On Connected $[k, k+1]$-Factors in Claw-Free Graphs

Guojun Li[*†]
Department of Mathematics
Shandong University
Jinan 250100, P. R. China
Binhai Zhu[‡]
Department of Computer Science
City University of HongKong
Hong Kong, P. R. China
and
Chuanping Chen[§]
Institute of Systems Science, Academia Sinia
Beijing 100080, P. R. China

Abstract It is shown in this paper that every 2-connected claw-free graph containing a k-factor has a connected $[k, k+1]$-factor, where $k \geq 2$.

1 Introduction

All graphs under consideration are undirected, finite and simple. Let $G = 3D(V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set

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$E(G)$. We denote by xy the edge joining the vertices x and y. Let g and f be two mappings from $V(G)$ to Z^+ such that $g(v) \leq f(v) \leq d_G(v)$ for all $v \in V(G)$. If a spanning subgraph F of G meets the condition that $g(v) \leq d_F(v) \leq f(v)$ for all $v \in V(G)$, then F is called a $[g, f]$-factor of G. If $g(v) = 3Df(v)$ for all $v \in V(G)$ then we call the $[g, f]$-factor an f-factor. If $g(v) = 3Da$ and $f(v) = 3Db$ for all $v \in V(G)$, then the $[g, f]$-factor is called $[a, b]$-factor, where a and b are constants. When $a = 3Db = 3Dk$, the $[a, b]$-factor is called a k-factor.

A graph is said to be claw-free if it contains no copy of $K_{1,3}$ as an induced subgraph. The concept of connected factor was first proposed by M.Kano[1]. It is easy to see that the problem of deciding whether a given graph contains a connected k-factor is NP-hard in general, as a connected 2-factor is a Hamiltonian cycle. It seems to us that connected factor problem is an interesting research topic since it is closely related to the hamilton problem and information networks. The following recent results on f-factors or on connected $[g, f]$-factors have been published.

Theorem 1 *[2] Let k be a positive integer, and G a graph of order n with $n \geq 4k - 5$, kn even, and minimum degree at least k, then G has a k-factor if the degree sum of each pair of nonadjacent vertices of G is at least n.*

Theorem 2 *[3] Let $k \geq 3$ be an integer, and G a connected graph of oder n with $n \geq 4k - 3$, kn even, and minimum degree at least k. If for each pair of nonadjacent vertices u, v of $V(G)$*

$$max\{d_G(u), d_G(v)\} \geq \frac{n}{2}$$

then G has a k-factor.

Theorem 3 *[4] Let k be a positive integer and G a connected graph of order n. If G has a k-factor F and, moreover, among any three independent vertices of G there are(at least)two with degree sum at least $n - k$, then G has a matching M such that $M \cup F$ is a connected $[k, k + 1]$-fasctor of G.*

Theorem 4 *[5] Let G be a graph of order n and g, f be two mappings from $V(G)$ to $Z^+ - \{0, 1\}$ with $g(v) \leq f(v) \leq d_G(v)$ for all $v \in V(G)$. If G has an $[g, f]$-factor and a Hamiltonian path, then G has a connected $[g, f+1]$-factor.*

Theorem 5 *[6] If G is a 2-connected claw-free graph, then G has a connected $[2, 3]$-factor.*

We are going to show that every 2-connected claw-free graph containing a k-factor has a connected $[k, k+1]$-factor for an integer $k \geq 2$.

2 Preliminary

In this section and next section, it is assumed that G is 2-connected, and contains a k-factor, but contains no connected $[k, k+1]$-factor, where $k \geq 2$. In the following, we use F to denote a fixed k-factor of G with $k \geq 2$. Then we try to construct a connected $[k, k+1]$-factor based on the k-facor F and on the connectivity of G. To do so, we need to introduce some notations.

