Edge-added Eccentricities of Vertices in a Graph

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Abstract

The eccentricity of a vertex v in a connected graph G is the distance between v and a vertex farthest from v. For a vertex v, we define the edge-added eccentricity of v as the minimum eccentricity of v in all graphs $G + e$, taken over all edges e in the complement of G. A graph is said to be edge-added stable (or just stable) if the eccentricity and the edge-added eccentricity are the same for all vertices in the graph. This paper describes properties of edge-added eccentricities and edge-added stable graphs.

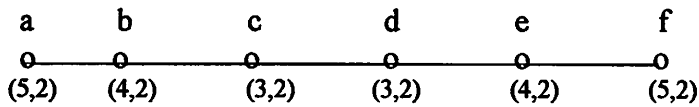
Introduction

There have been recent articles in the popular media about the so-called small world phenomenon [1]. This phenomenon is the observed fact that often, in a large group of people with any binary relationship among the people, the longest chain of relations needed to connect any two individuals is surprisingly short. Relationships mentioned in the articles include acquaintanceship, having appeared in the same motion picture or having co-authored a paper with one another. The phenomenon is partially explained by the observation that a single individual has the ability to "join" disparate groups of individuals by being acquainted with individuals in both groups. From a graph theoretic perspective, this suggests that it may be of interest to see how the addition of a single edge to a graph effects vertex eccentricities. While the foregoing may serve as a motivation, the investigation in this paper does not deal with the large random graphs which seem most suitable to model the situation. Instead, we follow the development suggested by [4].

The *distance*, $d_G(u, v)$, between two vertices u and v in a connected graph G is the length of the shortest path from u to v. The *eccentricity* $e_G(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is called the *radius* of G and denoted rad(G), while the maximum eccentricity is called the *diameter* and denoted

$$h(v) = \min\{e_{G+e}(v) \mid e \text{ is an edge in the complement of G}\}.$$

Clearly $h(v) \le e_G(v)$ for every vertex v. A graph is said to be *edge-added stable* (or just *stable*) if the eccentricity and the edge-added eccentricity are the same for all vertices in the graph. A vertex v is a *central* vertex if $e_G(v) = rad(G)$ and a *peripheral* vertex if $e_G(v) = diam(G)$. The *center* of G, C(G), is the subgraph induced by the central vertices of G and the *periphery* of G, P(G), is the subgraph induced by the peripheral vertices of G.



$(e_G(v)\ ,\ h(v)\ )$

Figure 1

Theorem 1 For every vertex v in a connected graph G, $h(v) \le rad(G) + 1$.

Proof A central vertex, w, is within a distance of rad(G) of any other vertex in the graph. Adding the edge vw to the graph G, if necessary, then puts v within rad(G) +1 of any other vertex in the graph.

It would seem that computing h(v) for a given vertex v would be computationally cumbersome because one would have to compute the eccentricity of v in the large number of graphs derived from G by adding any single edge in the complement. The following shows that h(v) can be computed by considering only the eccentricity of v in graphs derived from G by adding an edge from v to another vertex.

Lemma 1 Let v and w be vertices and e an edge in the complement of a connected graph G. Suppose that $d_{G+e}(v,w) < d_G(v,w)$. Then a shortest path from v to w in the graph $G+e$ must contain the edge e. Furthermore, there exists an edge f in the complement of G such that $d_{G+f}(v,w) \le d_{G+e}(v,w)$ and f is of the form f = vx for some vertex x.

Proof If a shortest path from v to w in G + e does not contain e then this shortest path exists in G as well and so $d_G(v,w) = d_{G+e}(v,w)$. Now assume that $d_{G+e}(v,w) < d_G(v,w)$ and that e =xy. As noted, a shortest path from v to w must contain e. We conclude, without loss of generality , that y is closer to v than is x, and that $d_G(w,x) < d_G(w,y)$. Let f = vx. The edge f cannot be an edge of G since otherwise the inequality $d_{G+e}(v,w) < d_G(v,w)$ could not hold. The conclusion follows.

