On Asymmetric Colorings of Integer Grids

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Abstract

Let $\nu(\mathbb{Z}^m)$ be the minimal number of colors enough to color the m-dimensional integer grid $\mathbb{Z}^m$ so that there would be no infinite monochromatic symmetric subsets. Banakh and Protasov [3] compute $\nu(\mathbb{Z}^m) = m + 1$. For the one-dimensional case this just means that one can color positive integers in red, while negative in blue, thereby avoiding an infinite monochromatic symmetric subset by trivial reason. This motivates the question what changes if we allow only colorings unlimited in both directions (in "all" directions for $m > 1$). In this paper we show that then $\nu(\mathbb{Z})$ increases in 1, whereas for higher dimensions the values $\nu(\mathbb{Z}^m)$ remain unaffected.

Furthermore we examine the density properties of a set $A \subseteq \mathbb{Z}^m$ that ensure the existence of infinite symmetric subsets or arbitrarily large finite symmetric subsets in A. In the case that A is a sequence with small gaps, we prove a multi-dimensional analogue of the Szemerédi theorem, with symmetric subsets in place of arithmetic progressions. A similar two-dimensional statement is known for collinear subsets (Pomerance [10]), whereas for two-dimensional arithmetic progressions even the corresponding version of van der Waerden's theorem is known to be false.

We also observe that $A \subseteq \mathbb{N}$ contains arbitrarily large symmetric subsets whenever the series $\sum_{a \in A} 1/a$ diverges (for arithmetic progressions this is the well-known unproven conjecture of Erdős). A natural two-dimensional analogue of the latter statement is false. We show a counterexample built upon Erdős's construction of a dense infinite B_2-sequence.

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1 Preliminaries

A set $B \subseteq \mathbb{Z}^m$ such that $B = g - B$ for some $g \in \mathbb{Z}^m$ is called *symmetric* (with respect to the center at rational point $\frac{1}{2}g$). We say that a *set* $A \subseteq \mathbb{Z}^m$ is *asymmetric*, if it contains no infinite symmetric subset. Clearly, A is asymmetric iff $A \cap (g - A)$ is finite for all $g \in \mathbb{Z}^m$.

An *r-coloring* of $\mathbb{Z}^m$ is a partition of $\mathbb{Z}^m$ onto r sets $A_1, \ldots, A_r$ that are called *color classes*. We call a set *monochromatic* if it is a subset of a color class. A *coloring* is *asymmetric* if all the color classes are asymmetric sets.

We define the norm $\|\cdot\|$ on $\mathbb{Z}^m$ by $\|(z_1, \ldots, z_m)\| = \max\{|z_1|, \ldots, |z_m|\}$. With a set $A \subseteq \mathbb{Z}^m$ we associate a monotone function

$$\alpha(R) = \min\{\, \|a + a'\| : a, a' \in A,\ \|a\|, \|a'\| > R\}\,.$$

Asymmetric subsets of $\mathbb{Z}^m$ admit the following characterization.

Lemma 1.1 *A set $A \subseteq \mathbb{Z}^m$ contains an infinite symmetric subset if and only if $\alpha(R) = O(1)$.*

Proof. If $\alpha(R) = O(1)$, then there are infinitely many pairs a, a' with $\|a + a'\|$ bounded by a constant c. As at least one of $(2c + 1)^m$ values of sum $a + a' = g$ must occur infinitely often, a symmetric set $A \cap (g - A)$ for this particular g is an infinite subset of A. Conversely, if $A \cap (g - A)$ is infinite for some g, then there are infinitely many pairs a, a' with $a + a' = g$ and, therefore, $\alpha(R) \leq \|g\|$ for all R. □

Given an r-coloring $\mathbb{Z}^m = A_1 \cup \ldots \cup A_r$, consider the function

$$\beta(R) = \min\{\, c : \text{there exist } a_1 \in A_1, \ldots, a_r \in A_r \text{ with } \|a_i\| > R,\ \|a_i - a_j\| \leq c \text{ for all } i, j\,\}.$$

We now have the following necessary condition on a coloring to be asymmetric.