Let H and T be two disjoint vertex subsets (or subgraphs) of G. We denote by $N_H(T, G)$ the set of all vertices in H adjacent in G to a vertex in T. Specially, when $T = 3D\{v\}$ and $H = 3DG$, we use $N(v, G)$ instead of $N_G(\{v\}, G)$. For a subset M of $V(G)$, we use $G[M]$ to denote the subgraph of G induced by M. M is called a clique of G if $G[M]$ is complete. Let F be a factor of G. We denote by C_x the component of F containing x, and by $\mathcal{F}$ the set of all components of F, and $|\mathcal{F}|$ the cardinality of $\mathcal{F}$. A connected subgraph H of G is called $\mathcal{F}$-connected if $F[V(H)]$ consists of some components of F, that is, $V(H)$ is partitioned into $U_1 \cup U_2 \cup \cdots \cup U_j$ such that every $U_i, 1 \leq i \leq j$, is the vertex set of some component of F. A connected $[k, k+1]$-subgraph H of G is called $\mathcal{F}$-maximum if H is an $\mathcal{F}$-connected subgraph of maximum order among all $\mathcal{F}$-connected subgraphs. We denote by $\mathcal{H}$ the set of all $\mathcal{F}$-maximum connected $[k, k+1]$-subgraphs of G. For a path P of G, we always assume that the path P has a fixed orientation. Denote by $IN(P)$ the set of internal vertices of P. For a vertex x in $IN(P)$, we use x_P^-

and x_P^+ to denote the predecessor and successor of x on P according to the fixed orientation. An inner vertex x in P is said to be *singular* if none of x_P^- and x_P^+ is contained in the component C_x; otherwise, x is said to be *non-singular*. Put

$$F_P = 3DE(P) \cup (\cup_{x \in IN(P)} E(C_x))$$

For a subgraph A and subgraphs $B_i, i \in I$, of G, let $H + A$ and $H + \cup_{i \in I} B_i$ denote the subgraphs of G induced by the subsets $E(H) \cup E(A)$ and $E(H) \cup (\cup_{i \in I} B_i)$, respectively.

Let $H \in \mathcal{H}$, $R = 3DG - H$, and a, b be two distinct vertices in H. If $G[V(R) \cup \{a, b\}]$ has a path of length at least 3 connecting a and b with all internal vertices nonsingular then a and b are said to be related by R on H, denoted by aRb. For $H \in \mathcal{H}$, we define a relation set on H, denoted by $\mathcal{D}(H)$, as follows:

$$\mathcal{D}(H) = 3D\{(a, b) | a, b \in V(H) \quad \text{and} \quad aRb\}$$

Lemma 1 *$\mathcal{D}(H) \neq \emptyset$ for every $H \in \mathcal{H}$, where $\mathcal{H}$ is the set of all $\mathcal{F}$-maximum connected $[k, k+1]$-subgraphs of G.*

Proof Let $H \in \mathcal{H}$ be a $\mathcal{F}$-maximum connected $[k, k+1]$-subgraph of G. Since G has no connected $[k, k+1]$-factor, $R = 3DG - H$ is a nonempty subgraph such that $F[V(R)]$ consists of components of F. To show that $\mathcal{D}(H) \neq \emptyset$, it suffices to verify that there are vertices a and b in H such that $G[V(R) \cup \{a, b\}]$ contains a path connecting a and b of length at least 3 and its all internal vertices nonsingular.

By the connectivity of G, for any component C of $F[V(R)]$, there exist two vertex disjoint paths X and Y connecting C and H, where X has endvertices $u \in V(H)$ and $s \in V(C)$, and Y has endvertices $v \in V(H)$ and $t \in V(C)$. Let Z denote the path in C connecting s and t. Then the coalition of X, Y and Z, denoted by P, has length at least 3, all internal vertices in R, and at least two internally nonsingular vertices (say s and t). It follows that there exist paths of length at least 3 with endvertices in H, and all internal vertices in R , and at least two internally nonsingular vertices. We choose such a path P with the minimum number of singular vertices.

The remainder to be proved is that number of singular vertices on such a path P equals to zero.