Lemma 2 Let e be an edge in the complement of a connected graph G such that $e_{G+e}(v) < e_G(v)$. Then there exists an edge f in the complement of G such that $e_{G+f}(v) \le e_{G+e}(v)$ and f is of the form f = vx for some vertex x.

Proof Let e = xy be an edge in the complement of G such that $e_{G+e}(v) < e_G(v)$. By Lemma 1, e is on a shortest path from v to any vertex w with the property that $d_{G+e}(v,w) < d_G(v,w)$. By our assumption that eccentricity decreases, we must have at least one vertex w which is closer to v in G+ e than in G. Following the proof of Lemma 1 we conclude, without loss of generality, that y is closer to v than is x. Let f = vx. It follows that $d_{G+f}(v,w) \le d_{G+e}(v,w)$ for all vertices w which are closer to v in G+ e than in G. The conclusion follows.

Theorem 2 For every vertex v in a connected graph G there exist an edge f in the complement of G of the form f = vx such that $h(v) = e_{G+f}(v)$.

Proof Let e be the edge in the complement of G which minimizes $e_{G+e}(v)$. We then apply Lemma 2, if necessary, and the result follows.

We call an edge f in the complement of G an *optimal edge for v in G* if $e_{G+f}(v) = h(v)$. In these terms, Theorem 2 says that each vertex v has an optimal edge of the form vx for some vertex x.

Theorem 3 Let x and y be adjacent vertices in the graph G. Then

$$|h(x) - h(y)| = 0 \text{ or } 1.$$

Proof Assume that $h(x) \le h(y)$. Let xw be an optimal edge for x in the graph G. Because x and y are adjacent in the graph G + xw

$$h(y) \le e_{G+xw}(y) \le e_{G+xw}(x) + 1 = h(x) + 1$$

The result then follows.

For a vertex v, let $A_G(v)$ denote the set of vertices which are farthest from v in G.

We call these the antipodal vertices of v. Whether $h(v) < e_G(v)$ or not depends on whether there is another vertex close to every vertex in the antipodal set of v.

Theorem 4 Let v be a vertex in a connected graph G. If there exist a vertex w such that

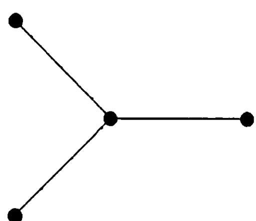
$$d(w,x) \leq e_G(v) - 2 \text{ for all } x \in A_G(v)$$

then $h(v) < e_G(v)$. Otherwise $h(v) = e_G(v)$.

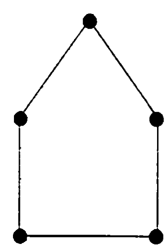
Proof If there exist a vertex w such that $d(w,x) \leq e_G(v) - 2$ for all $x \in A_G(v)$ then $e_{G+vw}(v) < e_G(v)$ and so $h(v) < e_G(v)$. Otherwise $e_{G+vw}(v) \geq e_G(v) - 1$ for all vertices w and we conclude by Theorem 2 that $h(v) = e_G(v)$.

Stable Graphs

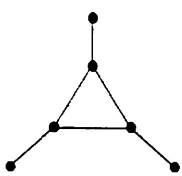
Stable graphs are those where $h(v) = e_G(v)$ for all vertices v. Some examples of stable graphs are given below.



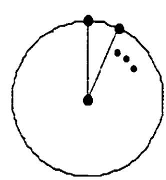
rad = 1
diam = 2



rad = 2
diam = 2



rad = 2
diam = 3



n-spoked wheel

rad = 1
diam = 2

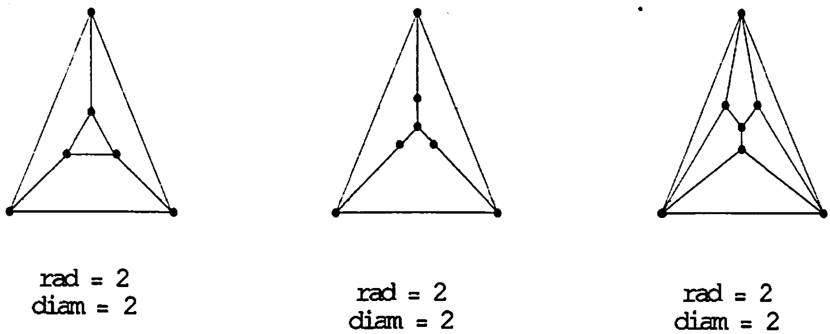


Figure 2

Other stable graphs include the complete bipartite graph $K_{m,n}$ for $m,n \geq 3$; the graph obtained from $2K_n$ by adding a perfect matching between vertices from the two copies of K_n $(n \geq 3)$; and the graph obtained by adding a perfect matching between the vertices of K_n and the leaves of the star $K_{1,n}$ for $n \geq 3$.

