Hamiltonian Cycles in N^2-Locally Connected Claw-Free Graphs

MingChu Li
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 3G3
Canada

Abstract

For a vertex v in a graph G, we denote by $N^2(v)$ the set $(N_1(N_1(v))-\{v\}) \cup N_1(v)=\{x \in V(G) : 1 \leq d(x,v) \leq 2\}$, where $d(x,v)$ denotes the distance between x and v. A vertex v is N^2-locally connected if the subgraph induced by $N^2(v)$ is connected. A graph G is called N^2-locally connected if every vertex of G is N^2-connected. A well-known result by Oberly and Sumner is that every connected locally connected claw-free graph on at least three vertices is Hamiltonian. This result was improved by Ryjacek using the concept of second-type neighborhood. In this paper, using the concept of N^2-locally connectedness, we show that every connected N^2-locally connected claw-free graph G without vertices of degree 1, which does not contain an induced subgraph H isomorphic to one of G_1, G_2, G_3, or G_4, is Hamiltonian, hereby generalizing the result of Oberly and Sumner (J. Graph Theory, 3 (1979) 351-356) and the result of Ryjacek (J. Graph Theory, 14(1990) 321-331).

1. Introduction

We deal with finite simple graphs in this paper. Our notation and terminology not mentioned here can be found in [1]. Let G be a graph. We denote by $\delta(G)$ the minimum degree of G. For a vertex v of G, the neighborhood of v, defined in the obvious sense, i.e., as the induced subgraph on the set of all vertices that are adjacent to v, will be called the neighborhood of the first type of v in G and denoted by $N_1(v,G)$ or briefly, $N_1(v)$. We say that an edge $xy \in E(G)$ is adjacent to v if $x \neq v \neq y$ and x or y is adjacent to v. We define the neighborhood of the second type of v in G (denoted by $N_2(v,G)$, or briefly, $N_2(v)$) as the edge-induced subgraph on the set of all edges that are adjacent to v. For a vertex v in G, we denote by $N^2(v,G)$ (or briefly, $N^2(v)$) the set $(N_1(N_1(v)) - \{v\}) \cup N_1(v)=\{x \in V(G) : 1 \leq d(x,v) \leq 2\}$, where $d(x,v)$ denotes the distance between x and v. For a vertex v of G, the subgraph of G induced by the set $N^2(v)$ is called a 2-order neighborhood and also denoted by $N^2(v)$. We say that a vertex v is locally connected if the $N_1(v)$ is a connected graph. G is called locally connected if every vertex of G is locally connected. A vertex v is N_2-locally connected if its second-type neighborhood $N_2(v)$ is connected. G is called N_2-locally connected if every vertex of G is N_2-locally connected. Analogously, a vertex v is N^2-locally connected if its 2-order neighborhood is connected. G is called N^2-locally connected if every vertex of G is N^2-locally connected. Obviously, every N_2-locally connected graph is N^2-locally connected, and every locally connected graph is N_2-locally connected. A graph G is called claw-free if it does not contain a copy of $K_{1,3}$ as an induced subgraph.

For a subgraph H of a graph G and a subset S of $V(G)$, we denote by $G - H$ and $G[S]$ the induced subgraphs of G by $V(G) - V(H)$ and S, respectively, and we denote by $N_H(S)$ the set of all vertices v in H adjacent to some vertex of S. Let $d_H(S)=|N_H(S)|$. For a cycle C with a fixed orientation of G, and two vertices x and y on C, we define the segment $C[x, y]$ to be the set of vertices on C from x to y (including x and y) and $C^-[y, x]$ to be a traversal of $C[x, y]$ in the opposite sense according to the orientation. Let $C(x, y) = C[x, y] - \{x, y\}$, and x^+ and x^- denote the successor and the predecessor of x according to the orientation, respectively.
We know the following result due to Oberly and Sumner.

Theorem 1 (Oberly and Sumner [6]). *Every connected locally connected claw-free graph on at least three vertices is Hamiltonian.*

Clark [4] showed the following result.

