F-factor and Vertex-disjoint F in a graph

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Abstract

Let G be a graph of order $n \geq 4k$ and let S be the graph obtained from K_4 by removing two edges which have a common vertex. In this paper, we prove the following theorem:

A graph G of order $n \geq 4k$ with $\sigma_2(G) \geq n + k$ has k vertex-disjoint S. This theorem implies that a graph G of order $n = 4k$ with $\sigma_2(G) \geq 5k$ has an S-factor.

1 Introduction

In this paper, all graphs considered are finite, undirected and without loops or multiple edges. For a graph G, $V(G)$, $E(G)$ and $\delta(G)$ denote the set of vertices and the set of edges and the minimum degree of G, respectively. For a non-complete graph G, let $\sigma_2(G)$ denote the minimum degree sum of a pair of nonadjacent vertices. For a given graph G and $v \in V(G)$, we write $N_G(x)$ the neighbourhood of $V(G)$ and $d_G(x) = |N_G(x)|$. For a subset S of $V(G)$, the subgraph induced by S is denoted by $< S >$. For a subgraph H of G, $G-H =< V(G)-V(H) >$ and for a vertex x of G, $G-x =< V(G)-\{x\} >$. For a graph G, n is always the order of G. With a slight abuse of notation, for a subgraph H of G and a vertex $v \in V(G)$, $N_H(v) = N_G(v) \cap V(H)$ and $d_H(v) = |N_H(v)|$. In addition, for a subgraph H of G and a subset S of $V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$, and when $S \cap V(H) = \emptyset$, $N_H(S) = \bigcup_{v \in S} N_H(v)$. Let F be a given connected graph. Suppose that $|V(G)|$ is a multiple of $|V(F)|$. A spanning subgraph of G is called an F-factor if its components are all isomorphic to F.

In the case where $|V(F)| = 4$, there are six graphs which satisfy the conditions that $|V(F)| = 4$ and F is connected, namely,

$C_4, K_4, P_4, K_{1,3}, K_4^-$, which is the graph obtained from K_4 by removing one edge (in this paper, we call it D), and the graph obtained from K_4 by removing two edges which have a common vertex (in this paper, we call it S).

For P_4, it is easy to prove the following: $\delta(G) \geq \frac{n}{2}$ is sufficient to have a P_4-factor.

For C_4, there is a famous conjecture of El-Zahar [3]: If $\delta(G) \geq \sum_{i=1}^{k} \lceil a_i/2 \rceil$, then G can be decomposed into cycles of length $a_1, \cdots, a_k$. This conjecture can apply to a C_4-factor in case that $a_1 = \cdots = a_k = 4$ and $\delta(G) \geq \frac{n}{2}$.

For K_4, by adapting Hajnal and Szemerédi's theorem [4] that for K_l, $\delta(G) \geq \frac{l-1}{l}n$ suffices, $\delta(G) \geq \frac{3}{4}n$ suffices.

For $K_{1,3}$, Egawa and Ota proved in [1] that $\delta(G) \geq \frac{n}{2}$ is sufficient unless G is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$ with $\frac{n}{4}$ being odd.

For D, the author proved the following.

Theorem 1 ([6]) *A graph G of order $4k$ with $\delta(G) \geq \frac{5}{2}k$ has a D-factor.*

Since S is a subgraph of D, we can also get the following corollary from Theorem 1.

Corollary 2 ([6]) *A graph G of order $4k$ with $\delta(G) \geq \frac{5}{2}k$ has an S-factor.*

What happen if we consider the minimum degree condition for a given graph G of order $n \geq 4k$ to have k vertex-disjoint F ?

In the case that F is $K_{1,3}$, Egawa and Ota proved in [2] that a graph G of order $n \geq 4k + 6$ with $\delta(G) \geq k + 2$ has k vertex-disjoint $K_{1,3}$.

In [7], the author proved the following theorem.

Theorem 3 ([7]) *A graph G of order $n \geq 4k$ with $\delta(G) \geq \frac{n+k}{2}$ has k vertex-disjoint D.*

For the case of S, as S is a subgraph of D, we can get the following.

Corollary 4 ([7]) *A graph G of order $n \geq 4k$ with $\delta(G) \geq \frac{n+k}{2}$ has k vertex-disjoint S.*

In this paper, we consider a condition $\sigma_2(G)$ instead of $\delta(G)$ and we prove the following theorem, which implies Corollary 4.

Theorem 5 *A graph G of order $n \geq 4k$ with $\sigma_2(G) \geq n+k$ has k vertex-disjoint S.*

The condition on $\sigma_2(G)$ is best possible in a sense. Consider the graph $G = \overline{K_{k-1}} + (\overline{K_{\frac{n-k+1}{2}}} + \overline{K_{\frac{n-k+1}{2}}})$. It is obvious that G contains at most $k-1$ vertex-disjoint triangles. So G does not have k vertex-disjoint S and $\sigma_2(G) = n+k-1$.

For the case that F is D, we propose the following conjecture.

Conjecture 1 *A graph G of order $n > 4k$ with $\sigma_2(G) \geq n+k$ has k vertex-disjoint D.*

The condition on $\sigma_2(G)$ is also best possible as a same example in Theorem 5. Note that the same conclusion does not hold when $n = 4k$. Consider the graph $G = K_1 + \overline{K_{\frac{3k}{2}}} + K_{\frac{5k}{2}-1}$. It is obvious that G does not have k vertex-disjoint D and $\sigma_2(G) = 5k = n+k$.

Theorem 5 implies the following corollary, which also implies Corollary 2.

Corollary 6 *A graph G of order $n = 4k$ with $\sigma_2(G) \geq 5k$ has an S-factor.*

2 Proof of Theorem 5

In [5], Justesen stated the following theorem.