Corollary 1.2 *If $\beta(R) = O(1)$, then for the coloring $\mathbb{Z}^m = A_1 \cup \ldots \cup A_r$ there is an infinite monochromatic symmetric set.*

Proof. The assumption $\beta(R) = O(1)$ implies that for any R there are $a_1 \in A_1, \ldots, a_r \in A_r$ with $\|a_i\| > R$ such that $\|a_i + a'\|$ is bounded by a constant c for all i and $a' = -a_1$. Infinitely often a' belongs to a particular color class A_i for which, therefore, $\alpha(R) \leq c$. By Lemma 1.1, A_i contains an infinite symmetric subset. □

Define $\nu(\mathbb{Z}^m)$ to be the minimal r such that there exists an asymmetric r-coloring of $\mathbb{Z}^m$.

Theorem 1.3 *[Banakh-Protasov [3]]* $\nu(\mathbb{Z}^m) = m + 1$.

We outline the proof not only for the sake of survey but also because this theorem motivates results of the next section.

Sketch-proof. To show that $\nu(\mathbb{Z}^m) \leq m+1$, define an $(m+1)$-coloring $\mathbb{Z}^m = A_1 \cup \ldots \cup A_{m+1}$ as follows. Consider an m-dimensional simplex S (a segment in $\mathbb{Z}^2$, a triangle in $\mathbb{Z}^2$, a tetrahedron in $\mathbb{Z}^3$ and so on). Choose a point p inside S. For a point $z \in \mathbb{Z}^m$, let $R(z)$ be a ray extending from p and passing through z. Let A_i consist of those lattice points z that $R(z)$ intersects i-th face of S. Clearly, each A_i is asymmetric.

Now we need to prove that in any m-coloring of $\mathbb{Z}^m$ one can find an infinite monochromatic symmetric set. The one-dimensional case is trivial, and the two-dimensional case is still not so hard. We outline the proof for the first non-trivial case of $m = 3$ that can be easily extended to higher dimensions.

Suppose, to the contrary, that there exists a 3-coloring of $\mathbb{Z}^3$ without infinite monochromatic symmetric sets. Let $G = \{-1, 0, 1\}^3$ be a discrete cube and $K = [-k, k]^3$ a continuous cube in $\mathbb{R}^3$. It follows from our assumption that if k is large enough, then the boundary ∂K of K contains no two lattice points of the same color and symmetric with respect to a center in G. Fix such a cube K with even integer k. Triangulate ∂K into isosceles right-angled triangles with vertices in all those lattice points of ∂K whose three coordinates are even. We can choose this triangulation symmetric with respect to the origin $(0, 0, 0)$.

Fix now a triangle T in $\mathbb{R}^2$ and assign each of three colors to one of the vertices of T. Define a mapping $h : \partial K \to T$ by the following two conditions.

1. h takes each lattice point of ∂K with all three coordinates even (i.e. each vertex of the triangulation) into the vertex of T with the same color.

2. On each element of the triangulation h is linear. In other words, for every triangle T' in triangulation of ∂K, there is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ that induces $h : T' \to T$.

Clearly, h is uniquely determined by these two conditions and is continuous. By the Borsuk-Ulam theorem (see e.g. [17]) applied to the map $h : \partial K \to T$, there exists a pair $(x, -x)$ of antipodal points on ∂K with $h(x) = h(-x)$. Let a, b, c be the vertices of the triangle containing the point x (if x lies on the border between two or more triangles, we just choose one of them). The triangle with vertices $-a$, $-b$, $-c$ contains the point $-x$. By the linearity of h, images $h(x)$ and $h(-x)$ belong to the convex hulls of sets

$\{h(a), h(b), h(c)\}$ and $\{h(-a), h(-b), h(-c)\}$ respectively. Consequently, the two convex hulls have nonempty intersection.

On the other hand, any point in $\{a, b, c\}$ is symmetric to any point in $\{-a, -b, -c\}$ with respect to a center in G. By our assumption, colors of $\{a, b, c\}$ and colors of $\{-a, -b, -c\}$ do not intersect and, hence, $\{h(a), h(b), h(c)\}$ and $\{h(-a), h(-b), h(-c)\}$ are disjoint sets of vertices of the triangle T. Therefore, convex hulls of these two sets are disjoint too, a contradiction. □

Some related results can be found in [1, 2, 13].

