On the Gracefulness of the Graph $C_m \cup P_n$

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Abstract On the gracefulness of graph $C_m \cup P_n$, Frucht and Salinas first proved $C_4 \cup P_n$ for $n \geq 3$ is graceful. and conjectured: $C_m \cup P_n$ is graceful if and only if $m+n \geq 7$. In this paper, we prove graphs $C_m \cup P_n$ is graceful, for $m=4k$, $n=k+2, k+3, 2k+1, \ldots, 2k+5$; $m=4k+1$, $n=2k, 3k+1, 4k+1$; $m=4k+2$, $n=3k, 3k+1, 4k+1$; $m=4k+3$, $n=2k+1, 3k, 4k$.

Key words: graceful labeling. vertex. edge. graph $C_m \cup P_n$

1.Introduction

Let $G=(V,E)$ be a simple graph and $f: V \to \{0,1,2,\cdots,|E|\}$ be an injection map. Define an induced map $f': E \to \{1,2,\cdots,|E|\}$ by $f'(uv) = |f(u)-f(v)|$ for $uv \in E$. If f' maps E onto $\{1,2,\cdots,|E|\}$, then f is said to be a graceful labeling of G and the graph G is said to be graceful. Let f be a graceful labeling of G. If there exists a constant B, such that $f(u) \leq B$, $f(v) \geq B$ for every $vu \in E$, then f is said to be a balance labeling of G, and B is called its character.

The graph $C_m \cup P_n$ is the disjoint union of cycle C_m and path P_n, it has m+n-1 edges, and m+n vertices. Let the vertices of C_m be denoted by $x_1, x_2, \cdots, x_m$ and the vertices of P_n be denoted by $y_1, y_2, \cdots, y_n$, successively. It is easily seen that graph $C_m \cup P_n$ is graceful if and only if both f and f' are injection.

In 1985, R.W.Frucht and L.C.Salinas proved $C_4 \cup P_n$ is graceful for m=4 and $n \geq 3$. They conjectured $C_m \cup P_n$ is graceful if and only if $m+n \geq 7$ see[1]; Z.Gao proved $C_m \cup P_n$ is graceful when $m \geq 5$ (see[2]).

For convenience sake, we shall use the following notations:

$[a,b] = \{x \in Z \mid a \leq x \leq b\}$, $[a,b]_2 = \{x \in Z \mid a \leq x \leq b$ and $x \equiv a \bmod 2\}$

where Z is ring of integers, $a,b \in Z$. Let $[x]$ denotes the greatest integer y such that $y \leq x$.

2.The main results

Lemma 1(see[3]) The cycle C_{4k} is balance graphs for all positive integers k. If f is a balance labeling of C_{4k}, then $[0, 4k] - \{f(x_i) \mid i \in [1, 4k]\} = k$ or $3k$.

We give the balance labeling of C_{4k} as follows:

$$f(x_i)=\begin{cases}(i-1)/2, & i\in[1,4k-1]_2\\ 4k+1-i/2, & i\in[2,2k]_2\\ 4k-i/2, & i\in[2k+2,4k]_2\end{cases}$$

Its character is $2k$. Set of vertex labels is $[0,4k]-\{3k\}$.

Theorem 1. If $n\in\{k-2,k+3,2k+1,\ldots,2k+5\}$, then graphs $C_{4k}\cup P_n$ are graceful for all positive integers k.

Proof. In the theorem upper labeling of C_{4k}, the labeling of the even vertex x_{2i} in C_{4k} plus n-1 is labeling of even vertex x_{2i} in graph $C_{4k}\cup P_n$. The labeling of odd vertex x_{2i-1} in C_{4k} is labeling of odd vertex x_{2i-1} in $C_{4k}\cup P_n$. It is easy to see that the set of vertex labels of $C_{4k}\cup P$ is $[0,2k-1]\cup[2k+n-1,4k+n-1]-\{3k+n-1\}$, and set of edge labels is $[n,4k+n-1]$.

In the path P_n, if set of vertex labels is $[2k,2k+n-2]\cup\{3k+n-1\}$ and set of edge labels is $[1,n-1]$, then $C_{4k}\cup P_n$ is graceful.

