REPRESENTING FINITE GROUPS AS REGULAR AUTOMORPHISM GROUPS OF COMBINATORIAL STRUCTURES

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Abstract

Given a regular action of a finite group G on a set V, we consider the problem of the existence of an incidence structure $\mathcal{I} = (V, \mathcal{B})$ on the set V whose *full* automorphism group $Aut(\mathcal{I})$ is the group G in its regular action. Using results on graphical and digraphical regular representations ([2, 7], [1]), we show the existence of such an incidence structure for all but four small finite groups.

1 Introduction

In our paper, we take up the problem of classifying finite groups G for which there exists an incidence structure $\mathcal{I}$ such that the full automorphism group of the combinatorial structure is the group G acting regularly on the vertices of the structure. We show the existence of such structures for all but very few finite groups of small orders.

This problem is closely related to the so called GRR-problem – the problem of existence of an (unoriented) graph $\Gamma = (V, E)$ with the property $Aut(\Gamma) = G$ for a given finite group G acting regularly

ARS COMBINATORIA 62(2002), pp. 51-64

on the set V. The complete list of finite groups admitting a GRR includes almost all finite groups with the exception of abelian groups of exponent at least 3, generalized dicyclic groups, and a list of thirteen sporadic small groups of order not larger than 32 (for a complete classification of finite groups allowing the existence of a GRR, see, for instance, [2, 7]).

The DRR-problem is the problem of classification of those finite groups that allow a regular representation as the full automorphism group of a digraph. The DRR-problem has been fully answered in [1], where the author showed that each finite group different from $\mathcal{Z}_2^2$, $\mathcal{Z}_2^3$, $\mathcal{Z}_2^4$, $\mathcal{Z}_3^2$ and the quaternion group admits a DRR. The proof of this result is considerably simpler than the classification of finite groups admitting GRR's, due to the fact that digraph automorphisms have to preserve oriented edges, and thus digraphs naturally tend to have fewer automorphisms than their underlying (unoriented) graphs.

In our paper, we do not have the advantage of oriented blocks. Nevertheless, we will be able to use Babai's results on DRR's as well as the classification of finite groups admitting GRR's.

Beside proving the results presented here, it is also our intention to demonstrate the unifying features of these topics and the techniques used.

2 Preliminaries

All the groups and group actions considered in our paper are finite. A (finite) group G acting faithfully on a (finite) set V is said to act *regularly* if its action on V is transitive and the stabilizer G_x of a (any) vertex $x \in V$ is trivial. In terms of the elements of V, this is equivalent to the condition that for each pair of elements $x, y \in V$ there exists a unique element $g \in G$ mapping x to y, $x^g = y$. Moreover, if the group G acts regularly on the set V, there exists a one-to-one correspondence $\varphi : V \longrightarrow G$ that allows one to identify the elements of V with the elements of G and the action of G on V can be represented as an action of G on its own elements via left translations $\varphi(a^g) = g \cdot \varphi(a)$, for all $a \in V$ and $g \in G$. Thus, a regular permutation representation of G can be thought of as an action of G on itself and we shall always assume this. We denote the left-regular permutation representation of G by G_L.

Although one could find several more general definitions of incidence structures, we shall use the following very simple definition : an *incidence structure* on a set V is any ordered pair $\mathcal{I} = (V, \mathcal{B})$, where $\mathcal{B}$ is a family of subsets of V, $\mathcal{B} \subseteq \mathcal{P}(V)$. Thus, an incidence structure is a set with a system of subsets called *blocks* and we do not allow for repeated appearances of elements in blocks or repeated appearances of the same block in $\mathcal{B}$. An *automorphism* of $\mathcal{I}$ is a permutation of the elements of V that preserves the blocks of $\mathcal{I}$, i.e., a permutation $\psi \in Sym_V$ is an automorphism of $\mathcal{I}$ if $\psi(B) \in \mathcal{B}$, for all $B \in \mathcal{B}$ (and $\psi(B') \notin \mathcal{B}$ for subsets B' of V that do not belong to $\mathcal{B}$). The group of all automorphisms of $\mathcal{I}$ will be denoted by $Aut(\mathcal{I})$. We compose the operators from right to left.