Theorem 5 For all stable graphs G,

$$rad(G) \leq diam(G) \leq rad(G) + 1.$$

Proof From the definition of a stable graph we have $h(v) = e_G(v)$ for all vertices v. It then follows from Theorem 1 that $e_G(v) \leq rad(G) + 1$. By taking v to be a vertex of maximum eccentricity we get
$diam(G) \leq rad(G) + 1$. The result now follows.

In view of Theorem 5, it is natural to ask whether there are stable graphs of every possible radius and diameter. The following construction answers this question in the affirmative.
For $n \geq 1$,

Let Γ_n be the graph whose vertices and edges are the following:
vertices: all n-tuples consisting of entries from the set {0,1,2}
edges: two n-tuples are adjacent in Γ_n if and only if they differ in exactly one coordinate.

Let Γ'_n be the graph whose vertices and edges are the following:
vertices: all (n+1)-tuples with first entry either the symbol "a" or "b",

and other entries chosen from the set {0,1,2}

edges: two (n+1)-tuples with first coordinate "a" are adjacent in Γ_n if and only if they differ in exactly one coordinate. An (n+1)- tuple with first coordinate "b" is adjacent only to the (n+1)-tuple whose first coordinate is "a" but otherwise matches the given (n+1)-tuple in the last n positions.

Note that Γ_1 is a triangle and Γ'_1 is the graph appearing in row 2 and column 1 of Figure 2.

Theorem 6 Γ_n and Γ'_n are stable graphs. Furthermore

$$rad(\Gamma_n) = diam(\Gamma_n) = n$$
$$rad(\Gamma'_n) = n+1, \quad diam(\Gamma'_n) = n+2$$

Proof Let $v = (v_1, v_2, \ldots, v_n)$ be an n-tuple in Γ_n. The distance in Γ_n between v and another n-tuple x is equal to the number of positions in which v and x differ. Thus every vertex in Γ_n has eccentricity n:

$$e_{\Gamma_n}(v) = n \text{ for all } v \in \Gamma_n.$$

Similarly, in Γ'_n we have $e_{\Gamma'_n}(\ (b, v_1, v_2, \ldots, v_n)\ ) = n+2$ and $e_{\Gamma'_n}(\ (a, v_1, v_2, \ldots, v_n)\ ) = n+1$. The statements concerning radius and diameter then follow.

For each vertex $v = (v_1, v_2, \ldots, v_n)$ in Γ_n there are 2^n antipodal vertices in $A_{\Gamma_n}(v)$ since each coordinate has three possible values. For a fixed vertex $v \in \Gamma_n$, we consider the distance between a vertex $w = (w_1, w_2, \ldots, w_n)$ and each vertex in $A_{\Gamma_n}(v)$. Let $x = (x_1, x_2, \ldots, x_n)$ be defined by the rule

$$x_i \neq v_i, w_i \text{ and } x_i \in \{0,1,2\}; \quad \text{for } 1 \leq i \leq n$$

where an arbitrary choice is made if necessary. Then $d(x,v) = n$ so that $x \in A_{\Gamma_n}(v)$ and $d(x,w) = n$. Thus x is an antipodal vertex of v a distance n from w. Since w was chosen arbitrarily we conclude by Theorem 4 that $h(v) = e_G(v)$ for all vertices v in Γ_n. Thus Γ_n is stable. The proof that Γ'_n is stable proceeds similarly.