2 Unlimited colorings

We call a *set* $A \subseteq \mathbb{Z}$ *unlimited* if it has neither the largest nor the least element. We say that a *coloring of* $\mathbb{Z}$ is *unlimited* if each color class is unlimited. The following theorem shows connections between notions of unlimitedness and symmetricalness.

Theorem 2.1

1. *For a 2-coloring of $\mathbb{Z}$, an infinite monochromatic symmetric set exists if and only if at least one color class is unlimited. In particular, it exists for any unlimited 2-coloring of $\mathbb{Z}$.*

2. *There exists an unlimited asymmetric 3-coloring of $\mathbb{Z}$.*

Proof. The first statement in direction "only if" is obvious. Suppose that one of color classes is unlimited. Then another color class is either finite or unlimited too. The first case is trivial. In the second case the existence of an infinite monochromatic symmetric set immediately follows from Corollary 1.2, since for any unlimited 2-coloring of $\mathbb{Z}$ we obviously have $\beta(R) = 1$ for all R.

The second statement follows from the stronger Theorem 2.2 below. □

Our arguments give even a bit more than stated in Theorem 2.1 (1). Namely, for any 2-coloring of $\mathbb{Z}$ with at least one unlimited color class and for any integer g, either g or $g + 1/2$ is the center of an infinite monochromatic symmetric subset. Notice also that if we allow only integers centers, then Theorem 2.1 (1) is not true any more. A counterexample is the partition of $\mathbb{Z}$ onto odd and even numbers.

The bound $\nu(\mathbb{Z}^m) \leq m + 1$ in the proof of Theorem 1.3 was shown with apparently limited coloring of $\mathbb{Z}^m$. By Theorem 2.1 (1), in the one-dimensional case this would be impossible to do in any other way. It turns out that in the case of dimension $m > 1$ the additional requirement for

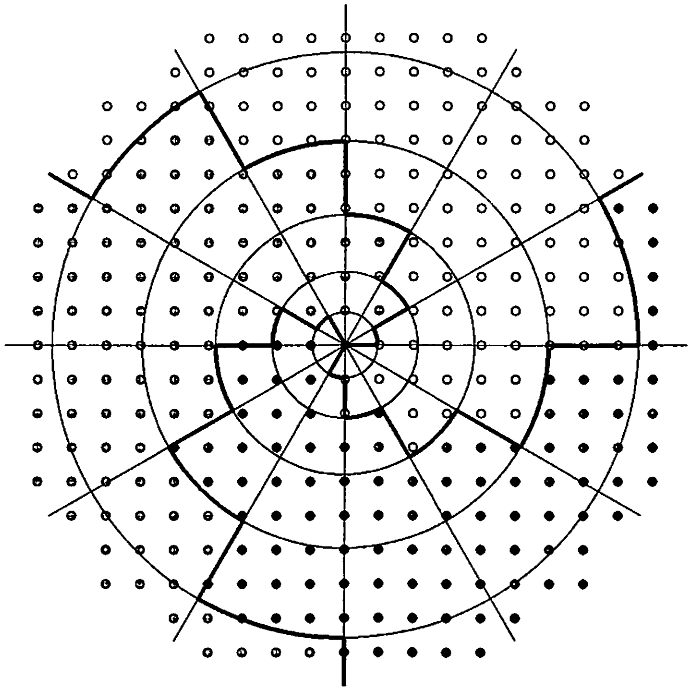

Figure 1: The unlimited asymmetric 3-coloring of $\mathbb{Z}^2$.

colorings to be unlimited does not change anything, even under the rather strong notion of limitlessness. Namely, let us call a *coloring of* $\mathbb{Z}^m$ *unlimited* if any ray passing through infinitely many (i.e. at least two) integer points intersects each color class. The construction of the following theorem can be easily extended to higher dimensions.

Theorem 2.2 *There is an unlimited asymmetric 3-coloring of* $\mathbb{Z}^2$.