Theorem 2 (Clark [4]). *Every connected locally connected claw-free graph on at least three vertices is vertex pancyclic.*

Let G be a connected claw-free graph and let G_1, G_2, G_3 and G_4 be these graphs with 8 vertices and maximum degree 4 in Figure 1. Furthermore, assume that G_i is the induced subgraph of G such that $N_1(x, G)$ of every vertex x of degree 4 in G_i is disconnected and G_i has exactly four vertices of degree 4 for $i = 1, 2$, and assume that G_i is the induced subgraph of G such that $N_1(x, G)$ of every vertex x of degree 3 or 4 in G_i is disconnected and $N_1(y, G)$ of only one vertex y of degree 2 in G_i adjacent to two vertices of degree 3 in G_i is disconnected for $i = 3, 4$.

Z. Ryjacek [8] improved Theorem 1 and showed the following result.

Theorem 3 (Ryjacek [8]). *Every connected N_2-locally connected claw-free graph G without vertices of degree 1, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure 1), is Hamiltonian.*

The author [5] proved the following result.

Theorem 4 (M.Li [5]). *Every connected N_2-locally connected claw-free graph G with $\delta(G) \geq 3$, which does not contain an induced subgraph H isomorphic to either G_1 or G_2 (Figure 1), is vertex pancyclic.*

In this paper, we improve Theorems 1 and 3 and prove the following result.

Theorem 5. *Every connected N^2-locally connected claw-free graph G without vertices of degree 1, which does not contain an induced subgraph H isomorphic to one of G_1, G_2, G_3, or G_4 (Figure 1), is Hamiltonian.*

Let G be a graph such that for every $x \in V(G)$, $G[N(x)]$ is either a clique or a union of two disjoint cliques. Then G is a line graph of a triangle-free graph by Lemma 1 in [7] and is equal to its closure defined by Ryjacek [7]. We say that G is a *closed claw-free graph*.
Z.Ryjacek [7] introduced a closure concept in claw-free graphs and proved

the following very important result which is used in the proof of our theorem.

Theorem 6 (Ryjacek [7]). *If G is a claw-free graph, then there is a closed claw-free graph $cl(G)$ (called the closure of G) such that*

(1). G is a spanning subgraph of $cl(G)$, and

(2). the length of a longest cycle in both G and $cl(G)$ is the same.

In [8], Ryjacek made the following conjecture.

Conjecture 7 [8]. *Every 3-connected N_2-locally connected claw-free graph is Hamiltonian.*

We now make the following two conjectures.

Conjecture 8. *Every 3-connected N^2-locally connected claw-free graph is Hamiltonian.*

Conjecture 9. *Every connected N^2-locally connected claw-free graph G with $\delta(G) \geq 3$, which does not contain an induced subgraph H isomorphic to one of G_1, G_2, G_3, or G_4 (Figure 1), is vertex pancyclic.*

Remarks. Obviously, Conjecture 7 is a weaker version of Conjecture 8. If Conjecture 9 is true, then it generalizes Theorem 4.

2. Proof of Theorem 5

Assume that G be a graph which satisfies the conditions of Theorem 5. By Theorem 6, we know that the closure $cl(G)$ of G under Theorem 6 is a closed claw-free graph and also satisfies the conditions of the theorem. Note that G is Hamiltonian if and only if $cl(G)$ is Hamiltonian. Thus, without loss of generality assume that G is a closed claw-free graph. If Theorem 5 is false, then for every longest cycle C in G, an edge x_0u can be found such that u is on C while x_0 is not on C. If u is locally or N_2-locally connected, then we can obtain a contradiction by a similar proof to main Theorems in [6] and [8]. Hence suppose that u is not N_2-locally connected. Since G is N^2-locally connected, we can find a shortest path P in $N^2(u)$ from x_0 to one of u^+ or u^-. Since u^+x_0, $u^-x_0 \notin E(G)$ and $G[u, u^+, u^-, x_0] \neq K_{1,3}$, $u^+u^- \in E(G)$. We may without loss of generality assume that P is a path from x_0 to u^+ and that $u^- \notin V(P)$. Let the longest cycle C and the edge x_0u be chosen so that the path P is shortest possible and let $x_0, x_1, ..., x_k = u^+$ be its vertices. From the minimality of P we immediately obtain $x_ix_j \notin E(G)$ for $|i-j| > 1$. We further have the following claim.