The Edge-Added Center and Periphery

The vertices of a connected graph G with minimum edge-added eccentricity are called *edge-added central* vertices. The subgraph induced by these vertices is called the *edge-added center* of the graph G and is denoted EAC(G). The vertices of G with maximum edge-added eccentricity are *called edge-added peripheral* vertices. The subgraph induced by these vertices is called the *edge-added periphery* of the graph G and is denoted EAP(G). The following figure shows a graph G with indicated vertex eccentricities and edge-added eccentricities along with the center, periphery, edge-added center and edge-added periphery.

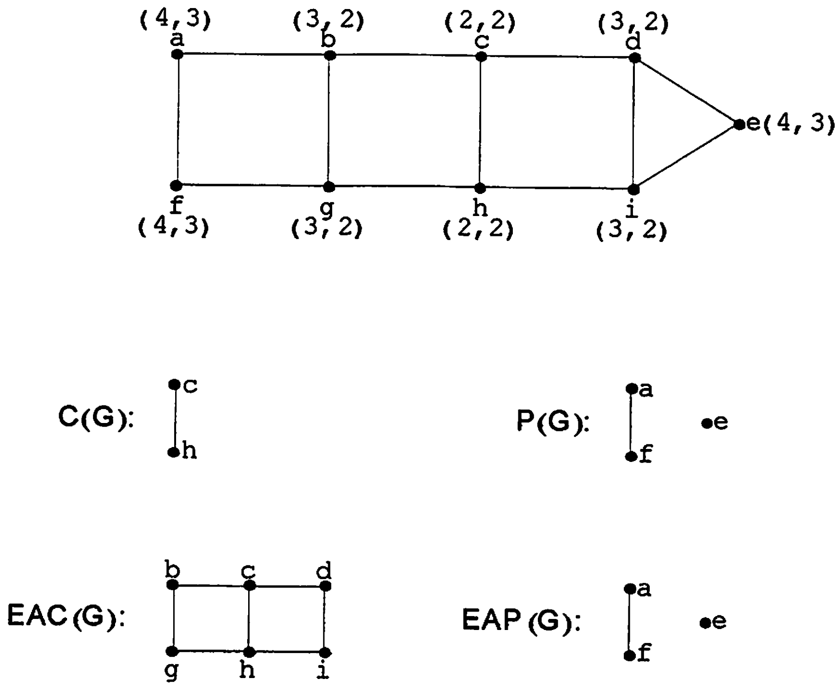


Figure 3

Hedetniemi [2] showed that every graph is the center of some connected graph; that is for every graph G, there exists a connected graph H such that $C(H) \cong G$. The following construction shows that any graph can be an edge-added center.

Theorem 7 Let G be any graph. Then there exists a connected graph H such that $EAC(H) \cong G$.

Proof We construct the graph H from the graph G by adding six vertices $v_1, v_2, v_3, v_4, v_5, v_6$, along with all possible edges from vertices in G to v_1, v_2, v_3. In addition we add the edges v_1v_4, v_2v_5, v_3v_6.

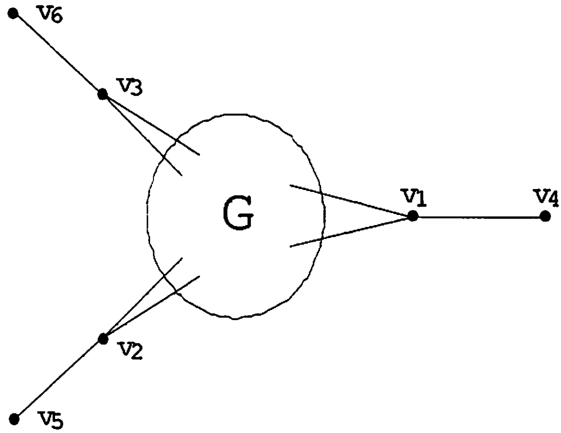


Figure 4

Then $h(v_1) = h(v_2) = h(v_3) = h(v_4) = h(v_5) = h(v_6) = 3$, while $h(v) = 2$ for all $v \in G$. Therefore $EAC(H) \cong G$.

Theorem 8 Let F and G be graphs and let n be a positive integer. Then there exists a connected graph H such that $EAC(H) \cong G$, $C(H) \cong F$ and $d(EAC(H), C(H)) = n$.