Proof. Fix the polar system of coordinates $\langle\rho,\phi\rangle$ in the plane and determine a 3-coloring by a sequence $0 = R_0 < R_1 < R_2 < \dots$ as follows. Color a point $x \in \mathbb{Z}^2$ with $R_{k-1} < \rho \leq R_k$ in red if $k\pi/6 < \phi \leq (k+4)\pi/6$, in blue if $(k+4)\pi/6 < \phi \leq (k+8)\pi/6$, and in green if $(k+8)\pi/6 < \phi \leq (k+12)\pi/6$ (see Figure 1). Select the sequence of R_k so that $R_k - R_{k-1} \to \infty$ for $k \to \infty$. This condition makes each color class unlimited and ensures for each color class that $\alpha(R) \to \infty$ for $R \to \infty$. By Lemma 1.1 the latter implies asymmetry of the coloring. □

3 Density properties of asymmetric sets

For integers a and b with $a \le b$, denote $[a,b] = \{a, a+1, \ldots, b-1, b\}$ and $[a] = [1,a]$, the latter for $a \ge 1$. Given a set $A \subseteq \mathbb{Z}$, we define its *upper density* by

$$d(A) = \limsup_{n\to\infty} \frac{|A \cap [-n,n]|}{2n+1}$$

and *unbiased upper density* by

$$d^*(A) = \limsup_{n\to\infty} \min\left\{\frac{|A \cap -[n]|}{n}, \frac{|A \cap [n]|}{n}\right\}.$$

Obviously, $d(A) \ge d^*(A)$.

Theorem 3.1

1. *If $A \subseteq \mathbb{Z}$ has density $d(A) > 1/2$ and g is an arbitrary integer, then A contains an infinite symmetric subset $B = g - B$ with $d^*(B) \ge 2d(A) - 1$.*

2. *There is an asymmetric set $A \subseteq \mathbb{Z}$ with $d^*(A) = 1/2$.*

Proof. 1. It is easy to see that $d^*(A \cap -A) = d(A \cap -A) \ge 2d(A) - 1$. This proves the statement for $g = 0$. The case of arbitrary g also follows, as $d(A) = d(g + A)$.

2. Given a sequence of integers

$$0 < a_1 < b_1 < u_1 < v_1 < a_2 < b_2 < u_2 < v_2 < \ldots,$$

let A consist of all segments $[a_k, b_k]$ and $-[u_k, v_k]$ for $k \ge 1$. Select the sequence a_k, b_k, u_k, v_k so that

$$u_k - b_k \to \infty, \qquad a_{k+1} - v_k \to \infty, \tag{1}$$

$$\frac{1}{v_k}\sum_{i=1}^{k}(b_i - a_i) \to \frac{1}{2}, \qquad \frac{1}{v_k}\sum_{i=1}^{k}(v_i - u_i) \to \frac{1}{2}, \tag{2}$$

for $k \to \infty$. This can be done, for example, by taking $u_k - b_k = a_{k+1} - v_k = k$ and $b_k - a_k = v_k - u_k = k^2$. The conditions (1) imply that for A the value $\alpha(R)$ goes to the infinity as R increases. By Lemma 1.1, the set A is asymmetric. The conditions (2) enforce the equality $d^*(A) = 1/2$. □

Theorem 3.2 *Let integers*

$$\ldots < a_{-2} < a_{-1} < a_1 < a_2 < \ldots$$

be elements of an unlimited set A listed in the ascending order.

1. *If $a_k - a_{k-1} = O(1)$ for $k > 1$, then A contains an infinite symmetric subset.*

2. *Let $f : \mathbb{N} \to \mathbb{N}$ be a monotone nondecreasing function that goes to the infinity as slowly as desired. There exists an unlimited asymmetric set A such that*

$$a_{k+1} - a_k \leq f(k) \text{ and } a_{-k} - a_{-(k+1)} \leq f(k) \text{ for } k \geq 1. \qquad (3)$$

Proof. The first statement is a simple corollary of Lemma 1.1.
To prove the second statement, set

$$a_1 = 1, \; a_{k+1} = a_k + f(k), \quad a_{-1} = 0, \; a_{-(k+1)} = -a_k - \lfloor f(k)/2 \rfloor.$$

It follows that

$$\lim_{R\to\infty} \alpha(R) = \lim_{k\to\infty} f(k)/2 = \infty.$$

By Lemma 1.1, the set A is asymmetric. It remains to check the relations (3). The first of them is obvious. The second is also true, as

$$a_{-k} - a_{-(k+1)} = f(k-1) - \lfloor f(k-1)/2 \rfloor + \lfloor f(k)/2 \rfloor =$$
$$\lceil f(k-1)/2 \rceil + \lfloor f(k)/2 \rfloor \leq \lceil f(k)/2 \rceil + \lfloor f(k)/2 \rfloor = f(k).$$

□

Remark 3.3 Theorem 3.2 (1) implies that if a set $A \subseteq \mathbb{Z}$ has nonempty intersection with segment $[k, k+c]$ for every integer k and a constant c, then A contains an infinite symmetric subset. The latter is true for any dimension. If a set $A \subseteq \mathbb{Z}^m$ has nonempty intersection with every ball of a constant radius, then A contains an infinite symmetric subset.