Assume to the contrary: let v be a singular vertex in $IN(P)$. Then P is of length at least 4 since $IN(P)$ contains at least two nonsingular vertices and one singular vertex. By the choice of P, v^-v^+ is not an edge of G (since otherwise the path $P' = 3DP - \{vv^-, vv^+\} + \{v^-v^+\}$ is still of at least two internally nonsingular vertices because v^- (resp.v^+) being a nonsingular vertex of P implies that v^- (resp.v^+) is also a nonsingular vertex of P', but of less singular vertices than that of P, a contradiction). Let w be a vertex adjacent to v in C_v. Since G is claw-free, either v^-w or v^+w, say v^+w, is an edge of G. Then the path $P - \{vv^+\} + \{vw, wv^+\}$ is of at least two internally nonsingular vertices, but of singular vertices less one than P, a final contradiction. ∎

Lemma 2 *Let $H \in \mathcal{H}$, and $(a,b) \in \mathcal{D}(\mathcal{H})$. If a vertex x of H is adjacent to a vertex in $V(G) \setminus V(H)$ in G, then $d_H(x) = 3Dk+1$, in particular, $d_H(a) = 3Dd_H(b) = 3Dk+1$.*

Proof Suppose that $d_H(x) = 3Dk$ and $xw \in E(G)$, where $x \in V(H)$ and $w \in V(G)\setminus V(H)$. Then $H' = 3DH + \{xw\} + C_w$ is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph, contradicting the $\mathcal{F}$-maximality of H. ∎

3 Main Result and Proof

The main result we are going to prove in this section is as follows.

Theorem 6 *Let $k \geq 2$, and G be a 2-connected claw-free graph containing a k-factor. Then G has a connected $[k, k+1]$-factor.*

Proof Let $H^* \in \mathcal{H}$, $(a^*, b^*) \in \mathcal{D}(H^*)$. Then there is a path in $G[V(R^*) \cup \{a^*, b^*\}]$, connecting a^* and b^*, where $R^* = 3DG - H^*$, of length at least 3 and of all internal vertices nonsingular. Let $P^* = 3Da^*sw_1w_2 \cdots w_rtb^*$ be a path with minimum length among all such paths. It is easy to see that a^*b^* is not an edge of H^* (otherwise $H' = 3DH^* - \{a^*b^*\} + F_{P^*}$ is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G). Now we first observe the following fact.

Claim 1 *If* $N(t,G) \cap N(b^*,H^*) \neq \emptyset$, *then* $N(s,G) \cap N(a^*,H^*) = 3D\emptyset$. *Symmetrically, the inverse statement holds. In particular, we may assume that* $N(s,G) \cap N(a^*,H^*) = 3D\emptyset$ *holds without loss of generality.*

Proof Suppose that $N(s,G) \cap N(a^*,H^*) \neq \emptyset$ and $N(t,G) \cap N(b^*,H^*) \neq \emptyset$. It suffices to derive a contradiction under this assumption. Let $x \in N(s,G) \cap N(a^*,H^*)$. If there exists a vertex $y \in N_G(x) \cap N_F(s)$, then we obtain a contradiction by considering $H' = 3DH^* - \{xa^*\} + F_P$, where $P = 3Da^*syx$. Hence $N_G(x) \cap N_F(s) = 3D\emptyset$. Since $sx \in E(G)$ and G is claw-free, $G[N_F(s)]$ is a clique. If two vertices $u, v \in N_F(s)$ are not adjacent in F, then we obtain a contradiction by considering $H' = 3DH^* + \{uv, sa^*, sx\} + C_s - \{a^*x, su\}$. Thus $F[N_F(s)]$ is also a clique of F. Note that C_s is a k-regular subgraph of F. It follows that $F[N_F(s) \cup \{s\}] = 3DC_s$. Because of the minimality of the length of P^*, the edge w_1w_2 of P^* is not an edge of F. Similar statements hold for t and w_r.

Choose $y \in N(t,G) \cap N(b^*,H^*)$. Then $H' = 3DH^* + F_{P^*} + \{sx, ty\} - \{xa^*, yb^*, sw_1, tw_r\} - \{w_iw_{i+1} \mid C_{w_i} = 3DC_{w_{i+1}}, w_iw_{i+1} \notin E(C_{w_i}), i = 3D1, 2, \cdots, r-1\}$ is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G, a contradiction. The claim 1 is proved. ∎

By Claim 1, we may assume that $N(s,G) \cap N(a^*,H^*) = 3D\emptyset$. Since G is claw-free, $N(a^*,H^*)$ must be a clique of G. Let $v_1, w_1 \in N(a^*,H^*)$. Put

$$V_1' = 3D\{a^*\} \cup N(a^*,H^*)$$

Then we note the following fact.