Theorem 7 ([5]) *A graph G of order $n \geq 3k$ with $\sigma_2(G) \geq n+k$ has k vertex-disjoint triangles.*

By Theorem 7, since G has k vertex-disjoint triangles, we can choose k vertex-disjoint subgraph $S_1, \cdots, S_k$ such that S_i is isomorphic to S or K_3. Now we choose $S_1, \cdots, S_{k'}, \cdots, S_k$, where S_i is isomorphic to S for $i = 1, \cdots, k'$, and S_i is isomorphic to K_3 for $i = k'+1, \cdots, k$ such that k' is as large as possible. If $k' = k$, then Theorem 5 holds. So we may assume $k' \leq k-1$. Let H be the subgraph of G induced by $\bigcup_{i=1}^{k'} V(S_i)$. Let M be the subgraph of G induced by $\bigcup_{i=k'+1}^{k-1} V(S_i)$. Note that, if $k' = k-1$, then M is empty. Let v be a vertex in $G - H - M - S_k$. Let v_1, v_2, v_3 be the distinct vertices in S_k. For a subgraph N of

G, $d_N = 3d_N(v) + d_N(v_1) + d_N(v_2) + d_N(v_3)$. Note that, since $vv_1, vv_2, vv_3 \notin E(G)$, $d_G \geq 3\sigma_2(G)$. Let $Z := G - H - S_k - \{v\}$ and also, let a_i, b_i, c_i, d_i be the vertex in S_i for each $i = 1, \cdots, k'$ such that $d_{S_i}(a_i) = 1$, $d_{S_i}(b_i) = 3$ and $d_{S_i}(c_i) = d_{S_i}(d_i) = 2$.

We prove the following claims.

Claim 1 *For any* $z \in Z$, $d_z \leq 3$.

Proof. There are two possibilities for z, namely $z \in M$ or $z \in Z - M$.

Suppose $z \in Z - M$. If $N_{S_k}(z) \neq \emptyset$, then $< \{z\} \cup V(S_k) >$ contains S. But this contradicts the maximality of k'. So, $d_z \leq 3$, the result follows.

Suppose $z \in M$. Then z is in some of $S_{i'}$, $k' + 1 \leq i' \leq k - 1$. Then $zv \notin E(G)$. For otherwise, $< \{v\} \cup V(S_{i'}) >$ contains S. But this contradicts the maximality of k'. So, $d_z \leq 3$, the result follows. ∎

Claim 2 $d_{S_i} \leq 15$ *for* $i = 1, \cdots, k'$.

Proof. If $d_{S_i}(v) \leq 1$, then the result easily follows. So, we may assume $d_{S_i}(v) \geq 2$.

Suppose $d_{S_i}(v) = 2$. Since $d_{S_i - \{a_i\}}(v) \geq 1$, $< \{v\} \cup (V(S_i) - \{a_i\}) >$ contains S. If $d_{S_k}(a_i) \geq 1$, then $< \{a_i\} \cup V(S_k) >$ contains S. But this contradicts the maximality of k'. So, $d_{S_k}(a_i) = 0$. Therefore, $d_{S_i}(v_1) + d_{S_i}(v_2) + d_{S_i}(v_3) \leq 9$. So, $d_{S_i} \leq 15$, the result follows.

Suppose $d_{S_i}(v) = 3$. By the same argument in the case $d_{S_i}(v) = 2$, $d_{S_k}(a_i) = 0$. Since c_i, d_i are symmetric, we may assume without loss of generality, $va_i, vb_i, vc_i \in E(G)$ or $va_i, vc_i, vd_i \in E(G)$ or $vb_i, vc_i, vd_i \in E(G)$. Suppose $va_i, vb_i, vc_i \in E(G)$ or $vb_i, vc_i, vd_i \in E(G)$. Then $d_{S_k}(d_i) = 0$. For otherwise, $< \{d_i\} \cup V(S_k) >$ contains S and $< v, a_i, b_i, c_i >$ contains S. But this contradicts the maximality of k'. Therefore, $d_{S_i}(v_1) + d_{S_i}(v_2) + d_{S_i}(v_3) \leq 6$. So, $d_{S_i} \leq 15$, the result follows. Suppose $va_i, vc_i, vd_i \in E(G)$. Then $d_{S_k}(b_i) = 0$. For otherwise, $< \{b_i\} \cup V(S_k) >$ contains S and $< v, a_i, c_i, d_i >$ contains S. But this contradicts the maximality of k'. Therefore, $d_{S_i}(v_1) + d_{S_i}(v_2) + d_{S_i}(v_3) \leq 6$. So, $d_{S_i} \leq 15$, the result follows.

Suppose $d_{S_i}(v) = 4$. Then $d_{S_i}(v_1) + d_{S_i}(v_2) + d_{S_i}(v_3) = 0$. For otherwise, if $d_{S_k}(u_i) \geq 1$ for some $u_i \in V(S_i)$, then $< \{u_i\} \cup V(S_k) >$

contains S and $< \{v\} \cup (V(S_i) - \{u_i\}) >$ contains S. But this contradicts the maximality of k'. Therefore, $d_{S_i}(v_1) + d_{S_i}(v_2) + d_{S_i}(v_3) = 0$. So, $d_{S_i} \leq 12$, the result follows. ∎

We can easily get the facts that $d_{S_k} = 6$ and $d_v = 0$. By Claims 1 and 2, we get the fact that $d_G \leq 15k' + 3(n - 4k' - 4) + 6 = 3n + 3k' - 6 \leq 3n + 3k - 9$. But this contradicts the fact that $d_G \geq \sigma_2(G) \geq 3n + 3k$. So, Theorem 5 follows. ∎

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