Theorem 3.2 (1) admits an interesting restatement. If $\mathbb{Z}$ is a finite union of sets congruent with A and A is unlimited, then A contains an infinite symmetric subset. This is not true even for $\mathbb{Z}^2$. Using the same idea as in the proof of Theorem 2.2, one can construct an unlimited set $A \subseteq \mathbb{Z}^2$ that together with itself rotated onto angles $\pi/2$, π, and $3\pi/2$ covers all $\mathbb{Z}^2$ but nonetheless is asymmetric.

4 Existence of arbitrarily large symmetric subsets

In this section we address the density properties of a set $A \subseteq \mathbb{Z}^m$ that guarantee the existence of finite but arbitrarily large symmetric subsets in A.

4.1 One-dimensional case

Lemma 4.1 *The largest symmetric subset of a k-element set $A \subseteq [n]$ has more than $k^2/(2n)$ elements.*

Proof. Since there are k^2 ordered pairs (a, a') of elements of A and at most $2n-1$ centers $|a+a'|/2$, at least $k^2/(2n-1)$ pairs have a common center g. It follows that the maximum subset of A symmetric with respect to the g has more than $k^2/(2n)$ elements. □

Remark 4.2 By the prime number theorem, for the set P of all primes we have $|P \cap [n]| \sim n/\ln n$. By Lemma 4.1 this implies that the largest symmetric subset of $P \cap [n]$ has $\Omega(n/\ln^2 n)$ elements. It is worth noting that this estimate, though given by a simple counting argument based only on the density of primes, is nearly best possible (up to the factor of $\ln\ln n$).

Indeed, let $N(g)$ denote the total number of solutions to the equation $g = p + p'$ in prime numbers. Obviously, $N(g)$ is the cardinality of the largest set of primes symmetric with respect to the center $\frac{1}{2}g$. By [16] (see also [11, corollary 2.36]) it holds $N(g) = O(f(g)\, g/\ln^2 g)$, where $f(g) = \prod_{p|g}(1-1/p)^{-1}$. By Landau's bound on the Euler function [9] we have $f(g) = g/\phi(g) = O(\ln\ln g)$ and, therefore, $N(g) = O(\frac{g\ln\ln g}{\ln^2 g})$. It immediately follows that the maximum cardinality of a symmetric set of primes within $[n]$ is $O(\frac{n\ln\ln n}{\ln^2 n})$ (and it is actually $\Omega(\frac{n\ln\ln n}{\ln^2 n})$ by [12]).

Sets of integers without symmetric subsets of cardinality more than 2 have been investigated in additive and combinatorial number theory, where were called *B_2-sequences* or *Sidon sets*. In other terminology, those are sets $A \subset \mathbb{N}$ with all sums $a+a'$ distinct for $a, a' \in A$ and $a \le a'$ (see [11, section 4.1] for survey and references). We refer to the following result.

Erdős's theorem [7] *There is a Sidon set A with*

$$\limsup_{n\to\infty} \frac{|A \cap [n]|}{\sqrt{n}} \ge 1/2.$$

The constant $1/2$ was improved to $1/\sqrt{2}$ in [8].

It follows that the property of existence of arbitrarily large symmetric subsets depends on the density of a set and has sharp threshold behavior.

Theorem 4.3

1. *If $\limsup_{n\to\infty} \frac{|A\cap[n]|}{\sqrt{n}} = \infty$, then A contains arbitrarily large symmetric subsets.*

2. *There is a set $A \subset \mathbb{N}$ with finite nonzero $\limsup_{n\to\infty} \frac{|A\cap[n]|}{\sqrt{n}}$ that does not contain any symmetric subset of cardinality more than 2.*

Proof. By Lemma 4.1, A has a symmetric subset of cardinality more than $|A \cap [n]|^2/(2n)$ for any n. As this number may be as large as we wish, the first statement follows. The second statement is given by Erdős's theorem. □

If we, from the perspective of Ramsey theory, consider a symmetric set of integers as an extension of an arithmetic progression, most questions become more tractable. For example, the existence of arbitrarily large symmetric sets of primes easily follows from their density, whereas the longest known arithmetic progression of primes has length 22 [14]. In the sequel we compare how existence of long arithmetic progressions and large symmetric subsets in a set A depends on density properties of A.

