Arc Signed Graphs of Oriented Graphs

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ABSTRACT. A *signed graph* is an unoriented graph with a given partition $E^- \bigcup E^+$ of its edge-set. We define the *arc signed graph* $A(G)$ of an oriented graph G (G has no multiple arcs, opposite arcs, and loops). The arc signed graphs are like to the line graphs. We prove both a Krausz-type characterization and a forbidden induced subgraph characterization (like the theorem of Beineke and Robertson on line graphs). Unlike line graphs, there are infinitely many minimal forbidden induced subgraphs for the arc signed graphs. Nevertheless, the arc signed graphs are polynomially recognizible. Also, we obtain a result similar to Whitney's theorem on line graphs.

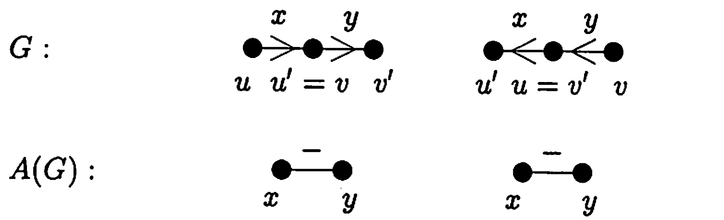
We will consider oriented graphs without multiple arcs, opposite arcs, and loops. A *signed graph* $H = (V, E^-, E^+)$ is an unoriented graph with a given partition $E^- \bigcup E^+$ of its edge-set $E(H)$. If $e = uv \in E^*$, where $* \in \{-,+\}$, then the edge e is called an $(*)$-*edge* (or e has the *type* $*$), and both the vertices u and v are called $(*)$-*adjacent.*

The *arc signed graph* $A(G)$ of an oriented graph $G = (V, X)$ is a signed graph with the vertex-set X (the arc-set of G) in which vertices $x = (u, u')$ and $y = (v, v')$ are

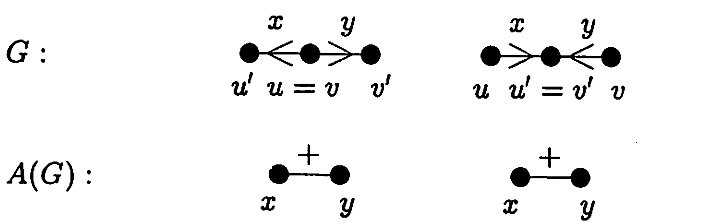
(i) $(-)$-adjacent if and only if either $u' = v$ and $u \neq v'$ or $u = v'$ and $u' \neq v$ (the arcs x and y have the *same* direction in G, see figure 1(a));

*Supported by RUTCOR, INTAS and the Belarus Government (Project INTAS-BELARUS 970093)

(ii) (+)-adjacent if and only if either $u = v$ and $u' \neq v'$ or $u' = v'$ and $u \neq v$ (the arcs x and y have *opposite* directions in G, see figure 1 (b)).



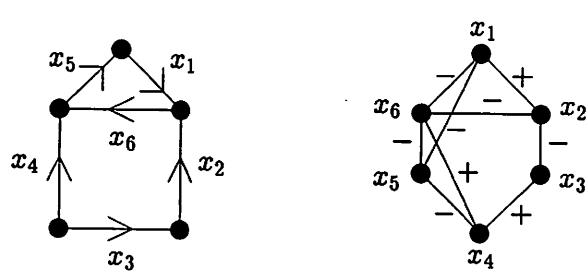
(a) x and y have the same direction in G



(b) x and y have opposite directions in G

Figure 1. The adjacency rules for the arc signed graph $A(G)$ of an oriented graph G

A signed graph H is called an *arc signed graph* if $H = A(G)$ for some oriented graph G. An example of an arc signed graph is shown in figure 2.



G $A(G)$

Figure 2. The arc signed graph $A(G)$ of an oriented graph G

1 Krausz-type characterization of the arc signed graphs

A signed graph is said to be *complete* if any two distinct vertices are adjacent (i.e., either $(-)$-adjacent or $(+)$-adjacent). A signed graph is called $(+)$-*complete* if any two distinct vertices are $(+)$-adjacent. A complete signed graph B is *bicomplete* if there exists a partition $V \cup W = V(B)$ such that

1) both V and W induce $(+)$-complete graphs;

2) any two vertices $v \in V$ and $w \in W$ are $(-)$-adjacent.

The sets V and W are called *parts* of B (one of them may be empty). Note that the unordered pair $\{V, W\}$ is uniquely defined.