The definition of an incidence structure we introduced here is a very general one and it includes many well-known combinatorial structures as special cases. Let $\mathcal{P}_k(V)$ denote the family of all k-subsets (sets of size k) of V. Any simple loopless unoriented *graph* $\Gamma = (V, \mathcal{E})$ is an incidence structure on V satisfying the additional regularity property that each block is of size 2, i.e., $\mathcal{E} \subseteq \mathcal{P}_2(V)$. Clearly, the graph automorphisms of Γ are the automorphisms of Γ considered as an incidence structure. Further, a *k-hypergraph* $\mathcal{H}$ on a set of vertices V is any incidence structure $\mathcal{H} = (V, \mathcal{B})$ with the property $\mathcal{B} \subseteq \mathcal{P}_k(V)$, i.e., $|B| = k$, for all blocks $B \in \mathcal{B}$. Finally, a $t-(v,k,\lambda)$ *design* $\mathcal{D}$ is an incidence structure $(V, \mathcal{B})$ such that $|V| = v$, $|B| = k$, for all $B \in \mathcal{B}$, and each t-subset of V appears in exactly λ blocks from $\mathcal{B}$.

We will say that a finite group G admits a *regular representation as the full automorphism group of an incidence structure* if there exists a block system $\mathcal{B}$ on G such that $Aut(G, \mathcal{B}) = G_L$ (i.e., the automorphisms of $(G, \mathcal{B})$ are exactly the left multiplications by the elements of G). Similarly, a finite group G admits a *graphical regular representation* (GRR) if there exists a system of blocks of size 2 (edges) on G such that the (full) automorphism group of the resulting incidence structure (graph) is the group G in its left-regular representation. It is clear what we mean when we say that G admits a regular representation on a k-hypergraph or a t-design.

Using the notation introduced, we can state the main problems of this paper.

Problem 1 *Classify the finite groups G that admit a regular representation as the full automorphism group of an incidence structure.*

Further refining the above problem, we obtain :

Problem 2 *For each finite group G, find all the positive integers k such that G admits a regular representation as the full automorphism group of a k-hypergraph.*

In what follows, we provide a complete answer to Problem 1, and a partial answer to Problem 2 for cyclic groups.

We conclude this section with some simple observations about the properties of automorphism groups of incidence structures.

It is well-known that the automorphism groups of a graph and its complement are equal : if $\Gamma = (V, \mathcal{E})$ and the complement $\Gamma^c = (V, \mathcal{P}_2(V) - \mathcal{E})$, then $Aut(\Gamma) = Aut(\Gamma^c)$. This observation can be generalized in following different ways.

Lemma 1 *Let $\mathcal{I} = (V, \mathcal{B})$ be an incidence structure and let k be a non-negative integer $0 \leq k \leq |V|$. Then*

(i) $Aut(V, \mathcal{B}) = Aut(V, (\mathcal{B} - \mathcal{P}_k(V)) \cup (\mathcal{P}_k(V) - \mathcal{B}))$,

(ii) $Aut(V, \mathcal{B}) = Aut(V, (\mathcal{B} - \mathcal{P}_k(V)) \cup \{B^c \mid B \in \mathcal{B} \cap \mathcal{P}_k(V)\})$,

(iii) $Aut(V, \mathcal{B}) = Aut(V, \mathcal{P}(V) - \mathcal{B})$.

As a specific consequence of part (ii) of the above lemma, let us deduce that for any incidence structure $(V, \mathcal{B})$ there exists an incidence structure $(V, \mathcal{B}')$ satisfying the property $|B| \leq |V|/2$, for all $B \in \mathcal{B}'$, such that $Aut(V, \mathcal{B}) = Aut(V, \mathcal{B}')$.