Let A be an infinite set of integers $a_1 < a_2 < a_3 < \dots$. The property of A to be "dense" in $\mathbb{N}$ can be formalized in several ways.

Density conditions.

1. $a_{k+1} - a_k = O(1)$.
2. $\limsup_{k\to\infty} k/a_k > 0$ (equivalently, $\limsup_{n\to\infty} |A \cap [n]|/n > 0$).
3. $\sum_{k=1}^{\infty} 1/a_k = \infty$.

Of this conditions, the second is weaker than the first, and the third is weaker than the second.

The existence of arbitrarily long arithmetic progressions in a set with property 1 is just another form of the van der Waerden theorem (see [4]). Szemerédi [18] proves that arbitrarily long arithmetic progressions exist even in sets with property 2. If the same is true for sets with property 3 is the famous open problem for whose solution Paul Erdős offered 3000$. Theorem 4.3 (1) can be viewed as an analogue of Szemerédi's theorem for symmetric sets. As for an analogue of the Erdős's problem, it turns out to be resolvable in affirmative.

Theorem 4.4 *If the series $\sum_{a\in A} 1/a$ diverges, then A contains arbitrarily large symmetric subsets.*

The theorem immediately follows from Theorem 4.3 (1) and the following

Lemma 4.5 *If $|A \cap [n]| = O(n/\ln^{1+\epsilon} n)$ with $\epsilon > 0$ constant, then the series $\sum_{a\in A} 1/a$ converges.*

For the sake of completeness, we prove the lemma in a stronger form than we actually need (a relaxation of the assumption to $|A \cap [n]| = O(\sqrt{n})$ would be sufficient for our purposes).

Proof. Assume that $|A \cap [n]| \leq f(n)$, where $f(n) = cn/\ln^{1+\epsilon} n$ with c constant. For k-th element a_k of A this gives inequality $f(a_k) \geq k$. As function $f(n)$ is monotone for sufficiently big n, we can consider its inversion $f^{-1}(k)$. For k big enough we have

$$\frac{1}{a_k} \leq \frac{1}{f^{-1}(k)} \leq \frac{c}{k \ln^{1+\epsilon} k}.$$

Since the series $\sum_{k=2}^{\infty} 1/(k \ln^{1+\epsilon} k)$ converges, the lemma follows. □

4.2 Higher dimensions

Let $A = \{a_1, a_2, \ldots\}$ be an infinite subset of $\mathbb{Z}^m$, with elements in a fixed order. Consider the following

Density conditions for higher dimensions.

1. $||a_{k+1} - a_k|| = O(1)$.

2. $\liminf_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} ||a_{k+1} - a_k|| < \infty$.

3. $\sum_{k=1}^{\infty} 1/||a_k|| = \infty$.

It is not hard to see that in the case of $m = 1$ these conditions coinside with density conditions in the preceeding subsection. Again, the first condition implies the second, which in its turn implies the third. Notice that the third condition does not depend on how elements of A are ordered.

In contrast with the one-dimensional case, for $m > 1$ condition 1 does not guarantee existence of arbitrarily long arithmetic progressions of vectors in A. There is an example of $A \subset \mathbb{Z}^2$ with each difference $a_{k+1} - a_k$ either $(0, 1)$ or $(1, 0)$ and such that no five of the a's are in arithmetic progression [6] (see also [11, theorem 6.13]). On the other hand, condition 1 and even condition 2 suffice to ensure the existence of arbitrarily large symmetric subsets.

Theorem 4.6 *If $A \subseteq \mathbb{Z}^m$ satisfies the above condition 2, then A contains arbitrarily large symmetric subsets.*

As opposed to the result of [6], there is a two-dimensional analogue of Szemerédi's theorem with collinear subsets instead of arithmetic progressions [15, 10]. In this respect our Theorem 4.6 can serve as yet another multi-dimensional analog of Szemerédi's theorem, with arithmetic progressions replaced by symmetric subsets.