Denote by O_n the edgeless signed graph of order $n \geq 1$. Let $K_3(-,-,-)$ be a complete signed graph of the order three without $(+)$-edges and $K_3(-,+,+)$ be a complete signed graph of the order three with exactly one $(-)$-edge.

A $(+)$-*odd cycle* (respectively, $(+)$-*even cycle*) is a cycle which contains an odd (respectively, an even) number of $(+)$-edges.

Claim A. *A signed graph H is bicomplete if and only if it does not contain O_2, $K_3(-,-,-)$ and $K_3(-,+,+)$ as induced subgraphs.*

Proof: Clearly, O_2, $K_3(-,-,-)$ and $K_3(-,+,+)$ are not bicomplete signed graphs. Hence they can not be induced subgraphs of a bicomplete signed graph.

Conversely, since H does not contain O_2, H is a complete signed graph. The absence of $K_3(-,+,+)$ implies that all $(+)$-edges of H constitute a signed subgraph H^+ which is a disjoint union of complete components. If H^+ has at least three components, then H has $K_3(-,-,-)$, a contradiction. So H^+ has at most two components. Thus, H is a bicomplete signed graph. □

Now we give a characterization of the arc signed graphs which is similar to Krausz' theorem [3] on line graphs.

Theorem 1. *A signed graph $H = (V, E^-, E^+)$ is an arc signed graph if and only if the following conditions hold:*

(1) *the set $\mathcal{B} = \{B_0, B_1, \ldots, B_t\}$ of all maximal (by inclusion) bicomplete subgraphs of H induces a partition of the edge-set of H;*

(2) *every vertex of H belongs to at most two subgraphs in $\mathcal{B}$;*

(3) *H does not contain induced $(+)$-odd cycles C_n of the order $n \geq 4$.*

Proof: Necessity. Let H be an arc signed graph, i.e., $H = A(G)$ for an oriented graph G. Without loss of generality we may assume that G has neither isolated vertices nor isolated arcs.

Let $V' = \{v_0, v_1, \ldots, v_t\}$ be the set of all vertices in G which have degrees at least two. For any vertex v_i the set X_i of all arcs incident to v_i constitutes a bicomplete subgraph B_i in H.

It easy to see that a set $Y \subseteq X(G)$ induces a maximal complete subgraph in H in the following two cases only:

a) $Y = X_i$ for some $i \in \{0, 1, \ldots, t\}$;

b) Y is a triangle in G.

We show that the case b) produces non-bicomplete subgraph in H. Indeed, if Y is an oriented triple (i.e., $Y = \{(u,v),(v,w),(w,u)\}$) then Y constitutes $K_3(-,-,-)$ in H. If Y is a transitive triple (i.e., $Y = \{(u,v),(v,w),(u,w)\}$) then Y constitutes $K_3(-,+,+)$ in H. Both $K_3(-,-,-)$ and $K_3(-,+,+)$ are not bicomplete graphs (Claim A). Thus, the necessity of (1) is proved.

Necessity of (2) follows from the fact that any arc of G belongs at most two sets $X_0, X_1, \ldots, X_t$.

Now we consider the condition (3). Let C_n $(n \geq 4)$ be an induced cycle in H. The vertices of C_n are arcs in G which constitute a cycle C (in G) of order n. If we consider C as an oriented graph itself then the total number s of sources and sinks in C is even (a *source* is a vertex with zero in-degree while a *sink* is a vertex with zero outdegree). Therefore C_n has an even number of (+)-edges (which equals to s), i.e., (3) holds.

Sufficiency. We will construct an oriented graph G such that $H = A(G)$. At first we define the underlying unoriented graph G^* of G.

To every maximal bicomplete subgraph $B_i \in \mathcal{B}$ we introduce a vertex v_i in $V(G^*)$ thus defining a set $V' \subseteq V(G^*)$. Vertices v_i and v_j $(0 \leq i \neq j \leq t)$ are adjacent in $V(G^*)$ if and only if B_i and B_j have a common vertex (in H).

Further we define $V'' = V(G^*) \backslash V'$. For any vertex $u \in V(H)$ which belongs to exactly one subgraph B_i, we introduce a new pendant vertex $v_u \in V''$ and an edge $v_i v_u \in E(G^*)$. Now the graph G^* is completely defined. The conditions (1) and (2) imply correctness of the definition of G^* and its uniqueness.

Finally, we transform G^* into G, i.e., we define an orientation of each edge in G^*. We may assume that G^* is a connected graph.