Remark 1 $N_{V_1'}(t,G) = 3D\emptyset$.

Proof Assume the contrary: there is a vertex $w \in N_{V_1'}(t,G)$. Then $H' = 3DH^* + F_{P'} - \{wa^*\}$, where $P' = 3DP^* - \{tb^*\} + \{tw\}$, is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph, a contradiction. ∎

Let

$$V_1 = 3DV_1' \cup \{b^*\}, \quad \bar{V}_1 = 3DV(H^*) - V_1$$

$$A_1 = 3D\{v_1w_1\}, \quad B_1 = 3D\{a^*w_1\}$$

Claim 2 $d_{H^*}(v_1) = 3Dk + 1$ *and* $|N_{\bar{V}_1}(v_1, H^*)| = 3Dk$.

Proof We first show that $d_{H^*}(v_1) = 3Dk+1$. Assume to the contrary that $d_{H^*}(v_1) = 3Dk$. By Lemma 2 that $d_{H^*}(a^*) = 3Dk+1$, there is another vertex y adjacent to a^*, but not to v_1 in H^*. Note that yv_1 is an edge of G. Then $H' = 3DH^* + \{v_1y\} - \{a^*y\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, contradicting with Lemma 2.

Now we are going to show that $|N_{\bar{V}_1}(v_1, H^*)| = 3Dk$. If $N(v_1, H^*) \cap N(a^*, H^*) \neq \emptyset$, then $H' = 3DH^* - \{a^*v_1\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, this contradiction implies that $N(v_1, H^*) \cap N(a^*, H^*) = 3D\emptyset$. If v_1b^* is an edge of H^*, then $H' = 3DH^* + F_{P^*} + \{v_1w_1\} - \{a^*w_1, b^*v_1\} - \{w_iw_{i+1} \mid C_{w_i} = 3DC_{w_{i+1}}, w_iw_{i+1} \notin E(C_{w_i}), i = 3D1, 2, \cdots, r-1\}$ is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G. This contradiction implies that v_1b^* is not an edge of H^*. Therefore, $N_{V_1}(v_1, H^*) = 3D\{a^*\}$. Recalling that $d_{H^*}(v_1) = 3Dk+1 =$, we get $|N_{\bar{V}_1}(v_1, H^*)| = 3Dk$. The claim 2 is proved. ∎

Let $H_1 = 3DH^* + A_1 - B_1$. By Claim 2, H_1 is a $\mathcal{F}$-connected subgraph of G such that

$$d_{H_1}(x) = 3D \begin{cases} k, & x = 3Da^* \\ k+2, & x = 3Dv_1 \\ d_{H^*}(x), & x \in V(H^*) - \{a^*, v_1\} \end{cases}$$

Since G is claw-free and $k \geq 2$, Claim 2 allows us to choose two vertices $v_2, w_2 \in N_{\bar{V}_1}(v_1, H^*)$ such that $d_{H^*}(v_2)$ is as large as possible and v_2 is adjacent either to a^* or to w_2 in G.

In fact, if v_2 is chosen such that $d_{H^*}(v_2)$ is as large as possible, but v_2 is not adjacent to both of a^* and w_2 in G, then a^*w_2 must be an edge of G because G is claw-free. If $d_{H^*}(w_2) = 3Dk$, then $H' = 3DH_1 + \{w_2a^*\} - \{a^*v_1\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, this contradiction gurantees that $d_{H^*}(w_2) = 3Dk+1$. Thus we should choose w_2 instead of v_2. Put

$$V_2' = 3DV_1' \cup N_{\bar{V}_1}(v_1, H^*).$$

Then we observe the following fact.

Remark 2 *1.* $N_{V_2'}(t, G) = 3D\emptyset$.

2. If $d_{H^*}(v) = 3Dk$ *for* $v \in N_{\bar{V}_1}(v_1, H^*)$, *then* $va^* \notin E(G)$.

Proof Suppose that there is a vertex $w \in N_{V_2'}(t, G)$. By remark 1, $w \in N_{\bar{V}_1}(v_1)$. Then we have a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph $H' = 3DH_1 + F_{P'} - \{v_1 w\}$, where $P' = 3DP^* - \{tb^*\} + \{tw\}$. This contradiction implies that $N_{V_2'}(t, G) = 3D\emptyset$.