Proof. Let $d_k = a_{k+1} - a_k$. The set A is detemined by the sequence of differences

$$W = (d_1, d_2, d_3, \ldots)$$

(the choice of a_1 is actually irrelevant). It will be convenient to adopt some terminology of the formal language theory. We will view elements of W as letters of the infinite alphabet $\mathcal{Z} = \mathbb{Z}^m$. Respectively, W itself will be viewed as an infinite word over $\mathcal{Z}$. A *pattern* is a word over the alphabet of variables $\{x_1, x_2, \ldots\}$. Pattern $x_{i_1} x_{i_2} \ldots x_{i_l}$ is *symmetric* if $i_j = i_{l+1-j}$ for all $j \leq l$. A subword u of a word w over $\mathcal{Z}$ is called an *occurrence* of a pattern $P = x_{i_1} \ldots x_{i_l}$ in w if u can be obtained from P by substituting nonempty words in place of each variable, with the same variable being everywhere replaced by the same word. For example, in word $z_1 z_1 z_2 z_3 z_1 z_2 z_2 z_1$ subword $z_1 z_2 z_3 z_1 z_2$ is an occurrence of pattern $x_1 x_2 x_1$.

Lemma 4.7 *Each occurrence of a symmetric pattern of length l in W corresponds to an $(l+1)$-element symmetric subset of A.*

Proof. Let $u = u_1 \ldots u_l$ be an occurrence in W of a symmetric pattern P of length l, where u_s is substituted in place of s-th variable of P. Let $s_1 < s_2 < \ldots < s_{l+1}$ be a sequence such that $u_i = d_{s_i} d_{s_i+1} \ldots d_{s_{i+1}-1}$. Then $\{a_{s_1}, a_{s_2}, \ldots, a_{s_{l+1}}\}$ is a symmetric subset of A. This can be shown by easy induction. Really, assume that a_{s_2} and a_{s_l} are symmetric with respect to center $\frac{1}{2}g$, that is, $a_{s_2} + a_{s_l} = g$. As $u_1 = u_l$, we have $a_{s_2} - a_{s_1} = a_{s_{l+1}} - a_{s_l}$. Consequently, a_{s_1} and $a_{s_{l+1}}$ are symmetric with respect to $\frac{1}{2}g$ too. □

In the rest of the proof we show that W contains each pattern of an infinite sequence (P_j), where $P_1 = x_1$ and $P_j = P_{j-1} x_j P_{j-1}$ for $j > 1$. By Lemma 4.7 this will prove the theorem.

We extend the norm $\|\cdot\|$ from $\mathcal{Z} = \mathbb{Z}^m$ to the set of all finite and infinite words over the alphabet $\mathcal{Z}$. Given a finite word $w = z_1 z_2 \ldots z_k$, let $\|w\| = \frac{1}{k} \sum_{i=1}^{k} \|z_i\|$. Given an infinite word W, we define $\|W\| = \liminf_{k\to\infty} \|w_k\|$, where w_k is the prefix of length k of W. The following lemma generalizes a theorem of Coudrain and Schützenberger [5] on unavoidability of the patterns P_j to the case of an infinite alphabet.

Lemma 4.8 *Every infinite word W with finite norm $\|W\|$ contains occurrences of all the patterns P_j for $j \geq 1$.*

Lemma 4.8 directly follows from Lemma 4.9 below, as the condition $\|W\| < \infty$ implies that $\|w_k\| \leq c$ for a constant c and infinitely many k.

Lemma 4.9 *For any j and c there exists a number $k(j,c)$ such that the following condition is true. Every finite word w with $\|w\| \leq c$ and length at least $k(j,c)$ contains an occurrence of pattern P_j.*

In its turn, we will derive Lemma 4.9 from the next

Lemma 4.10 *Let ϵ be an arbitrary positive real. For all c and k there exists a number $n = n(\epsilon, c, k)$ such that the following condition is true. Every finite word w with $\|w\| \leq c$ and length at least $n(\epsilon, c, k)$ contains a subword uvu, where u has length k and norm $\|u\| \leq c+\epsilon$ and v is nonempty.*