To this end, we construct a vertex labeling of path P_n as follows:

1. When $n=k+2$ $(k\geq 1)$

$f(v_i)$	*range of i*	*set of vertex labels*
4k+1	1	{4k+1}
2k-1+(i-1)/2	$[3,k+2]_2$	[2k,2k-1+[(k+1)/2]]
3k+1-i/2	$[2,k+2]_2$	[3k-[k/2],3k]

It is easy to see that the vertex labeling satisfies the requirement.

When k is odd, $\{f'(v_{2i-1}v_{2i})\mid 1\leq i\leq (k+1)/2\}=\{k+1\}\cup\{k+3-2i\mid 2\leq i\leq (k+1)/2\}=\{k+1\}\cup[2,k-1]_2$,

$\{f'(v_{2i}v_{2i+1})\mid 1\leq i\leq (k+1)/2\}=\{k+2-2i\mid 1\leq i\leq (k+1)/2\}=[1,k]_2$.

Therefore, the set of edge labels of P_{k+2} is $[1,k+1]$.

When k is even,

$\{f'(v_{2i-1}v_{2i})\mid 1\leq i\leq (k+2)/2\}=\{k+3-2i\mid 1\leq i\leq (k+2)/2\}=[1,k+1]_2$

$\{f'(v_{2i}v_{2i+1})\mid 1\leq i\leq k/2\}=\{k+2-2i\mid 1\leq i\leq k/2\}=[2,k]_2$

Therefore, the set of edge labels of P_{k+2} is also $[1,k+1]$.

2. When $n=k+3$ $(k\geq 1)$

$f(v_i)$	*range of i*	*set of vertex labels*
3k+1-(i-1)/2	$[1,k+3]_2$	[3k-[k/2],3k+1]
4k+2	2	{4k+2}
2k-2+i/2	$[4,k+3]_2$	[2k,2k-1+[(k+1)/2]]

Obviously, the set of vertex labels of P_{k+3} is $[2k,3k+1]\cup\{4k+2\}$. Just as with

P_{k+2} .we can see that the set of edge labels of P_{k+2} is [1,k+2].

3.When n=2k+2 (k≥1)

$f(y_i)$	range of i	set of vertex labels
2k+(i-1)/2	$[1,2k+1]_2$	[2k,3k]
4k+1-i/2	$[2,2k]_2$	[3k+1,4k]
5k+1	2k+2	{5k+1}

The table shows that the set of vertex labels of path P_{2k+2} is $[2k,4k] \cup \{5k+1\}$. Since

$\{f'(y_{2i-1}y_{2i}) \mid 1 \leq i \leq k+1\} = \{2k+2-2i \mid 1 \leq i \leq k\} \cup \{2k+1\} = [2,2k]_2 \cup \{2k+1\}$

and $\{f'(y_{2i}y_{2i+1}) \mid 1 \leq i \leq k\} = \{2k+1-2i \mid 1 \leq i \leq k\} = [1,2k-1]_2$

Thus the set of vertex labels of P_{2k+2} is $[1,2k+1]$.

To sum up, since both f and f' are injections, it follows that the theorem holds.

4. When $n=2k+1$ $(k \geq 1)$

$f(y_i)$	range of i	set of vertex labels
2k+(i-1)/2	$[1,2k-1]_2$	[2k,3k-1]
4k-i/2	$[2,2k]_2$	[3k,4k-1]
5k	2k+1	{5k}

5. $n=2k+3$ $(k \geq 1)$

4k+1-(i-1)/2	$[1,2k+1]_2$	[3k+1,4k+1]
2k-1+i/2	$[2,2k+2]_2$	[2k,3k]
5k+2	2k+3	{5k+2}

6. $n=2k+4$ $(k \geq 1)$

4k+2	1	{4k+2}
2k-1+(i-1)/2	$[3,2k+1]_2$	[2k,3k-1]
4k+2-i/2	$[2,2k+4]_2$	[3k,4k+1]
5k+3	2k+3	{5k+3}

7. $n=2k+5$ $(k \geq 1)$

4k+3	1	{4k+3}
2k+(i-1)/2-1	$[3,2k+5]_2$	[2k,3k+1]
4k+3-i/2	$[2,2k+2]_2$	[3k+2,4k+2]
5k+4	2k+4	{5k+4}

it is easy to verify that f is a graceful labeling of the $C_{4k} \cup P_n$.