Suppose that there is a vertex $v \in N_{\bar{V}_1}(v_1, H^*)$ such that $d_{H^*}(v) = 3Dk$ and $va^* \in E(G)$. Then we obtain a $\mathcal{F}$-connected $[k, k+1]$-subgraph $H' = 3DH_1 + \{va^*\} - \{a^* v_1\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, contradicting with Lemma 2. ∎

Then we rechoose $b^* \in N_{H^*}(t, G)$ which is not adjacent to v_2 in $G =$ if possible, i.e., if $N_{H^*}(t, G) \subseteq N_{H^*}(v_2, G)$ then any vertex in $N_{H^*}(t, G)$ can be chosen as b^*; otherwise, we choose $b^* \in N_{H^*}(t, G) \setminus N_{H^*}(v_2, G)$. Put

$$V_2 = 3DV_2' \cup \{b^*\}, \quad \bar{V}_2 = 3DV(H^*) - V_2,$$

$$A_2 = 3D \begin{cases} A_1 \cup \{v_0 v_2\} & v_0 v_2 \in E(G) \\ A_1 \cup \{v_2 w_2\} & v_0 v_2 \notin E(G) \end{cases}$$

$$B_2 = 3D \begin{cases} B_1 \cup \{v_0 v_1\} & v_0 v_2 \in E(G) \\ B_1 \cup \{v_1 w_2\} & v_0 v_2 \notin E(G) \end{cases}$$

where $v_0 = 3Da^*$. Then we observe the following fact.

Claim 3 $d_{H^*}(v_2) = 3Dk + 1$ *and* $|N_{\bar{V}_2}(v_2, H^*)| = 3Dk$.

Proof Assume to the contrary that $d_{H^*}(v_2) = 3Dk$. By the choice of v_2, we have $d_{H^*}(v) = 3Dk$ for all $v \in N_{\bar{V}_1}(v_1, H^*)$. Remark 2 ensures that va^* is not an edge of G for all $v \in N_{\bar{V}_1}(v_1, H^*)$. Because G is claw-free, we see that $N_{\bar{V}_1}(v_1, H^*)$ is a clique of G. Let x and y be two vertices of $N_{\bar{V}_1}(v_1, H^*)$. If xy is not an edge of H^*, then $H' = 3DH_1 + \{xy\} - \{v_1 x\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, this contradiction implies that $N_{\bar{V}_1}(v_1, H^*)$ is also a clique of H_1. Note that $d_{H^*}(v) = 3Dk$ for all $v \in N_{\bar{V}_1}(v_1, H^*)$ and $|N_{\bar{V}_1}(v_1, H^*)| = 3Dk$. We see that v_1 is a cut vertex of H_1 (or H^*). By the connectivity of G, G contains a path P connecting $u \in N_{\bar{V}_1}(v_1, H^*)$ and $v \in V(H^*) - N_{\bar{V}_1}(v_1, H^*) \cup \{v_1\}$ with all internal vertices not in H^*. By Lemma 2, P is of length exact 1, that is, P is just an edge uv. Furthermore, we have $d_{H_1}(v) = 3Dk + 1$, since otherwise $H' = 3DH_1 + \{uv\} - \{uv_1\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, a contradiction.

On the other hand, since $H_1 - \{v_1u\}$ is connected, $H_1 - \{v_1u\}$ contains a path from v to u, say $P' = 3Dvv^+ \cdots u^- u$. If $d_{H_1}(v^+) = 3Dk+1$ then $H' = 3DH_1 - \{vv^+, v_1u\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, this contradiction implies that $d_{H_1}(v^+) = 3Dk$. Now that $d_{H_1}(v) = 3Dk+1$ and $d_{H_1}(v^+) = 3Dk$, there exists a vertex w which is adjacent to v, but not to v^+ in H_1. Now we consider the four vertices u, v, w and v^+. Since v^+ has degree k in H_1, we can guarantee that uv^+ is not an edge of G by the same reason as we prove that v must be of degree $k+1$. Because $G[\{u, v, w, v^+\}] \neq K_{1,3}$, one of wu or wv^+ is an edge of G. But then we can get a contradiction by considering $H' = 3DH_1 + \{wu\} - \{uv_1, vw\}$, if wu is an edge of G; or $H' = 3DH_1 + \{wv^+\} - \{uv_1, vw\}$, if wv^+ is an edge of G.