Proof. Let $N = (2k(c+\epsilon)+1)^m$. This is an upper bound on the number of $z \in \mathbb{Z}^m$ with $\|z\| \leq k(c+\epsilon)$. Set $n = \lceil 2kN^k \frac{c+\epsilon}{\epsilon} \rceil$ and verify that this value fits our requirements. Suppose that a word w has length at least n and norm at most c. Let $t = \lfloor n/k \rfloor$ and $w = w_1 \ldots w_t w_{t+1}$, where each subword w_i for $i \leq t$ has length k (the suffix w_{t+1} may be empty). Let $I = \{ i \leq t : \|w_i\| \leq c+\epsilon \}$. We claim that $|I| \geq 2N^k$. This follows from the estimate

$$nc \geq n\|w\| \geq k \sum_{i=1}^{t} \|w_i\| > k(t - |I|)(c+\epsilon)$$

and our choice of n.

There are less than N^k pairwise distinct words with length k and norm at most $c+\epsilon$. This implies that I contains at least three indices i, j, p, for which $w_i = w_j = w_p$. It is now clear that w contains a subword uvu, where $u = w_i$ has the required properties and v is nonempty. □

To complete the proof of Theorem 4.6, it remains to prove Lemma 4.9. We proceed by induction on j. For $j = 1$ the statement is trivial. Assume that it is true for $j-1$. Then for an arbitrary chosen $\epsilon > 0$, the value $k(j,c) = n(\epsilon, c, k(j-1, c+\epsilon))$ will satisfy the required condition. Really, suppose that a word w has length not less than $n(\epsilon, c, k(j-1, c+\epsilon))$. By Lemma 4.10, w contains a subword uvu, where u has length $k(j-1, c+\epsilon)$ and norm at most $c+\epsilon$. By the induction hypothesis, u contains an occurrence of pattern P_{j-1}. Therefore, we have two such occurrences in w. As these occurrences are separated by the nonempty subword v, we obtain an occurrence of pattern P_j in w. □

In the contrast with Theorem 4.4, the density condition 3 cannot guarantee existence of large symmetric subsets even in the two-dimensional case.

Theorem 4.11 *There exists a set $A \subset \mathbb{Z}^2$ with the series $\sum_{a\in A} 1/\|a\|$ divergent but without symmetric subsets of cardinality more than 4.*

Proof. By the Erdős theorem cited in Subsection 4.1, there is a set $B \subset \mathbb{N}$ with

$$\limsup_{n\to\infty} \frac{|B \cap [n]|}{\sqrt{n}} \geq 1/2 \tag{4}$$

but without any symmetric subset of cardinality more than 2. Let $A = B^2$. As easily seen, A contains no symmetric subset of cardinality more than 4.

Let $b_1, b_2, \ldots$ be elements of B listed in the ascending order. Notice the equality

$$\sum_{a\in A} \frac{1}{\|a\|} = \sum_{k=1}^{\infty} \frac{2k-1}{b_k}$$

with meaning, in particular, that both series diverge or converge simultaneously. Therefore, we need to show that the series $\sum_{k=1}^{\infty} k/b_k$ diverges.

From (4) we conclude that there is an infinite sequence (n_i) such that

$$\sqrt{n_i}/2 > n_{i-1}^2 + n_{i-1} \quad \text{and} \quad |B \cap [n_i]| \geq \sqrt{n_i}/2.$$

For $i \geq 1$, let $k_i = \max\{k : b_k \leq n_i\}$. As easily seen,

$$n_i \geq k_i = |B \cap [n_i]| \geq \sqrt{n_i}/2.$$

This implies

$$\sum_{n_{i-1}<b_k\leq n_i} \frac{k}{b_k} \geq \frac{1}{n_i} \sum_{k=k_{i-1}+1}^{k_i} k \geq \frac{1}{n_i} \sum_{k=n_{i-1}+1}^{\lceil\sqrt{n_i}/2\rceil} k =$$

$$\frac{(\lceil\sqrt{n_i}/2\rceil + n_{i-1}+1)(\lceil\sqrt{n_i}/2\rceil - n_{i-1})}{2n_i} \geq \frac{n_i/4 + \sqrt{n_i}/2 - n_{i-1}^2 - n_{i-1}}{2n_i} \geq \frac{1}{8}$$

for all $i > 1$. The divergence follows. □

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