Now, consider the case when $\mathcal{I} = (V, \mathcal{B})$ possesses a regular subgroup G of the full automorphism group $Aut(\mathcal{I})$. Then V can be identified with G, the action of G on itself is the left regular action of left multiplication, and the blocks are subsets of G. Let us assume from now on that the identification has been completed and that $\mathcal{I} = (G, \mathcal{B})$, $\mathcal{B} \subseteq \mathcal{P}(G)$, and $G_L \leq Aut(\mathcal{I})$. Then $Aut(\mathcal{I})$ acts transitively on the vertices of $\mathcal{I}$, and for each block $B = \{a_1, a_2, \ldots, a_k\} \in \mathcal{B}$, the sets $g \cdot B = \{ga_1, ga_2, \ldots, ga_k\}$ belong to $\mathcal{B}$ for all $g \in G$. Thus, $\mathcal{B}$ is a union of orbits of G in its induced action on $\mathcal{P}(G)$,

$\mathcal{B} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \ldots \cup \mathcal{O}_r$, where each of the orbits $\mathcal{O}_i$ is contained in one of the families $\mathcal{P}_k(G)$. It follows from Lemma 1(i) that whenever $\mathcal{P}_k(G) \subseteq \mathcal{B}$, $Aut(G, \mathcal{B}) = Aut(G, \mathcal{B} - \mathcal{P}_k(G))$, i.e., including all of $\mathcal{P}_k(G)$ into $\mathcal{B}$ does not effect the resulting automorphism group. In particular, since blocks of sizes $0, 1, |G| - 1$ and $|G|$ each comprise a single orbit of G on $\mathcal{P}(G)$, including (or not including) blocks of these sizes does not effect the resulting automorphism group. Let us therefore assume from now on that $2 \leq |B| \leq |G| - 2$, for all $B \in \mathcal{B}$.

Each of the orbits $\mathcal{O}_i$ then contains at least one block that includes the identity 1_G. If we choose one such block B_i for each orbit $\mathcal{O}_i$, we see that $\mathcal{B} = B_1^G \cup B_2^G \cup \ldots \cup B_r^G$. Thus, each incidence structure with a regular automorphism group G can be represented by a family $\mathcal{B}_i$, $1 \leq i \leq r$, of subsets of G, such that $1_G \in B_i$, $1 \leq i \leq r$, and $\mathcal{B} = \bigcup_{i=1}^r B_i^G$. The notation we just introduced allows for a simple generalization of Sabidussi's characterization of vertex-transitive graphs that possess a regular automorphism group [6].

Lemma 2 *Let $\mathcal{I} = (V, \mathcal{B})$ be a vertex transitive incidence structure. Then $\mathcal{I}$ admits a regular subgroup G of the full automorphism group $Aut(\mathcal{I})$ if and only if there exists a family of sets $B_r \in \mathcal{P}(G)$, $1 \leq r \leq k$, each of which contains 1_G, such that $\mathcal{I}$ is isomorphic to $(G, \bigcup_{r=1}^k B_r^G)$.*

The last two results of this section summarize the solutions to the GRR- and DRR-problems. The first three abelian groups listed in Lemma 3 were found by W. Imrich [3]. The ten non-abelian groups were found by L.A. Nowitz and M.E. Watkins [5] and by Watkins alone [7], and the list was proved to be complete by C.D. Godsil in [2].

A *generalized dicyclic group* G is a group generated by an abelian subgroup A and an element $b \notin A$ satisfying the properties $b^4 = 1$, $b^2 \in A - \{1_G\}$ and $b^{-1}ab = a^{-1}$ for all $a \in A$.