Claim 1. x_{k-1} is not adjacent to u.

Suppose that $x_{k-1}u \in E(G)$. By the choice of C and x_0, we obtain that x_{k-1} is on C. Clearly, $ux^+_{k-1}, ux^-_{k-1} \notin E(G)$, otherwise, let $ux^-_{k-1} \in E(G)$. Then replacing on C the edge $x^-_{k-1}x_{k-1}$ by the path $x^-_{k-1}ux_{k-1}$ and the path u^-uu^+ by the edge u^+u^- we obtain a cycle C' of same length as C, and such that if we denote $u' = x_{k-1}$ then u' is a neighbor of u on C' and in $N_2(u)$ exists a path from x_0 to u' shorter than P. This is a contradiction with the choice of C and P. Similarly, $ux^+_{k-1} \notin E(G)$. Since $G[x_{k-1}, x^-_{k-1}, u, x^+_{k-1}] \neq K_{1,3}$, $x^+_{k-1}x^-_{k-1} \in E(G)$. Replacing on C the path $x^+_{k-1}x_{k-1}x^-_{k-1}$ by the edge $x^+_{k-1}x^-_{k-1}$ and the edge uu^+ by the path $ux_{k-1}u^+$ we again obtain a contradiction. So Claim 1 is proved.

Claim 2. $3 \leq k \leq 4$.

Since $x_0u^+, x_0u^- \notin E(G)$, $k \geq 2$. If $k \geq 5$, then $x_2, x_3 \notin N_1(u)$, otherwise, if $x_2 \in N_1(u)$, then we have that $G[u, x_2, x_k, x_0] = K_{1,3}$, a contradiction. Similarly, $x_3 \notin N_1(u)$. It follows that x_1, x_4 are in $N_1(u)$ (otherwise x_2, or $x_3 \notin N^2(u)$) and $k \geq 6$ since $x_{k-1}u \notin E(G)$. Hence $\{u, x_0, x_4, x_k\}$ induces $K_{1,3}$, a contradiction.

If $k = 2$, then $x_1 \in V(C)$. Considering $\{x_1, x_1^+, x_1^-, x_0\}$ we have $x_1^+x_1^- \in E(G)$, and using this edge we easily get a cycle C' of length at least $|V(C)| + 1$, a contradiction.

Claim 3. x_{k-1} is on C.

Let $x_{k-1} \notin V(C)$. If $k = 3$, then we easily see that $x_1 \in V(C)$ and so $x_1^+x_1^- \in E(G)$. Replacing on C the path $x_1^-x_1x_1^+$ by the edge $x_1^-x_1^+$ and the edge uu^+ by the path $ux_0x_1x_2u^+$ we obtain a cycle C' longer than C, a contradiction. Thus $k = 4$ by Claim 2.

We have that at least one of x_1, x_2 is on C since otherwise we easily construct a longer cycle than C. If $x_1 \notin V(C)$, then $x_2 \in V(C)$ and $x_2^-x_2^+ \in E(G)$. Replacing on C the path $x_2^-x_2x_2^+$ by the edge $x_2^-x_2^+$ and the edge uu^+ $(=ux_4)$ by the path $ux_0x_1x_2x_3u^+$, we obtain a cycle longer than C. Thus $x_1 \in V(C)$, and then $x_1^-x_1^+ \in E(G)$ since $G[x_1, x_1^-, x_1^+, x_0] \neq K_{1,3}$. If $x_2 \notin V(C)$, then replacing on C the path $x_1^-x_1x_1^+$ by the edge $x_1^-x_1^+$ and the edge uu^+ $(=ux_4)$ by the path $ux_0x_1x_2x_3u^+$, we get a contradiction. Thus $x_2 \in V(C)$, and then $x_2^+x_2^- \in E(G)$ since $G[x_2, x_2^+, x_2^-, x_3] \neq K_{1,3}$. If x_1 and x_2 are consecutive, then, e.g., $x_2 = x_1^-$, G has a longer cycle $C[x_1^+, u]x_0x_1C^-[x_2^-, u^+]x_3x_2x_1^+$ than C. Thus x_1 and x_2 are not consecutive. Replacing on C the path $x_1^-x_1x_1^+$ by the edge $x_1^-x_1^+$, the path $x_2^-x_2x_2^+$ by the edge $x_2^-x_2^+$ and the edge uu^+ by the path $ux_0x_1x_2x_3u^+$, we get a cycle C' longer than C, a contradiction. Thus Claim 3 is true.