Now we start to show that $|N_{\bar{V}_2}(v_2, H^*)| = 3Dk$. We first observe that v_2 is not adjacent to any vertex in $V_2 \setminus \{v_1, b^*\}$ in H^* since otherwise $H' = 3DH_1 - \{v_1v_2\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, contradicting with Lemma 2. Since $d_{H_1}(v_2) = 3Dk+1$, we only need to show that v_2b^* is not an edge of H^*. Assume to the contrary that v_2b^* is an edge of H^*. Since $d_{H^*}(b^*) \geq k \geq 2$, there is a vertex $y (\neq v_2)$ which is adjacent to b^* in H^*. It is easy to see that $y \notin V_2$ since otherwise we can get a contradiction by considering $H' = 3DH_1 - \{v_1v_2\}$. Now we consider the four vertices b^*, y, t and v_2. If tv_2 is an edge of G, then $H' = 3DH_1 + F_{P'} - \{v_1v_2\}$, where $P' = 3DP^* - \{tb^*\} + \{tv_2\}$, is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G, a contradiction. If yv_2 is an edge of G, then $H' = 3DH_1 + F_{P^*} + \{yv_2\} - \{yb^*, v_1v_2\}$ is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G, a contradiction. Hence both of v_2y and v_2t are not edges of G. Since $G[\{b^*, y, t, v_2\}] \neq K_{1,3}$, ty must be an edge of G, contradicting the choice of b^*. Claim 3 is proved. ∎

By Claim 3, $H_2 = 3DH^* + A_2 - B_2$ is a $\mathcal{F}$-connected subgraph of G such that

$$d_{H_2}(x) = 3D \begin{cases} k, & x = 3Da^* \\ k+2, & x = 3Dv_2 \\ d_{H^*}(x), & x \in V(H^*) - \{a^*, v_2\} \end{cases}$$

and we can choose two vertices $v_3, w_3 \in N_{\bar{V}_2}(v_2, H^*)$ such that $d_{H^*}(v_3)$ is as large as possible and v_3 is adjacent either to v_1 or to w_3 in G.

In general, let $n \geq 1$ be an integer. Suppose that we have got $V_1', V_2', \cdots, V_n'$ such that

$$V_i' = 3DV_{i-1}' \cup N_{\bar{V}_{i-1}}(v_{i-1}, H^*), i = 3D1, 2, \cdots, n$$

and

$$V_i = 3DV_i' \cup \{b^*\}, \bar{V}_i = 3DV(H^*) - V_i, i = 3D1, 2, \cdots, n$$

such that for $i = 3D1, 2, \cdots, n$

1. $N_{V_i'}(t, G) = 3D\emptyset$;
2. If $d_{H^*}(v) = 3Dk$ for $v \in N_{\bar{V}_i}(v_i, H^*)$, then $vv_{i-1} \notin E(G)$;
3. $d_{H^*}(v_i) = 3Dk + 1$;
4. $|N_{\bar{V}_i}(v_i, H^*)| = 3Dk$.

and have got for $i = 3D1, 2, \cdots, n$

$$A_i = 3D \begin{cases} A_{i-1} \cup \{v_{i-2}v_i\} & v_{i-2}v_i \in E(G) \\ A_{i-1} \cup \{v_i w_i\} & v_{i-2}v_i \notin E(G) \end{cases}$$

$$B_i = 3D \begin{cases} B_{i-1} \cup \{v_{i-2}v_{i-1}\} & v_{i-2}v_i \in E(G) \\ B_{i-1} \cup \{v_{i-1}w_i\} & v_{i-2}v_i \notin E(G) \end{cases}$$

where $v_0 = 3Da^*$, and $H_i = 3DH^* + A_i - B_i$ is a $\mathcal{F}$-connected subgraph such that

$$d_{H_i}(x) = 3D \begin{cases} k, & x = 3Da^* \\ k+2, & x = 3Dv_i \\ d_{H^*}(x), & x \in V(H^*) - \{a^*, v_i\} \end{cases}$$

where $v_i \in N_{\bar{V}_{i-1}}(v_{i-1}), i = 3D2, \cdots, n$ has degree as large as possible, and one of $v_i v_{i-2}$ or $v_i w_i$ is an of G.