Lemma 3 ([2, 7]) *Let G be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group or G is isomorphic to one of the 13 groups : $\mathcal{Z}_2^2$, $\mathcal{Z}_2^3$, $\mathcal{Z}_2^4$, $\mathcal{D}_3$, $\mathcal{D}_4$,*

$\mathcal{D}_5$, $\mathcal{A}_4$, $\mathcal{Q} \times \mathcal{Z}_3$, $\mathcal{Q} \times \mathcal{Z}_4$,
$\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \rangle$,
$\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac \rangle$.

Recall that for a *digraph* $\Gamma = (V, E)$, E is a set of *ordered pairs* of vertices from V. A *digraph automorphism* ψ of Γ is then a permutation of V preserving the ordered pairs : $(\psi(u), \psi(v)) \in E$, for all $(u, v) \in E$. The automorphism group of Γ is the group of all digraph automorphisms, and a group G *admits a DRR* if there exists a set E of ordered pairs of elements from G such that $Aut(G, E) = G_L$.

Lemma 4 ([1]) *The finite group G admits a DRR if and only if G is neither the quaternion group $\mathcal{Q}$ nor any of $\mathcal{Z}_2^2$, $\mathcal{Z}_2^3$, $\mathcal{Z}_2^4$, $\mathcal{Z}_3^2$.*

3 Incidence structures from digraphs

In this section, we use Lemma 4 to obtain a classification of finite groups G admitting regular representations via incidence structures. Obviously, any finite group G that admits a GRR allows for the existence of an incidence structure with G_L being its full automorphism group, as any graph is an incidence structure. Thus we could simply focus on finite groups that do not admit a GRR. However, we will take another route and use the connection between digraphs and incidence structures. Let us point out that digraphs in general are not incidence structures, and, in particular, the automorphism group of a digraph Γ may be a *proper* subgroup of the automorphism group of the underlying (undirected) graph.

The essential ingredients of the proofs of Lemma 3 and Lemma 4 are the concepts of Cayley graph and Cayley digraph. Given a (finite) group G and a set X of elements of G, $X \subset G$, closed under taking inverses, $X = X^{-1}$, and not containing 1_G, the *Cayley graph* $\Gamma = C(G, X)$ is the graph $(G, \mathcal{E})$ where $\mathcal{E} = \{\ \{g, g \cdot x\} \mid g \in G, x \in X\}$. Alternately, the set of edges of Γ is the set of (unordered) pairs $\{a, b\}$ of elements of G satisfying the property $a^{-1} \cdot b \in X$. If we drop the symmetry requirement $X = X^{-1}$ from the definition and take the set E to be the set of ordered pairs $\{(g, g \cdot x) \mid g \in G, x \in X\}$, we obtain the *Cayley digraph* $CD(G,X) = (G,E)$. A well-known classical result of Sabidussi characterizes Cayley (di)graphs as exactly the vertex-transitive (di)graphs with a regular subgroup of the full automorphism group. As a consequence of this result, if a given a finite group G admits a GRR or a DRR, it must be a Cayley graph or Cayley digraph.

Thus, Lemmas 3 and 4 claim that for each finite group G other than the specified exceptional groups there exist subsets $X = X^{-1}$ and X' of G with the property $Aut(C(G,X)) = Aut(CD(G,X')) = G_L$. We will use these Cayley digraphs to construct the incidence structures for Problem 1.

The following result that will be used for most of the proofs, can be deduced from [4], Corollary 2.4 or [1], Fact 2.4.

Lemma 5 *Let $\Gamma = C(G,X)$ ($\Gamma' = CD(G,X')$) be a Cayley (di)graph, and let Y be a generating set of G such that each automorphism φ of $C(G,X)$ ($CD(G,X)$) that fixes the identity 1_G, fixes the set Y pointwise (i.e., $\varphi(y) = y$, for all $y \in Y$). Then $C(G,X)$ ($CD(G,X)$) is (di)graphical regular representation.*

We say that a generating set X of a group G is *irreducible* if no proper subset of X generates G. Let $irrgen(G)$ denote the maximal size of an irreducible generating set X of G. This characteristic turns out to be surprisingly important in the context of regular representations.