Claim B. *If G^* is a tree then any orientation of an arbitrary edge $x \in E(G^*)$ indices a unique orientation G of G^* such that $A(G) = H$.*

Proof: In case of $E(G^*) = \{x\}$ the statement is trivial. Otherwise, $x = v_i v$, where $v_i \in V'$, i.e., v_i corresponds to a maximal bicomplete subgraph $B_i \in \mathcal{B}$ (v does not necessarily correspond to some $B_j \in \mathcal{B}$). Let B_i has parts V_i and W_i, and $x \in V_i$.

Firstly, let $x = (v_i, v)$ be a chosen orientation of x. Then each vertex $y \in V(B_i)\backslash\{x\}$ must be oriented either $y = (v_i, *)$ in case of $y \in V_i$ or $y = (*, v_i)$ in case of $y \in W_i$. Indeed, if $y \in V_i$ then xy is a (+)-edge of H. So x and y must have opposite directions in G (see figure 1(b)). Similarly, if $y \in W_i$ then xy is a (−)-edge of H. So x and y must have the same direction in G (see figure 1(a)).

Now let $x = (v, v_i)$ be a chosen orientation of x. Then each vertex $y \in V(B_i)\backslash\{x\}$ must be oriented either $y = (*, v_i)$ in case of $y \in V_i$ or $y = (v_i, *)$ in case of $y \in W_i$.

Since G^* is a tree, for any $z \in E(G^*)$ there is a unique path $P = (u_1, u_2, \ldots, u_k)$ with $x = u_1u_2$ and $z = u_{k-1}u_k$. An orientation of x induces a unique orientation of u_2u_3. In general, orienting $u_{i-1}u_i$ $(i = 2, 3, \ldots, k-1)$ induces a unique orientation of u_in_{i+1}. Therefore $z = u_{k-1}u_k$ obtain a uniquely defined orientation.

Thus, we have defined an orientation G of G^* (induced by an orientation of x). By the construction, $H = A(G)$. □

By Claim B, if G^* is a tree then the proof is finished.

Now let G^* be an arbitrary connected graph. Choose a spanning tree T^* in G^* and an orientation T of T^* induced by an orientation of an edge $x \in E(T^*)$.

Consider an arbitrary edge $y = v_0v_1 \in E(G^*)\backslash E(T^*)$. The orientation of an edge $e_0 = v_0w \in E(T^*)$ (in T) induces an orientation y_0 of y. Similarly, the orientation of an edge $e_1 = v_1w' \in E(T^*)$ (in T) induces an orientation y_1 of y. The situation $y_0 \neq y_1$ will be called a *conflict*. We show that the condition (3) of Theorem 1 implies the absence of conflicts.

Claim C. *An orientation of an arbitrary edge $x \in E(G^*)$ induces an orientation the whole set $E(G^*)$ without conflicts.*

Proof: Suppose that there is a conflict in an edge $y = v_0v_1$. Consider a unique cycle $C = (v_0, v_1, \ldots, v_k)$, $k \geq 2$, which contains y in $T^* + y$. It is clear that an orientation of any edge x in C produces a conflict in any edge $z \in E(C)\backslash\{x\}$. Such a cycle C is called a *conflict cycle.*

Denote $y_i = v_iv_{i+1}$, $i = 0, 1, \ldots, k$ (indices are taken by $\mod (k+1)$). Let B_i be a maximal bicomplete subgraph in H which corresponds to the vertex v_i, $i = 0, 1, \ldots, k$.

Firstly we consider the case $k = 2$, i.e., C is a triangle. The condition (1) implies that the set $Y = \{y_0, y_1, y_2\} = V(C)$ does not induce a bicomplete subgraph in H (since $Y \cap B_0 = \{y_0, y_2\}$, $Y \neq B_0$, and two bicomplete subgraphs in H do not have a common edge). By Claim A, the number of (+)-edges in $H(Y)$ is either 0 or 2. If $H(Y)$ does not contain (+)-edges then C is an oriented triple and C is not a conflict cycle, a contradiction. Similarly, if $H(Y)$ has exactly two (+)-edges then C is a transitive triple and C is also not a conflict cycle, a contradiction.

We have proved that $k \geq 3$.

We claim that $A(C)$ is an induced cycle in H. Suppose that there exists a chord $z = y_i y_j$ of $A(C)$. Let B be a maximal bicomplete subgraph of H which contains z. The vertex y_i is contained in the bicomplete subgraphs B_i, B_{i+1} and B. The condition (2) implies that $B \in \{B_i, B_j\}$ (since $B_i \neq B_{i+1}$). Similarly, y_j belongs to B_j, B_{j+1} and B. By (2), $B \in \{B_i, B_j\}$. Then $\{B_i, B_{i+1}\} \cap \{B_j, B_{j+1}\} \neq \emptyset$. We arrive to a contradiction: the vertices $v_0, v_1, \ldots, v_k$ of C corresponds to pairwise distinct maximal bicomplete subgraphs $B_0, B_1, \ldots, B_k$ of H.