Now that $|N_{\bar{V}_n}(v_n, H^*)| = 3Dk \geq 2$, we can choose two distinct vertices $v_{n+1}, w_{n+1} \in N_{\bar{V}_n}(v_n, H^*)$ such that $d_{H^*}(v_{n+1})$ is as large as possible, and one of $v_{n+1}v_{n-1}$ or $v_{n+1}w_{n+1}$ is an edge of G. Put

$$V_{n+1}' = 3DV_n' \cup N_{\bar{V}_n}(v_n, H^*).$$

Then the following fact still remains.

Remark 3 *1.* $N_{V'_{n+1}}(t,G) = 3D\emptyset$.

2. If $d_{H^*}(v) = 3Dk$ *for* $v \in N_{\bar{V}_n}(v_n, H^*)$*, then* $vv_{n-1} \notin E(G)$.

Proof The proof of this remark is the same as that of Remark 2. So we omit it here. ∎

Then we choose $b^* \in N_{H^*}(t,G)$ which is not adjacent to v_{n+1} in G if possible, i.e., if $N_{H^*}(t,G) \subseteq N_{H^*}(v_{n+1},G)$ then any vertex in $N_{H^*}(t,G)$ can be chosen as b^*; otherwise, we choose $b^* \in N_{H^*}(t,G) \setminus N_{H^*}(v_{n+1},G)$. Put

$$V_{n+1} = 3DV'_{n+1} \cup \{b^*\}, \quad \bar{V}_{n+1} = 3DV(H^*) - V_{n+1},$$

$$A_{n+1} = 3D \begin{cases} A_n \cup \{v_{n-1}v_{n+1}\} & v_{n-1}v_{n+1} \in E(G) \\ A_n \cup \{v_{n+1}w_{n+1}\} & v_{n-1}v_{n+1} \notin E(G) \end{cases}$$

$$B_{n+1} = 3D \begin{cases} B_n \cup \{v_{n-1}v_n\} & v_{n-1}v_{n+1} \in E(G) \\ B_n \cup \{v_n w_{n+1}\} & v_{n-1}v_{n+1} \notin E(G) \end{cases}$$

Then we still have the following fact.

Claim 4 $d_{H^*}(v_{n+1}) = 3Dk+1$ *and* $|N_{\bar{V}_{n+1}}(v_{n+1}, H^*)| = 3Dk$.

Proof The proof of this claim is the same as that of Claim 3. For the completeness, we duplicate it here.

Assume to the contrary that $d_{H^*}(v_{n+1}) = 3Dk$. By the choice of v_{n+1}, we have $d_{H^*}(v) = 3Dk$ for all $v \in N_{\bar{V}_n}(v_n, H^*)$. Remark 3 ensures that vv_{n-1} is not an edge of G for all $v \in N_{\bar{V}_n}(v_n, H^*)$. Because G is claw-free, we see that $N_{\bar{V}_n}(v_n, H^*)$ is a clique of G. Let x and y be two vertices of $N_{\bar{V}_n}(v_n, H^*)$. If xy is not an edge of H^*, then $H' = 3DH_n + \{xy\} - \{v_n x\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, contradicting with Lemma 2. Hence $N_{\bar{V}_n}(v_n, H^*)$ is also a clique of H_n. Note that $d_{H^*}(v) = 3Dk$ for all $v \in N_{\bar{V}_n}(v_n, H^*)$ and $|N_{\bar{V}_n}(v_n, H^*)| = 3Dk$. We see that v_n is a cut vertex of H_n (or H^*). By the connectivity of G, G contains a path P connecting $u \in N_{\bar{V}_n}(v_n, H^*)$ and $v \in V(H^*) - N_{\bar{V}_n}(v_n, H^*) \cup \{v_n\}$ with all internal vertices not in H^*.

By Lemma 2, P is of length exact 1, that is, P is just an edge uv. Furthermore, we have that $d_{H_n}(v) = 3Dk + 1$, since otherwise

$H' = 3DH_n + \{uv\} - \{uv_n\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, contradicting with Lemma 2.