A *generalized dihedral group* G has an abelian subgroup A of index 2 (called the *kernel* of G) and an element $b \in G - A$ such that $b^2 = 1$ and $bab = a^{-1}$ for all $a \in A$.

Next lemma deals with "almost all" finite groups.

Lemma 6 *Let G be a finite group that is neither cyclic, nor generalized dihedral, nor $\mathcal{Z}_3^2$ nor the quaternion group $\mathcal{Q}$. Then G can be represented as a regular (full) automorphism group of an incidence structure $(G,\mathcal{B})$.*

Proof. Let G be a finite group that does not belong to any of the excluded cases. For every such a group, Lemmas 3.4 and 3.5 in [1] assert the existence of an irreducible generating set $X = \{x_1, x_2, \ldots, x_d\}$, $d \geq 2$, containing no involutions, such that for some

$m \in \{1, 2, \ldots, d\}$, the set X together with the set $K = L \cup L^{-1}$, $L = \{x_i^{-1} x_{i+1} \mid 1 \le i \le d-1\} \cup \{x_m x_1\}$ give rise to a DRR for G : $Aut\,(CD(G, X \cup K)) = G_L$.

In the remaining part of this proof, we use this DRR for G to construct an incidence structure $(G, \mathcal{B})$ such that $Aut(G, \mathcal{B}) = G_L$.

Let

$$\begin{aligned} \mathcal{B} &= \{\, \{a, ax_i\} \mid a \in G, 1 \le i \le d \,\} \cup \\ & \quad \{\, \{a, ax_i, ax_{i+1}\} \mid a \in G, 1 \le i \le d-1 \} \cup \\ & \quad \{\, \{a, ax_m^{-1}, ax_1, ax_m x_1\} \mid a \in G \,\}. \end{aligned}$$

Clearly, the blocks from the first set are of size 2, blocks from the second set are of size 3, and blocks from the third set are of size 4.

It is also clear that the set $\mathcal{B}$ is invariant under left multiplication by elements of G, and thus, $G_L \le Aut(G, \mathcal{B})$. To complete the proof, we will show that $Aut(G, \mathcal{B}) \le Aut\,(CD(G, X \cup K))$ ($= G_L$).

Any automorphism φ of $(G, \mathcal{B})$ must send blocks of size 2 to blocks of size 2. We have chosen the blocks of size 2 to be exactly the (unoriented) edges of the Cayley graph $C(G, X)$, and therefore, $Aut(G, \mathcal{B}) \le Aut(C(G, X))$.

The rest of the proof proceeds by contradiction. Suppose that there exists an automorphism φ of $(G, \mathcal{B})$ that does not belong to $Aut(CD(G, X \cup K))$. Then, there exist elements $a \in G$ and $y \in X \cup K$ such that the arc (a, ay) does not get mapped by φ to an arc of the digraph $CD(G, X \cup K)$, i.e., $\varphi(a)^{-1}\varphi(ay) \notin X \cup K$. Due to the fact that left-translations by elements of G belong to $Aut(G, \mathcal{B})$, we may compose φ with the left multiplications $\delta_a, \delta_{\varphi(a)^{-1}}$, and the composition $\delta_{\varphi(a)^{-1}} \circ \varphi \circ \delta_a$ will belong to $Aut(G; \mathcal{B})$ and map 1_G to 1_G. Thus, we may assume without loss of generality that $a = 1$, $\varphi(a) = 1$, and $\varphi(y) = \varphi(1)^{-1}\varphi(1 \cdot y) \notin X \cup K$.

