots n-1$, then the n edges form an $\mathcal{F}$- orthogonal subgraph. Obviously, any $\mathcal{F}$-orthogonal subgraph can be obtained this way. Therefore, the number of $\mathcal{F}$-orthogonal subgraphs is n^n.

On the other hand, the number of strictly monotone sequences of length i from $\{1, 2, \ldots, n\}$ is $\binom{n}{i}$. Denote the number of ordered i-partition of $\{0, 1, \ldots, n-1\}$ by $p_{n,i}$. Then $p_{n,i} =$

$$\sum_{k_1+\cdots+k_i=n, k_j>0} \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-k_2-\cdots-k_{i-1}}{k_i}$$

$$= \sum_{k_1+\cdots+k_i=n, k_j>0} \frac{n!}{k_1!k_2!\cdots k_i!}.$$

Therefore,

$$n^n = \sum_{i=1}^{n} \binom{n}{i} \sum_{k_1+\cdots+k_i=n, k_j>0} \frac{n!}{k_1!k_2!\cdots k_i!}$$

$$= \sum_{i=1}^{n} \sum_{k_1+\cdots k_i=n, k_j>0} \binom{n}{i} \frac{n!}{k_1!k_2!\cdots k_i!},$$

by Proposition 3.

This completes the proof. ∎

The second application is about permutations.

Proposition 16 *If n is even and p is an arbitrary permutation of $\{1, 2, \ldots, n\}$ such that $p(j) = i_j$, then there exists a pair $\{i_j, i_k\}$ such that*

$$i_j - i_k \equiv k - j (mod \;\; n) \quad or \quad j + i_j \equiv k + i_k (mod \;\; n).$$

Proof: Choose $s = x_1, x_2, \ldots, x_n$ and $P = (\{i_1\}, \{i_2\}, \ldots, \{i_n\})$. Then $E(H(s, P)) = \{x_j y_{j+i_j} | j = 1, 2, \ldots, n\}$. By Proposition 9(2), there exist integers j and k such that $y_{j+i_j} = y_{k+i_k}$. This implies that $j + i_j \equiv k + i_k \pmod{n}$ or $i_j - i_k \equiv k - j \pmod{n}$. ∎

We note that Proposition 16 is not true when n is odd.

5 Further directions

We consider the following problem:

"Describe the subgraphs of $K_{n,n}$ which are suborthogonal to all 1-factorizations of $K_{n,n}$."

Proposition 17 *The subgraph suborthogonal to all 1-factorizations of $K_{n,n}$ is isomorphic to $K_{1,k}$ for $1 \leq k \leq n$.*

Proof: It is obvious that any subgraph of $K_{n,n}$ which is isomorphic to $K_{1,k}$ is suborthogonal to any 1-factorization of $K_{n,n}$.

On the other hand, let H be a subgraph of $K_{n,n}$ which is suborthogonal to any 1-factorization of $K_{n,n}$. If H is not a star, then there are two independent edges in H, say e_1 and e_2 (note that H cannot be a triangle). But any two independent edges lie in a 1-factor and every 1-factor extends to a 1-factorization of $K_{n,n}$. Therefore, H is not orthogonal to this 1-factorization. ∎

Next we pose a more general problem.

Problem 4 *Let $k \geq 2$. Which subgraph H with $|E(H)| \leq k$ of $K_{n,n}$ is suborthogonal to a $(\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil)$- factorization $\{F_1, F_2, \ldots, F_k\}$ of $K_{n,n}$.*

Note that when $k = n$, this is the Problem 1. For $k = n + 1$ and $|E(H)| = n$, the following result answers the problem.

Proposition 18 *Every n-edge subgraph of $K_{n,n}$ is suborthogonal to some $(0,1)$-factorization $\{F_1, F_2, \ldots, F_{n+1}\}$ of $K_{n,n}$.*

Proof: Let H be an n-edge subgraph of $K_{n,n}$. If $H \not\cong K_{1,t} \uplus K_{n-t,1}$, then H is orthogonal to some 1-factorization $\{F_1, F_2, \ldots F_n\}$ of $K_{n,n}$ by Theorem 1. We choose $e \in F_1 - E(H)$, then $\{\{e\}, F_1 - \{e\}, F_2, \ldots, F_n\}$ is a desired $(0,1)$-factorization.

Let $H \cong K_{1,t} \uplus K_{n-t,1}$. We may assume $K_{1,t} = \{x_1; y_2, \ldots, y_{t+1}\}$ and $K_{n-t,1} = \{y_1; x_{t+1}, \ldots, x_n\}$. There exists a 1-factorization $\{F_2, \ldots, F_n\}$ in $(X - \{x_1\}, Y - \{y_1\})$ such that $x_i y_i \in F_i$ for $i = 2, 3, \ldots, n$. Now we obtain a $(0,1)$-factorization $\{F_1', F_2', \ldots, F_{n+1}'\}$ as following:

$F_i' = (F_i - \{x_i y_i\}) \cup \{x_1 y_i, y_1 x_i\}$ for $i \neq 1, t+1$; $F_1' = \{y_1 x_{t+1}\}$; $F_{t+1}' = (F_{t+1} - \{x_{t+1} y_{t+1}\}) \cup \{x_1 y_{t+1}\}$; and $F_{n+1}' = \{x_1 y_1, \ldots, x_n y_n\}$.

It is easy to see that H is suborthogonal to $\{F_1', F_2', \ldots, F_{n+1}'\}$. ∎

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