Theorem 2. When $n=2k, 3k-1, 4k-1$, the graphs $C_{4k+1} \cup P_n (k \geq 1)$ are graceful.

Proof In the cycle C_{4k+1}, the vertices $x_1, x_2, \cdots x_{4k}$ are assigned labels as follows .

$$f(x_i) = \begin{cases} (i-1)/2, & if \quad i \in [1,4k-1]; \\ (4k+1+n-i)/2, & if \quad i \in [2,4k]. \end{cases}$$

We can prove that the set of edge labels (except edges $f'(x_{4k}, x_{4k+1})$ and $f'(x_{4k+1}, x_1)$) on C_{4k+1} is $[n+2, 4k+n]$. and the set of vertex labels (except vertex $f(x_{4k+1})$) is $[0, 2k-1] \cup [2k+n+1, 4k+n]$.

When $n = 2k\ (k \geq 1)$

Let $f(x_{4k+1})$=2k. then {$f'(x_{4k}, x_{4k+1})$, $f'(x_{4k+1}, x_1)$}={2k, 2k+1}

in the path P_{4k}.

$$f(y_i) = \begin{cases} 2k-1-(i-1)/2, & i \in [1,2k-1]; \\ 4k+1-i/2, & i \in [2,2k]. \end{cases}$$

We can see that the set of edge labels of P_{2k} is [1,2k-1],and the set of vertex labels is $[2k-1, 4k]$.

When $n = 4k-1\ (k \geq 1)$

let $f(x_{4k+1})$ be 4k.Then the labels of two edges incident with x_{4k+1} are 4k and 2k.

If $k = 1$,vertices x_1, x_2, x_3 of P_3 are assigned 3,2,5,successively.

If k≥2, we divide the vertices of P_{4k-1} into two groups {$y_1, y_2, \cdots y_{2k}$} and {$y_{2k+1}, y_{2k+2}, \cdots y_{4k-1}$}, and set of edge labels corresponding to each group is a continuous integer sequence. To this end, we construct a labeling of vertices in P_{4k-1}:

$f(y_i)$	*range of i*	*set of vertex labels*
6k-1-(i-1)/2	[1,2k-1],	[5k,6k-1]
2k+(i-1)/2	[2k+1,4k-1];	[3k,4k-1]
2k-i/2	[2,2k];	[2k, 3k-1]
6k-i/2	[2k+2,4k-2]	[4k+1,5k-1]

It is easy to show that the set of vertex labels of P_{4k-1} is [2k,6k-1]-{4k},and that the set of edge labels is [1,4k]-{4k,2k}.

When n=3k-1 (k≥1)

in the C_{4k+1}, let $f(x_{4k+1})$ be 3k.Then labels of two edges incident with x_{4k+1} are 2k and 3k.

(a) When k is even

if $k = 2$,the vertices in P_5 are labeled 8,9,4,7,5, successively.

If k≥4, we divide the edges of P_{3k-1} into three parts . The first part is from y_1 to y_k, the second part is from y_k to y_{2k}, and the third part is form y_{2k} to y_{3k-1}. Its vertex labeling is as follows:

$f(y_i)$	*range of i*	*set of vertex labels*
5k-1-(i-1)/2	$[1,k-1]_2$	[5k-k/2,5k-1]
3k+k/2,	k+1	{3k+k/2}
5k-(i-1)/2	$[k+3,2k-1]_2$	[4k+1,5k-1-k/2]
2k+1+(i-1)/2	$[2k+1,3k-3]_2$	[3k+1,3k+k/2-1]
3k+k/2+1	3k-1	{3k+k/2+1}
2k-1+i/2	$[2,2k]_2$	[2k,3k-1]
5k+1-i/2	$[2k+2,3k-2]_2$	[4k+2-k/2,4k]

Then, the set of edge labels of the first, second and third parts are [2k+1,3k-1],[k,2k-1] and [1,k-1], respectively.

(b) When k is odd

for k = 1, see [2].