First, we observe that y cannot belong to X. To see this, consider all the blocks of size 3 containing 1_G. Each such block must be of the form $\{1_G, x_i, x_{i+1}\}$ or $\{x_i^{-1}, 1_G, x_i^{-1}x_{i+1}\}$ or $\{x_{i+1}^{-1}, x_{i+1}^{-1}x_i, 1_G\}$. We have already argued that φ is a graph automorphism of $C(G, X)$, and, as such, must preserve distances (=length of a minimal path) in $C(G, X)$. All the vertices x_i, $1 \le i \le d$, are of distance 1 from 1_G in $C(G, X)$, while all the elements in K, are of distance 2 (due to the irreducibility of X). Thus, φ must preserve $X \cup X^{-1}$. If we observe now that all the blocks of size 3 that contain 1_G and are of the second or third form contain an element of distance 2 from 1_G, we can conclude that φ must map blocks of the first form (namely, $\{1_G, x_i, x_{i+1}\}$) to blocks of that form again, hence, φ preserves X set-wise. It follows that y cannot belong to X.

Assume then that $y \in K$. If $y = x_i^{-1}x_{i+1}$, consider the image of the unique block of size 3 that contains both 1_G and $y = x_i^{-1}x_{i+1}$, $\{x_i^{-1}, 1_G, x_i^{-1}x_{i+1}\}$. The image of this block under φ must be again a block of size 3 that contains 1_G. Since φ preserves $X \cup X^{-1}$ as well as X, φ also preserves X^{-1}. We conclude that φ must map $y = x_i^{-1}x_{i+1}$ to the third vertex of some block of size 3 that contains 1_G and some x_j^{-1}. All such vertices belong to K, and the assumption $y = x_i^{-1}x_{i+1}$ leads to $\varphi(y) \in K$. A similar argument can be used to show that $\varphi(y) \in K$, for all $y = x_{i+1}^{-1}x_i$.

The remaining possibilities for y are $y = x_m x_1$ or $y = x_1^{-1}x_m^{-1}$. Consider the blocks of size 4 that contain 1_G. There are exactly 4 such blocks, and φ must map these blocks among themselves. Taking advantage of the irreducibility of X and the fact that φ must preserve distances in $C(G, X)$, it is easy to show again that $\varphi(y) = y^{\pm 1} \in K$ for this possibility as well.

As each choice for y leads to a contradiction, we have to conclude that there is no $y \in X \cup K$ such that $\varphi(y) \notin X \cup K$, and

$$G_L \leq Aut(G, \mathcal{B}) \leq Aut(CD(G, X \cup K)) = G_L$$

□

We have shown the existence of regular representations via incidence structures for most of the finite groups. Let us consider the remaining cases excluded in the above lemma. We start with cyclic groups.

Lemma 7 *A cyclic group $\mathcal{Z}_i$ admits a regular representation on an incidence structure if and only if $i \neq 3, 4, 5$. Furthermore, any cyclic group $\mathcal{Z}_n$, $n > 5$, can be regularly represented as the full automorphism group of some 3-hypergraph $\mathcal{H}$.*

Proof. As $\mathcal{Z}_1$ and $\mathcal{Z}_2$ clearly admit a regular representation of the form $(\mathcal{Z}_i, \emptyset)$, $i = 1, 2$, we start by proving that none of the three groups $\mathcal{Z}_3$, $\mathcal{Z}_4$ and $\mathcal{Z}_5$ admit a regular representation.

For each of these groups consider the left regular action of $\mathcal{Z}_i$ on the power set $\mathcal{P}(\mathcal{Z}_i)$, and write each of the power sets as the disjoint union of orbits of $\mathcal{Z}_i$, $\mathcal{P}(\mathcal{Z}_i) = \{\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{k_i}\}$, $3 \leq i \leq 5$.

All the subsets of $\mathcal{Z}_3$ are of size $0, 1, |\mathcal{Z}_3| - 1$, $|\mathcal{Z}_3|$. This leaves no blocks to effect the resulting automorphism group of an incidence structure on $\mathcal{Z}_3$ (see the notes following Lemma 1), and $Aut(\mathcal{Z}_3, \mathcal{B}) = \mathcal{S}_3$ whenever $\mathcal{Z}_3 \leq Aut(\mathcal{Z}_3, \mathcal{B})$.