Thus, $A(C)$ is an induced cycle of the length $k+1 \geq 4$ in H.

It can be seen that C is an conflict cycle if and only if its image $A(C) = (y_0, y_1, \ldots, y_k)$ (in H) is a (+)-odd cycle. Indeed, the number of (+)-edges in $A(C)$ is equal to the total number of sources and sinks in C. By (3), the number of (+)-edges in $A(C)$ is even. Hence C is not a conflict cycle, a contradiction. □

By Claim C, the orientation G of G^* induced by any edge of G^* has no conflicts. By the construction, G is well defined, and $H = A(G)$. □

For an oriented graph G, let G° be the *converse* of G, i.e., $V(G^\circ) = V(G)$ and $X(G^\circ) = \{(u,v)\colon (v,u) \in X(G)\}$.

The following result is an analogue of Whitney's theorem [4] on line graphs.

Corollary 2. *If H is a connected arc signed graph then there are at most two (up to isomorphism) oriented graphs G_1 and G_2 without isolated vertices such that $H \cong A(G_1)$ and $H \cong A(G_2)$. Moreover $G_1 \cong G_2^\circ$.*

2 A forbidden induced subgraph characterization of arc signed graphs

Here we give a characterization of the arc signed graphs which is an analogue of Beineke's theorem [1] (it was also proved by Robertson) on line graphs.

Theorem 3. *A signed graph H is an arc signed graph if and only if H does not contain induced subgraphs $H_1, H_2, \ldots, H_{28}$ (figure 3) and induced (+)-odd cycles C_n of the order $n \geq 4$.*

Proof: Necessity. It can be directly checked that the set of all maxmal bicomplete subgraphs of H_i ($i = 1, 2, \ldots, 28$) does not satisfy either (1) or (2) of Theorem 1. By Theorem 1, $H_1, H_2, \ldots, H_{28}$ are not arc signed graphs. Also, by Theorem 1, induced (+)-odd cycles C_n ($n \geq 4$) are not arc signed graphs.

Sufficiency. We prove that $H = (V, E^-, E^+)$ satisfies to both the conditions (1) and (2) of Theorem 1.

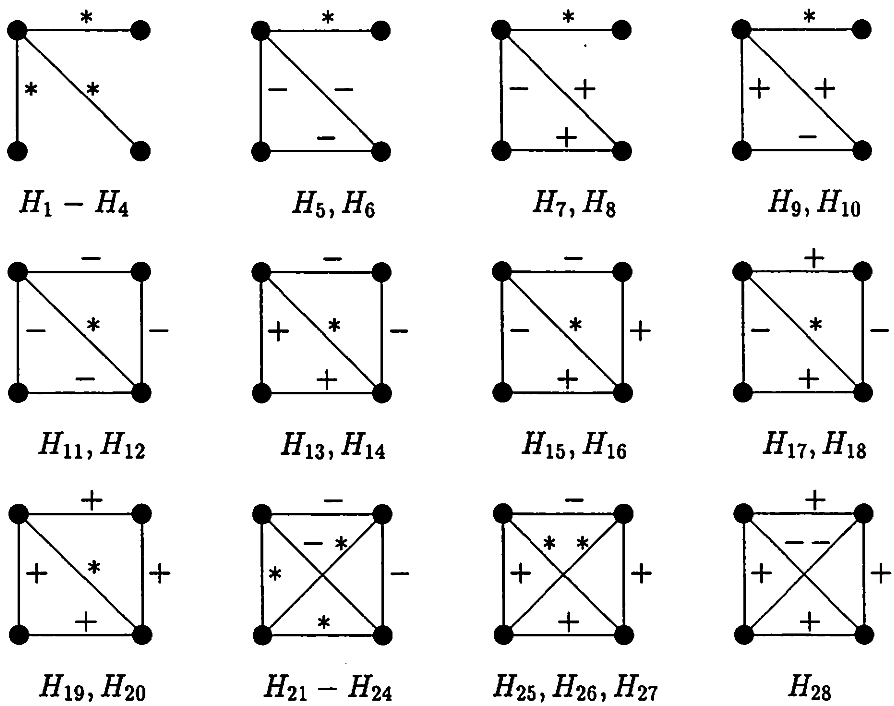


Figure 3. The symbol $*$ may be either $-$ or $+$

Claim D. *The set $\{H_1, H_2, \ldots, H_{28}\}$ (figure 3) can be divided into the following subsets:*