By Claim 2, we consider two cases.

Case 1. $k = 3$.

Then we easily get the following claim.

Claim 4. x_1 is on C, $x_2^+x_2^- \notin E(G)$ and $x_1^-x_1^+ \in E(G)$.

Note that $x_2 \in V(C)$ by Claim 3. If $x_1 \notin V(C)$, then $x_2^+x_2^- \in E(G)$. If $x_2 = x_3^+$, then replacing on C the path $ux_3x_3^+$ by the path $ux_0x_1x_3^+$, we obtain a cycle of length $|V(C)| + 1$ in G. Thus $x_2 \neq x_3^+$. Replacing on C the path $x_2^-x_2x_2^+$ by the edge $x_2^-x_2^+$ and the edge ux_3 by the path $ux_0x_1x_2x_3$ we obtain a cycle longer than C, a contradiction. Thus $x_1 \in V(C)$. Thus $x_1^-x_1^+ \in E(G)$ since $G[x_1, x_1^-, x_1^+, x_0] \neq K_{1,3}$.

If $x_2^+x_2^- \in E(G)$, then $x_2 \neq x_1^+$ since otherwise G has a cyclea $C[x_1^+, u]x_0C^-[x_1, x_3(= u^+)]x_1^+$ of length $|V(C)| + 1$. We have that $x_2 \neq x_1^-$ since otherwise G has a cycle $C[x_1^+, u]x_0x_1C^-[x_2^-, u^+(= x_3)]x_2x_1^+$ of length

$|V(C)|+1$. If $x_2 = x_3^+$, then G has a cycle $C[x_1^+, u]x_0x_1x_2x_3C[x_2^+, x_1^-]x_1^+$ of length $|V(C)|+1$. Thus $x_2 \neq x_3^+$. Replacing on C the path $x_1^-x_1x_1^+$ by the edge $x_1^-x_1^+$, the path $x_2^-x_2x_2^+$ by the edge $x_2^-x_2^+$ and the edge uu^+ ($=ux_3$) by the path $ux_0x_1x_2u^+$, we obtain a cycle of length $|V(C)|+1$. This contradiction shows $x_2^-x_2^+ \notin E(G)$.

Claim 5. We can assume that the edge x_1x_2 is on C.

Otherwise, since $\{x_1, x_2^-, x_2^+, x_2\}$ would not induce $K_{1,3}$, x_2^- or x_2^+ is in $N_1(x_1)$ (say $x_2^-x_1 \in E(G)$). Replacing on C the path $x_1^-x_1x_1^+$ by the edge $x_1^-x_1^+$ and the edge $x_2^-x_2$ by the path $x_2^-x_1x_2$ we get a cycle C' of the same length as C, and x_1x_2 is an edge on C'. So the claim holds.

Claim 6. We have $x_1^-{=}x_2$.

Otherwise, $x_2 = x_1^+$, deleting the edges $x_1x_1^+$ and ux_3 and adding the edge x_3x_2 and the path x_1x_0u we obtain a cycle C' longer than C, a contradiction. Thus Claim 6 is proved.

Claim 7. $ux_3^+, ux_2^-, u^-x_3^+, x_2^-x_1, x_2^-x_1^+, x_1^+x_3^+ \notin E(G)$ but $x_3^+x_2, x_3x_2^- \in E(G)$.

If $ux_3^+ \in E(G)$, then deleting from C the edges x_2x_1, u^-u, ux_3 and $x_3x_3^+$, and adding the edges x_2x_3, x_3u^- and x_3^+u and the path ux_0x_1, we would get a cycle C' longer than C, a contradiction. Similarly, x_2x_1, ux_2^-, $u^-x_3^+, x_2^-x_1^+, x_1^+x_3^+ \notin E(G)$. Obviously, $|C[x_3, x_2]| \geq 3$. Since $\{x_3, u, x_3^+, x_2\}$ and $\{x_2, x_1, x_2^-, x_3\}$ can not induce $K_{1,3}$, we obtain $x_3^+x_2, x_3x_2^- \in E(G)$. Thus Claim 7 is proved.