The only "relevant" orbits for $\mathcal{Z}_4$ are orbits of 2-sets, i.e., orbits of undirected edges. It is well-known that the automorphism group of any undirected graph on $\mathcal{Z}_4$ that allows a regular subgroup is either the dihedral $\mathcal{D}_4$ or full symmetric $\mathcal{S}_4$.

Finally, the only relevant orbits for $\mathcal{Z}_5$ are

$$\mathcal{O}_1 = \{\ \{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,0\}\},$$

$$\mathcal{O}_2 = \{\ \{0,2\}, \{1,3\}, \{2,4\}, \{3,0\}, \{4,1\}\},$$

$$\mathcal{O}_3 = \{\ \{0,1,2\}, \{1,2,3\}, \{2,3,4\}, \{3,4,0\}, \{4,0,1\}\},$$

$$\mathcal{O}_4 = \{\ \{0,1,3\}, \{1,2,4\}, \{2,3,0\}, \{3,4,1\}, \{4,0,2\}\},$$

where $\mathcal{O}_3$ is the orbit of complements of the 2-sets in $\mathcal{O}_1$ and $\mathcal{O}_4$ is the orbit of complements of the 2-sets in $\mathcal{O}_2$. Therefore, $Aut(\mathcal{Z}_5, \mathcal{O}_1) = Aut(\mathcal{Z}_5, \mathcal{O}_3) = Aut(\mathcal{Z}_5, \mathcal{O}_1 \cup \mathcal{O}_3) = \mathcal{D}_5$ and $Aut(\mathcal{Z}_5, \mathcal{O}_2) = Aut(\mathcal{Z}_5, \mathcal{O}_4) = Aut(\mathcal{Z}_5, \mathcal{O}_2 \cup \mathcal{O}_4) = \mathcal{D}_5$. It follows that $\mathcal{Z}_5 \leq Aut(\mathcal{Z}_5, \mathcal{B})$ implies $\mathcal{D}_5 \leq Aut(\mathcal{Z}_5, \mathcal{B})$ for all $\mathcal{B} \subseteq \mathcal{P}(\mathcal{Z}_5)$.

Now, let $i \geq 6$, and take $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathcal{B}_1 = \{\ \{j, j+1, j+2\} \mid 1 \leq j \leq i\}$ and $\mathcal{B}_2 = \{\ \{j, j+1, j+3\} \mid 1 \leq j \leq i\}$ (with the addition modulo i). We claim that $Aut(\mathcal{Z}_i, \mathcal{B}) = \mathcal{Z}_i$, for all $i \geq 6$.

Clearly, $\mathcal{Z}_i \leq Aut(\mathcal{Z}_i, \mathcal{B})$. To show the other inclusion, let $\varphi \in Aut(\mathcal{Z}_i, \mathcal{B})$. We claim that $\varphi \in Aut(C(\mathcal{Z}_i, \{1, -1\}))$. Suppose the opposite: if $\varphi \notin Aut(C(\mathcal{Z}_i, \{1, -1\}))$, then there must exist an $a \in \mathcal{Z}_i$, $-\varphi(a) + \varphi(a+1) \notin \{1, -1\}$ or $-\varphi(a) + \varphi(a-1) \notin \{1, -1\}$. Since $\mathcal{Z}_i$ acts transitively on $(\mathcal{Z}_i, \mathcal{B})$, the existence of such a φ also implies the existence of $\psi \in Aut(\mathcal{Z}_i, \mathcal{B})$ with the properties $\psi(0) = 0$ and $\psi(\{1, -1\}) \neq \{1, -1\}$. There are exactly 6 blocks containing the element 0, namely, $\{0,1,2\}$, $\{-1,0,1\}$, $\{-2,-1,0\}$, $\{0,1,3\}$, $\{-1,0,2\}$, $\{-3,-2,0\}$. The morphism ψ fixes 0 and thus must send a block containing 0 to a block containing 0. This means, in particular, that

$\psi(\{1,-1\}) \subseteq \{1,-1,2,-2,3,-3\}$. However, there are 3 blocks containing containing both 0 and 1 and 3 blocks containing both 0 and -1, while there are only 2 blocks containing both 0 and one of the elements $2,-2,3,-3$. It follows that $\psi(1), \psi(-1) \notin \{2,-2,3,-3\}$, a contradiction.