On the other hand, since $H_n - \{v_n u\}$ is connected, $H_n - \{v_n u\}$ contains a path from v to u, say $P' = 3Dvv^+ \cdots u^- u$. If $d_{H_n}(v^+) = 3Dk + 1$ then $H' = 3DH_n - \{vv^+, v_n u\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$, this contradiction implies that $d_{H_n}(v^+) = 3Dk$. Now that $d_{H_n}(v) = 3Dk+1$ and $d_{H_n}(v^+) = 3Dk$, there exists a vertex w which is adjacent to v, but not to v^+ in H_n. Now we consider the four vertices u, v, w and v^+. Since v^+ has degree k in H_n, we can guarantee that uv^+ is not an edge of G by the same reason as we prove that v must be of degree $k+1$. Because $G[\{u, v, w, v^+\}] \neq K_{1,3}$, one of wu or wv^+ is an edge of G. But then we can get a contradiction by considering $H' = 3DH_n + \{wu\} - \{uv_n, vw\}$, if wu is an edge of G; or $H' = 3DH_n + \{wv^+\} - \{uv_n, vw\}$, if wv^+ is an edge of G.

Now we start to show that $|N_{\bar{V}_{n+1}}(v_{n+1}, H^*)| = 3Dk$. We first observe that v_{n+1} is not adjacent to any vertex in $V_{n+1} \setminus \{v_n, b^*\}$ in H^* since otherwise we can get a contradiction by considering $H' = 3DH_n - \{v_n v_{n+1}\} \in \mathcal{H}$ with $d_{H'}(a^*) = 3Dk$. Since $d_{H_n}(v_{n+1}) = 3Dk + 1$, we only need to show that $v_{n+1}b^*$ is not an edge of H^*. Assume to the contrary that $v_{n+1}b^*$ is an edge of H^*. Since $d_{H^*}(b^*) \geq k \geq 2$, there is a vertex $y (\neq v_{n+1})$ which is adjacent to b^* in H^*. It is easy to see that $y \notin V_{n+1}$ since otherwise we can get a contradiction by considering $H' = 3DH_n - \{v_n v_{n+1}\}$. Now we consider the four vertices b^*, y, t and v_{n+1}. If tv_{n+1} is an edge of G, then $H' = 3DH_n + F_{P'} - \{v_n v_{n+1}\}$, where $P' = 3DP^* - \{tb^*\} + \{tv_{n+1}\}$, is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G, a contradiction. If yv_{n+1} is an edge of G, then $H' = 3DH_n + F_{P^*} + \{yv_{n+1}\} - \{yb^*, v_n v_{n+1}\}$ is a larger $\mathcal{F}$-connected $[k, k+1]$-subgraph of G, a contradiction. Hence both of $v_{n+1}y$ and $v_{n+1}t$ are not edges of G. Since $G[\{b^*, y, t, v_{n+1}\}] \neq K_{1,3}$, ty must be an edge of G, contradicting the choice of b^*. Claim 4 is proved. ∎

By Claim 4, $H_{n+1} = 3DH^* + A_{n+1} - B_{n+1}$ is a $\mathcal{F}$-connected subgraph of G such that

$$d_{H_{n+1}}(x) = 3D \begin{cases} k, & x = 3Da^* \\ k+2, & x = 3Dv_{n+1} \\ d_{H^*}(x), & x \in V(H^*) - \{a^*, v_{n+1}\} \end{cases}$$

and we can choose two vertices $v_{n+2}, w_{n+2} \in N_{\bar{V}_{n+1}}(v_{n+1}, H^*)$ such that $d_{H^*}(v_{n+2})$ is as large as possible and v_{n+2} is adjacent either to v_n or to w_{n+2} in G.

Repeating the above procedure infinitely, we can get a sequence $\{V_n\}_n^\infty$ such that $|V_n| = 3Dnk + 3$ for all $n \geq 1$. This contradicts the finiteness of G. Proof of the theorem is completed. ∎

As a remark, we can show that every connected claw-free graph containing a k-factor has a connected $[k, k+2]$-factor by utilizing the technique for proving Theorem 6. This result has been proved by B. G. Xu and Z. H. Liu [7].

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