Now we complete the proof of Case 1.

Since G is a closed claw-free graph, $N_1(x_1)$, $N_1(x_2)$, $N_1(u)$ and $N_1(x_3)$ are disconnected since otherwise, e.g., $N_1(x_2)$ is connected, we have that $N_1(x_2)$ is a clique. Thus $u^+x_1 \in E(G)$, which contradicts the choice of P. Note that if $u^-x_1^+ \in E(G)$, then $N_1(u^-)$ and $N_1(x_1^+)$ are disconnected since otherwise, e.g.,$N_1(u^-)$ is connected, we have $ux_1^+ \in E(G)$ since G is closed. Thus G has a cycle $C[u^+, x_1]x_0uC[x_1^+, u^-]u^+$ of length $|V(C)|+1$, a contradiction.

If $ux_1 \in E(G)$, then the induced subgraph of G on the set $\{x_0, x_1, x_2, x_3, u, u^-, x_3^+, x_1^+\}$ is isomorphic to either G_1 or G_2 (Figure 1), a contradiction.

If $ux_1 \notin E(G)$, then $N_1(x_0)$ is disconnected since G is a closed claw-free graph. Thus $\{u, x_0, x_1, x_2, x_3, u^-, x_3^+, x_1^+\}$ would induce the subgraph isomorphic to either G_3 or G_4 (Figure 1). So Case 1 is proved.

Case 2. $k=4$

Then $x_3 \in V(C)$ by Claim 3 and we have the following claim.

Claim 8. x_2 is on C.

Otherwise, we have $x_3^-x_3^+ \in E(G)$. If $x_1 \notin V(C)$, then replacing on C the path $x_3^- x_3 x_3^+$ by the edge $x_3^- x_3^+$ and the edge ux_4 by the path $ux_0x_2x_3x_4$ we obtain a cycle C' longer than C. Hence $x_1 \in V(C)$. It follows $x_1^- x_1^+ \in E(G)$. Replacing the path $x_1^- x_1 x_1^+$ by the edge $x_1^- x_1^+$, the path $x_3^- x_3 x_3^+$ by the edge edge $x_3^- x_3^+$ and the edge ux_4 by the path $ux_0x_1x_2x_3x_4$ we obtain a cycle C' longer than C.

Claim 9. x_2 is not adjacent to u but x_1 is adjacent to u in G.

If $ux_2 \in E(G)$, since x_0x_2, x_2x_4, x_0x_4 are not edges, $\{u, x_4, x_2, x_0\}$ would induce $K_{1,3}$, a contradiction. Since $x_2, x_3 \notin N_1(u)$, $x_1 u \in E(G)$, otherwise, $x_2, x_3 \notin N^2(u)$. So the claim holds.

Claim 10. x_1 is on C.

Otherwise, by a similar proof to Claims 5 and 6 we can assume that x_2x_3 is the edge on C and $x_2^- = x_3$. By Claim 9, $ux_1 \in E(G)$. Thus we are in Case 1, where x_1 plays the role of x_0. So Claim 10 holds.

Claim 11. We can assume that the edge x_1x_2 is on C (say $x_1^- = x_2$).