If k≥3, the basic idea is that the edges of P_{3k-1} are divided into three parts, the first part is from y_1 to y_k, the second part is from y_k to y_{2k+2}, the third part is from y_{2k+2} to y_{3k-1}, and the set of edge labels of every part is a continuous integer sequence. Let

$f(y_i)$	*range of i*	*set of vertex labels*
5k-1-(i-1)/2	$[1,2k+1]_2$	[4k-1,5k-1]
2k+(i-1)/2	$[2k+3,3k-2]_2$	[3k+1,3k+(k-1)/2-1]
2k-1+i/2	$[2,k-1]_2$	[2k,2k-1+(k-1)/2]
3k+1+(k-1)/2	k+1	{3k+1+(k-1)/2}
2k-2+i/2	$[k+3,2k+2]_2$	[2k+(k-1)/2,3k-1]
5k-i/2	$[2k+4,3k-3]_2$	[4k+1-(k-1)/2, 4k-2]
3k+(k-1)/2	3k-1	{3k+(k-1)/2}

Then the set of edge labels corresponding to the three parts are [2k+1,3k-1], [k-2,2k-1] and [1,k-3], respectively. To sum up, the theorem holds

Theorem 3. If n∈{3k,3k+1,4k-1}, then graphs $C_{4k+2}\cup P_n$ (k ≥ 1) are graceful.

Proof. In the cycle C_{4k+2}, the vertex labeling f is defined as follows.

$f(x_i)$	*range of i*	*set of vertex labels*
(i-1)/2	$[1,4k+1]_2$	[0,2k]
n+4k-2-i/2	$[2,4k]_2$	[n+2k+2,n+4k+1]
n-1	4k+2	{n-1}

The set of edge labels of C_{4k+2} is [n+1,n+4k+1] ∪ {n+1-2k}, the set of vertex labels is [0,2k] ∪ [n-2k+2,n+4k+1] ∪ {n-1}. If the set of edge labels of P_n is

$[1,n]-\{n+1-2k\}$ and the set of vertex labels is $[2k+1,n+2k+1]-\{n+1\}$,then $C_{4k+2}\bigcup P_n$ is graceful .To satisfy the condition ,we construct following a labeling in path P_n.

(1) When $n=4k+1$ $(k\geq 1)$

$f(y_i)$	*range of i*	*set of vertex labels*
$6k+2-(i-1)/2$	$[1,2k-1]_2$	$[5k+3,6k+2]$
$2k+1+(i-1)/2$	$[2k+1,4k+1]_2$	$[3k+1,4k+1]$
$2k+i/2$	$[2,2k]_2$	$[2k+1,3k]$
$6k+3-i/2$	$[2k+2,4k]_2$	$[4k+3,5k+2]$

(2) When $n=3k$ $(k\geq 1)$

$2k+1+(i-1)/2$	$[1,2k-1]_2$	$[2k+1,3k]$
$2k+2+(i-1)/2$	$[2k+1,3k]_2$	$[3k+2,3k+2+[(k-1)/2]]$
$5k+2-i/2$	$[2,3k]_2$	$[4k+2-[k/2],5k+1]$

(3) When $n=3k+1$ $(k\geq 2)$

$4k+3+(i-1)/2$	$[1,2k-1]_2$	$[4k+3,5k+2]$
$3k+1-i/2$	$[2,2k]_2$	$[2k+1,3k]$
$3k+1$	$2k+1$	$\{3k+1\}$
$2k+2+(i-1)/2$	$[2k+3,3k+1]_2$	$[3k+3,3k+2+[k/2]]$
$5k+3-i/2$	$[2k+2,3k+1]_2$	$[4k+3-[(k-1)/2],4k+2]$

When $k=1$, the vertices in P_4 are labeled 4,6,7,3, successively.

Theorem 4 .When $n=2k+1,3k,4k$, the graphs $C_{4k+3}\bigcup P_n$ $(k\geq 1)$ are graceful.

Proof. Except for the vertex x_{4k+3} label of the other vertices in C_{4k+3} as follows :

$$f(x_i)=\begin{cases}(i-1)/2, & i\in[1,4k-1]_2\\ n+4k+3-i/2, & i\in[2,4k-2]_2\\ p, & i=4k-3\end{cases}$$

Then the set of vertex labels in C_{4k+3} is $[0,2k]\cup[2k+n+2,n+4k+2]$,and the set of edge labels is $[n+2,n+4k+2]$. If f is a graceful labeling of $C_{4k+3}\bigcup P_n$ then f must satisfy $f(V(P_n))\cup\{p\}=[2k+1,n+2k+1]$, and

$f'(E(P_n))\cup\{f'(x_1x_{4k+3}),f'(x_{4k+2}x_{4k+3})\}=[1,n+1]$.