Proof We construct a graph H by defining

$$\begin{aligned} V(H) = V(G) \cup V(F) &\cup \{ x_i \mid 1 \le i \le n+1 \} \\ &\cup \{ y_i \mid 1 \le i \le n+1 \} \\ &\cup \{ z_i \mid 1 \le i \le n-1 \} \\ &\cup \{ z_i \mid n+1 \le i \le 3n+1 \} \end{aligned}$$

$$
\begin{aligned}
E(H) = E(G) \cup E(F) &\cup \{ x_i x_{i+1} \mid 1 \le i \le n \} \\
&\cup \{ y_i y_{i+1} \mid 1 \le i \le n \} \\
&\cup \{ z_i z_{i+1} \mid 1 \le i \le n-2 \} \\
&\cup \{ z_i z_{i+1} \mid n+1 \le i \le 3n \} \\
&\cup \{ x_1 v \mid v \in G \} \cup \{ y_1 v \mid v \in G \} \cup \{ z_1 v \mid v \in G \} \\
&\cup \{ z_{n-1} v \mid v \in F \} \cup \{ z_{n+1} v \mid v \in F \}
\end{aligned}
$$

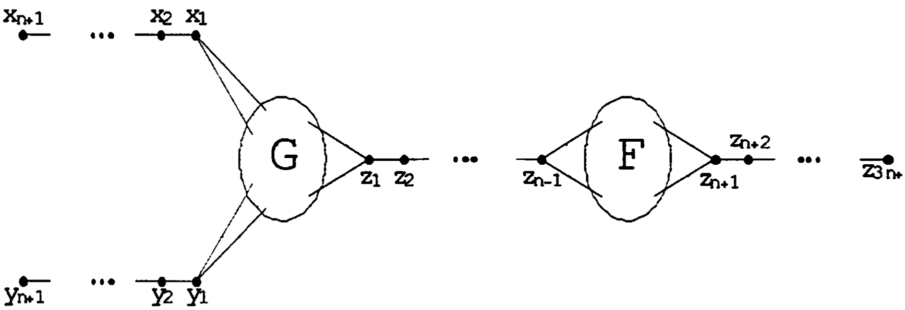


Figure 5

From the construction of the graph H, we have the following properties:

$e_H(v) = 2n+1$ for $v \in V(F)$

$e_H(v) > 2n+1$ for $v \in V(H) - V(F)$

$h(v) = n+1$ for $v \in V(G)$

$h(v) > n+1$ for $v \in V(H) - V(G)$

Thus we conclude that $EAC(H) \cong G$, $C(H) \cong F$ and $d(\,EAC(H), C(H)\,) = n$.

Theorem 9 Let G be any graph. Then there exists a connected graph H such that $EAP(H) \cong G$.

Proof We construct the graph H from the graph G by adding eight vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$, and the following edges:

v_1v_2, v_2v_3, v_3v_4, v_5v_6, v_6v_7, v_7v_8

and all edges from v_4 and v_5 to vertices in G

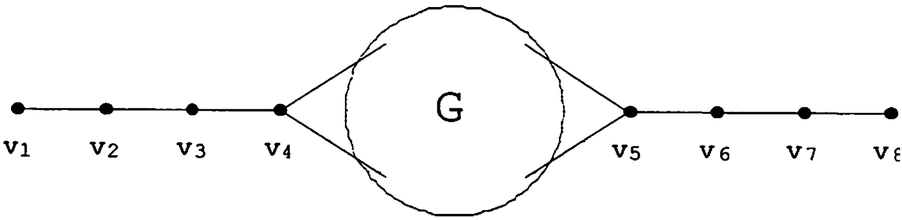


Figure 6

From the construction of the graph H we have the following properties:

$h(v) = 4$ for $v \in V(G)$

$h(v) < 4$ for $v \in V(H) - V(G)$

Thus we conclude that $EAP(H) \cong G$.

The constructions in Theorems 7-9 indicate that paths play an important role in the concept of edge-added eccentricity. It is therefore natural to look more closely at the edge-added eccentricity of vertices in paths. Let P_n denote the path on n vertices with the vertex labeling $v_1, v_2, \ldots, v_n$.