Otherwise, without loss of generality assume that $x_2 \in C[x_4^+, x_1^-]$ (and the proof of the other case: $x_2 \in C[x_1^+, u^-]$ is similar). Since $x_1^+ u, ux_2 \notin E(G)$ and $\{x_1, x_1^+, x_2, u\}$ would not induce $K_{1,3}$, $x_1^+ x_2 \in E(G)$. Obviously, $x_2 x_1^-$ is not a edge on C, otherwise, replacing on C the path $x_1^+ x_1 x_1^- x_2$ by the path $x_1^+ x_1^- x_1 x_2$ we obtain a cycle C' of the same length as C and x_1x_2 is on C'. Obviously, $x_2^- x_2^+ \notin E(G)$, otherwise, replacing on C the path $x_2^- x_2 x_2^+$ by the edge $x_2^- x_2^+$ and the edge $x_1 x_1^+$ by the path $x_1 x_2 x_1^+$ we again get a contradiction. Since $\{x_2, x_2^+, x_2^-, x_1\}$ would not induce $K_{1,3}$, we have that either $x_2^- x_1$ or $x_2^+ x_1$ is an edge. If $x_2^- x_1 \in E(G)$, then replacing on C the edge $x_2^- x_2$ by the path $x_2^- x_1 x_2$ and the path $x_1^- x_1 x_1^+$ by the edge $x_1^- x_1^+$ we obtain a cycle C' of the same length as C and the edge x_1x_2 is on C'. So $x_2^+ x_1 \in E(G)$. Replacing on C the edge $x_2 x_2^+$ by the path $x_2 x_1 x_2^+$ and the path $x_1^- x_1 x_1^+$ by the edge $x_1^- x_1^+$ we obtain a cycle C' of the same length as C and the edge x_1x_2 is on C'. So the Claim holds.

Claim 12. $x_4x_3x_2$ is not a path on C.

Assume that $x_4x_3x_2$ is a path on C. If $(u^-)^- x_3 \in E(G)$, then we obtain a cycle

$$C[x_1^+, (u^-)^-] x_3 x_4 u^- u x_0 x_1 x_2 x_1^+$$

longer than C, a contradiction. Thus $(u^-)^- x_3 \notin E(G)$. Similarly, u, u^-, x_1^+, $(x_1^+)^+ \notin N_1(x_3)$. It follows from Theorem 6 that $N_1(x_2)$, $N_1(x_4)$, $N_1(u)$ and $N_1(x_1)$ are disconnected. Since $x_2x_4 \notin E(G)$ and G is a closed claw-free graph, $N_1(x_3)$ is disconnected. Hence $\{u, x_0, x_1, x_2, x_3, x_4, u^-, x_1^+\}$ induces the subgraph isomorphic to either G_4 or G_3, a contradiction. Thus Claim 12 is proved.

Now we complete the proof of Case 2.

By Claim 3, we have $x_3 \in V(C)$. Since G is a closed claw-free graph and $x_0u^+, x_0x_2, x_1x_3, x_2x_4 \notin E(G)$, $N_1(u)$ and $N_1(x_i)$ $(i = 1, 2, 3)$ are disconnected. If $N_1(x_4)$ is connected, then $N_1(x_4)$ is a clique. Specifically, $ux_3, x_3x_4^+ \in E(G)$. If $x_3^-u \in E(G)$, then replacing $x_3^-x_3$ by $x_3^-ux_3$ and u^-uu^+ by u^-u^+ we are in Case 1. So $x_3^-u \notin E(G)$. Similarly, $x_3^+u \notin E(G)$. Considering a claw centered at x_3, we have $x_3^-x_3^+ \in E(G)$. However, replacing $x_3^-x_3x_3^+$ by $x_3^-x_3^+$ and $u^-ux_4x_4^+$ by $u^-x_4ux_3x_4^+$ we are again in Case 1. It shows that $N_1(x_4)$ is disconnected. Thus $\{u, x_1, x_2, x_3, x_4, x_0, x_1^+, u^-\}$ induces a subgraph isomorphic to either G_3 or G_4. So Case 2 is proved. Therefore, we have completed the proof of Theorem 5.

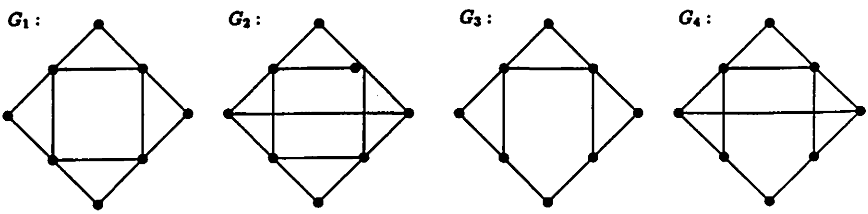

Figure 1

Acknowledgments

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