$S_1 = \{H_1, H_2, H_3, H_4\}$: $K_{1,3}$ with arbitrary types of edges;
$S_2 = \{H_5, H_6, \ldots, H_{10}\}$: $\overline{K_1 \cup P_3}$ with the (+)-even 3-cycle;
$S_3 = \{H_{11}, H_{12}, \ldots, H_{20}\}$: $K_4 - e$ with the (+)-even 4-cycle;
$S_4 = \{H_{21}, H_{22}, \ldots, H_{28}\}$: K_4 which contains a (+)-even 3-cycle.

Proof: The proof is straightforward and hence omitted.

Suppose that the condition (1) of Theorem 1 does not hold. It can be seen that H contains two maximal bicomplete graphs B_1 and B_2 ($B_1 \neq B_2$) with a common edge uv.

Since $V(B_1) \not\subseteq V(B_2)$, there is a vertex $w \in V(B_1) \backslash V(B_2)$. By the maximality of B_2, the set $V(B_2) \cup \{w\}$ does not induce a bicomplete subgraph. By Claim A, there exists a set $Z \subseteq V(B_2) \cup \{w\}$ which induces a subgraph $F \in .\{O_2, K_3(-,-,-), K_3(-,+,+)\}$. Clearly, $w \in Z$.

Case 1: $F \cong O_2$

Then $Z = \{w, x\}$, where $x \in V(B_2)$.

For an edge e of H, put $m(e) = 1$ if and only if $e \in E^+(H)$, and $m(e) = 0$ if and only if $e \in E^-(H)$. Since B_1 is a bicomplete subgraph, $uw, vw \in E(B_1)$ and, by Claim A,

$$m_1 = m(uv) + m(uw) + m(vw) \text{ is odd.}$$

Similarly, $ux, vx \in E(B_2)$ and

$$m_2 = m(uv) + m(ux) + m(vx) \text{ is odd.}$$

Therefore $m_1 + m_2$ is even and

$$m(uw) + m(vw) + m(ux) + m(vx) \text{ is even.}$$

Thus, $H(u, v, w, x)$ belongs to S_3 (Claim D), a contradiction.

Case 2: $F \in \{K_3(-,-,-), K_3(-,+,+)\}$.

Denote $Z = \{w, x, y\}$. Clearly, $\{x, y\} \neq \{u, v\}$. We may assume that $u \notin \{x, y\}$. Then $\{u, w, x, y\}$ induces a complete graph K_4 with a (+)-even triangle F, i.e., $H(u, v, w, x) \in S_4$ (Claim D), a contradiction.

Now we have shown that the condition (1) of Theorem 1 holds.

Suppose that the condition (2) of Theorem 1 is not true, i.e., there exists a vertex $u \in V(H)$ which belongs to three (pairwise distinct) maximal bicomplete subgraphs B_1, B_2 and B_3.

Consider vertices $v \in V(B_1) \backslash \{u\}$, $w \in V(B_2) \backslash \{u\}$ and $x \in V(B_3) \backslash \{u\}$. By the condition (1) of Theorem 1, there are no two distinct bicomplete subgraphs with at least two common vertices. Hence v, w and x are pairwise distinct. Moreover, each of the sets $\{u, v, w\}$, $\{u, v, x\}$, and $\{u, w, x\}$ induces a non-bicomplete subgraph. So each of these sets induces either P_3 or (+)-even triangle (by Claim A).

Put $Q = \{v, w, x\}$ and $U = \{u, v, w, x\}$.

If $E(Q) = \emptyset$ then $H(U) \in S_1$.

If $E(Q) = \{vw\}$ then the triangle $T_1 = \{u, v, w\}$ is (+)-even (by Claim A) and $H(U) \in S_2$.

If $E(Q) = \{vw, wx\}$ then both T_1 and $T_2 = \{u, w, x\}$ are (+)-even. Hence the 4-cycle (u, v, w, x) is (+)-even and $H(U) \in S_3$.

If $E(Q) = \{vw, wx, xv\}$ then $H(U) \in S_4$ (since T_1 is (+)-even).

Thus, H satisfies to both (1) and (2) of Theorem 1. By Theorem 1, H is an arc signed graph. □

Corollary 4. *It is possible to check in polynomial time if a given signed graph H is an arc signed graph and, if so, to construct an oriented graph G such that $H = A(G)$.*

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