Theorem 10 Let n be a positive integer written in the form $n = 4k - \sigma$ with $\sigma = 0,1,2$ or 3 and with $k > 3$. For each vertex v_s, of P_n we describe the edge-added eccentricity and the optimal edge(s) e in the complement of P_n such that $e_{P_n+e}(v_s) = h(v_s)$. We assume that

$1 \le s \le 2k - \left\lceil \frac{\sigma}{2} \right\rceil$, that is, that v_s is in the first half of P_n.

For $1 \le s \le (k - \sigma)$:

$$h(v_s) = k + 1 + \left\lfloor \frac{(k-\sigma) - s}{2} \right\rfloor$$

$e = v_s v_t$ where $3k-\sigma+1-\left\lceil \frac{(k-\sigma)+1-s}{2} \right\rceil \le t \le 3k-\sigma+1-\left\lfloor \frac{(k-\sigma)+1-s}{2} \right\rfloor$

For $(k-\sigma)+1 \le s \le k+1$:

$$h(v_s) = k$$

$e = v_s v_t$ where $3k-\sigma+1 \le t \le 3k-\sigma+1-(s-((k-\sigma)+1))$

For $k+2 \le s \le 2k-\left\lceil \frac{\sigma}{2} \right\rceil$:

$$h(v_s) = s-1$$

$e = v_s v_t$ where $3k-\sigma-(s-(k+2)) \le t \le 3k+3+3(s-(k+2))$ and $t \le 4k-\sigma$.

Idea of Proof These results follow from the observation that the edge e should connect v_s with a vertex roughly two-thirds of the distance from v_s to v_n.

Corollary We let n be as in Theorem 10. Then the edge-added central vertices of P_n are

$$\{ v_{k+1}, v_k, v_{k-1}, \ldots, v_{k-\sigma+1} \} \cup \{ v_{3k-\sigma}, v_{3k-\sigma+1}, \ldots, v_{3k} \}$$

and the edge-added peripheral vertices of P_n are

$v_{\frac{n+1}{2}}$ if n is odd

$v_{\frac{n}{2}}$ and $v_{\frac{n+2}{2}}$ if n is even.

Trees

We investigate the basic properties of edge-added eccentricities in trees and describe the location and nature of the edge-added center. An example of a tree with edge-added eccentricities already calculated is shown below.

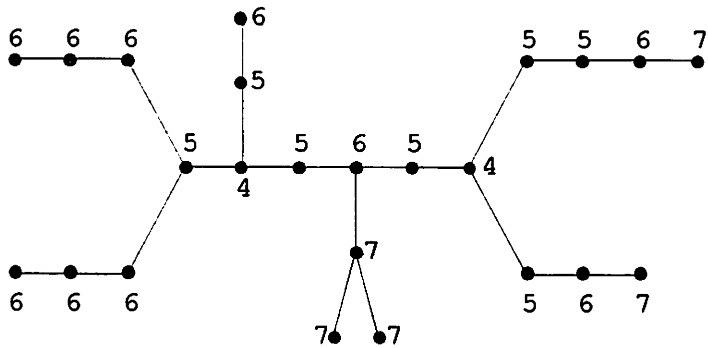


Figure 7

Let $C(T)$ denote the center of the tree T, and $T_1, T_2, \ldots, T_n$ be the components of $T - C(T)$. We begin with some basic facts about trees.

For any vertex v, define the antipodal set of v, $A_T(v)$, as the set of vertices in T which are farthest from v.

$$A_T(v) = \{w \in T \mid d_T(v,w) \geq d_T(v,x), \text{ for all } x \in T\}$$

For a set of vertices, S, we define the antipodal set of S, denoted by $A_T(S)$, as union of the antipodal sets of the vertices in S

$$A_T(S) = \bigcup_{v \in S} A_T(v).$$

Lemma 3 The antipodal set of the center, $A_T(C(T))$, contains vertices which lie in two or more of the components $T_1, T_2, \ldots, T_n$.