(1) When $n=2k+1$ $(k\geq 1)$

let p be $2k+1$.Then the labels of remaining two edges in C_{4k+3} are $2k+1$, $2k+2$.

In P_{2k+1}, let

$f(y_i)$	*range of i*	*set of vertex labels*
$4k+2-(i-1)/2$	$[1,2k+1]_2$	$[3k+2,4k+2]$

2k−1+i/2	$[2,2k]_2$	[2k+2,3k+1]

Then the set of edge labels of P_{2k+1} is [1,2k].

(2) When $n = 4k\ (k \geqslant 1)$

let p be $4k+1$. Then the labels of remaining two edges in C_{2p+3} are $2k+1$ and $4k-1$. In P_{4k}, let

$f(y_i)$	*range of i*	*set of vertex labels*
6k+1-(i-1)/2	$[1,2k-1]_2$	[5k+2,6k+1]
2k+1+(i-1)/2	$[2k+1,4k-1]_2$	[3k+1,4k]
2k+i/2	$[2,2k]_2$	[2k+1,3k]
6k+2-i/2	$[2k+2,4k]_2$	[4k+2,5k+1]

It is easy to see that the set of edge labels from y_1 to y_{2k} and from y_{2k} to y_{4k} and [2k+2,4k] and [1,2k], respectively.

(3) When $n = 3k\ (k \geqslant 1)$

let $f(x_{2k+3})=3k+1$. Then the labels of the remaining two edges in C_{2p+3} are $2k+1$ and $3k+1$. The vertex labeling of P_{3k} is given by two cases.

(a) When k is odd

if $k = 1$, then labels of vertices in P_3 are 3,5,6, successively.

When $k \geqslant 3$:

$f(y_i)$	*range of i*	*set of vertex labels*
2k−1−(i-1)/2	$[1,2k-1]_2$	[2k+1,3k]
5k−3-(i-1)/2	$[2k+1,3k-2]_2$	[4k+4-(k-1)/2,4k−3]
3k−3−(k-1)/2	3k	{3k+3-(k-1)/2}
5k−2-i/2	$[2,k-1]_2$	[5k+2-(k-1)/2,5k−1]
3k−4−(k-1)/2	k−1	{3k+4-(k-1)/2}
5k−3-i/2	$[k+3,2k-2]_2$	[4k+4,5k+1-(k-1)/2]
2k−2−i/2	$[2k,3k-1]_2$	[3k+2,3k+2-(k-1)/2]

The set of edge labels is partitioned in to three segments by the labeling, i.e. from y_1 to y_k, from y_k to y_{2k-1} and from y_{2k-1} to y_{3k}. The set of edge labels, corresponding to each parts, are [2k+2,3k], [k+2,2k] and [1,k−1], respectively.

(b) When k is even

if $k=2$, then labels of vertices in P_6 are 5,11,8,6,10,9,successively

When $k \geqslant 4$

$f(y_i)$	*range of i*	*set of vertex labels*
2k−1−(i-1)/2	$[1,k-1]_2$	[2k+1,2k+k/2]
3k−1−k/2	k+1	{3k+1+k/2}

$2k+(i-1)/2$	$[k+3,2k+1]_2$	$[2k+1+k/2,3k]$
$5k-2-(i-1)/2$	$[2k+3,3k-1]_2$	$[4k-3-k/2,4k-1]$
$5k-2-i/2$	$[2,2k]_2$	$[4k+2,5k-1]$
$2k-1+i/2$	$[2k+2,3k-2]_2$	$[3k+2,3k+k/2]$
$3k+2-k/2$	$3k$	$\{3k+2-k/2\}$

The set of vertex labels in P_{3k} is composed of three segments, from v_1 to v_k, from v_k to v_{2k+1} and from v_{2k+1} to v_{3k}. The set of edge labels in every segment is a continuous integer sequence. They are $[2k+2,3k]$, $[k,2k]$ and $[1,k-1]$.

From above construction, we obtain that both f and f' are injections. Hence the theorem holds.

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