Proof If, for example, $A_T(C(T)) \subseteq T_1$, then consider the vertex $x \in T_1$ which is adjacent to a central vertex. Every path from a central vertex to one of its antipodal vertices must pass through x. Therefore the eccentricity of x can be no more than the eccentricity of a central vertex, contradicting the fact that x is not central.

Lemma 4 Among the optimal edges for a vertex v in the component T_i is an edge of the form vv' for some vertex v' with $v' \in C(T)$ or $v' \in T_j$, for some other component T_j with $j \neq i$.

Proof By Lemma 3, there are vertices of $A_T(C(T))$ contained in a component T_k, with $k \neq i$. If v and v' are both vertices in T_i then

$$h(v) \;=\; e_{T+vv'}(v) \;\geq\; rad(T)+1$$

But adding to T the edge from v to a central vertex produces a graph in which v has eccentricity no greater than $rad(T)+1$. Thus adding an edge from v to a central vertex does at least as well in reducing the eccentricity of v as does adding an edge from v to any other vertex of T_i.

Lemma 5 Let v be a vertex in the component T_i and suppose there are two other components T_j, T_k ; $(j,k \neq i)$ which contain vertices of $A_T(C(T))$. Then $h_T(v) = rad(T)$ or $h_T(v) = rad(T)+1$.

Proof The optimal edge for v, under these circumstances, is of the form vv' with $v' \in C(T)$. If $C(T)$ consists of a single vertex, then $h_T(v) = rad(T)+1$. If $C(T)$ consists of two adjacent vertices then $h_T(v) = rad(T)$ or $h_T(v) = rad(T)+1$.

Theorem 11 If there are three or more other components T_i, T_j, T_k ; $(i,j,k \text{ distinct})$ which contain vertices of $A_T(C(T))$ then $C(T) \subseteq EAC(T)$.

Proof By the previous lemma $h_T(v) = rad(T)$ or $h_T(v) = rad(T)+1$ for every vertex $v \notin C(T)$. Since the edge-added eccentricity of a central vertex can be no more than $rad(T)$ the result follows.

Note that when $A_T(C(T))$ is contained in exactly two components T_j, T_k , as is the case in Figure 7 the situation is quite different. Under these conditions it is possible for $EAC(T)$ to be partitioned into two connected components, one within T_j and one within T_k. This happens because the optimal edges for vertices in T_j extend into T_k and vice-versa.

We note that the ordinary eccentricities of vertices in a tree have a convexity property: the eccentricity of any vertex on a path between vertices a and b cannot exceed the maximum of the eccentricities of a and b. The edge-added eccentricities of vertices in the components T_i have a similar property.

Theorem 12 Let v and w be vertices in a component T_i with v contained in the path from w to $C(T)$. If y is any vertex on the path between v and w then

$$h_T(y) \le \max\{h_T(v),\quad h_T(w)\}$$

Proof Let x be the vertex in the component T_i which is adjacent to a central vertex. We estimate the edge-added eccentricity of y by calculating distances from y to other vertices in the tree. Let x_1 be a vertex in T_i with the property that a path from x_1 to x contains v and y. Then

$$d_T(y, x_1) < d_T(v, x_1).$$

By Lemma 4, the vertex v has an optimal edge, e_v, extending into $C(T)$ or some other component T_j. Thus

$$d_T(y, x_1) < d_{T+e_v}(v, x_1) \le h_T(v). \qquad *$$

Let x_2 be a vertex in T with the property that a path from x_2 to x does not contain y. Then

$$d_T(y, x_2) < d_T(w, x_2)$$

since any path in T from w to x_2 must pass through y. Again, Lemma 4 implies that w has an optimal edge, $e_w = wz$, extending into $C(T)$ or some other component T_j. We conclude that

$$d_{T+yz}(y, x_2) \le d_{T+wz}(w, x_2) \le h_T(w). \qquad *$$

It follows from the two starred equations that

$$d_T(y, x_1) \le \max(h_T(v), h_T(w)) \quad \text{and}$$

$$d_{T+yz}(y, x_2) \le \max(h_T(v), h_T(w))$$

from which the conclusion follows.

We note that Theorem 12 implies that, in some sense, vertices in $EAP(T)$ are more likely to come from $C(T)$ or $P(T)$.

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