Since $Aut(C(\mathcal{Z}_i,\{1,-1\})) = \mathcal{D}_i$, we have shown that $Aut(\mathcal{Z}_i,\mathcal{B}) \leq \mathcal{D}_i$. On the other hand, the reflection $r \in \mathcal{D}_i$ that fixes 0 sends the block $\{0,1,3\}$ to the "non-block" $\{0,-1,-3\}$, and thus, $\mathcal{Z}_i \leq Aut(\mathcal{Z}_i,\mathcal{B}) < \mathcal{D}_i$. It follows that $\mathcal{Z}_i = Aut(\mathcal{Z}_i,\mathcal{B})$, for all $i \geq 6$. □

In the discussion preceding Lemma 6 we have already mentioned the relevance of the maximal size of an irreducible generating set to the existence of a regular representation of a group. The next lemma further demonstrates this relation.

Lemma 8 *Let G be a finite group that admits an irreducible generating set X of size $k \geq 3$. Then G is regularly representable as the full automorphism group of some incidence structure $(G,\mathcal{B})$ with the property*

$$max\ \{|B|\ : B \in \mathcal{B}\} = k+1.$$

Proof. Let G be a finite group with an irreducible generating set $X = \{x_1, x_2, \ldots, x_k\}$, $k \geq 3$. We define the blocks of elements from G as follows :

$$\mathcal{B} = \{\{a, ax\} | a \in G, x \in X\} \cup \{\{a, ax_1, ax_2, \ldots, ax_j\} | a \in G, 2 \leq j \leq k\}$$

$$\cup \{\ \{a, ax_1, ax_1x_2x_3\}\ |\ a \in G\}.$$

We show that $Aut(G,\mathcal{B}) = G_L$.

It is easy to see that the automorphism group contains G in its regular representation. We shall prove that the stabilizer of 1_G is trivial. First notice that blocks of size 2 in $\mathcal{B}$ are the edges of the Cayley digraph $CD(G,X)$, and so any automorphism of the incidence structure $(G,\mathcal{B})$ must be a graph automorphism of $CD(G,X)$. Let $\psi \in Aut(G,\mathcal{B})$, $\psi(1_G) = 1_G$, and consider the image of the block $\{1_G, x_1, x_2\}$ under ψ. The only blocks of size 3 that contain 1_G are the blocks $\{1_G, x_1, x_2\}$, $\{x_1^{-1}, 1_G, x_1^{-1}x_2\}$, $\{x_2^{-1}, x_2^{-1}x_1, 1_G\}$, $\{1_G, x_1, x_1x_2x_3\}$, $\{x_1^{-1}, 1_G, x_2x_3\}$ and $\{x_3^{-1}x_2^{-1}x_1^{-1}, x_3^{-1}x_2^{-1}, 1_G\}$. Each of these blocks other than $\{1_G, x_1, x_2\}$ contains an element of distance at least 2 from 1_G in $CD(G, X)$ (note that we are using the irreducibility of X). However, ψ, being a graph automorphism of $CD(G, X)$, must preserve distances, and thus, $\psi(\{1_G, x_1, x_2\}) = \{1_G, x_1, x_2\}$. A similar argument can be used to show this to be true for all blocks $\{1_G, x_1, x_2, \ldots, x_j\}$, $2 \leq j \leq k$, which yields $\psi(x_j) = x_j$, for all $3 \leq j \leq k$. Finally, if $\psi(x_1)$ were equal to x_2, then $\{1_G, x_1, x_1x_2x_3\}$ would have to map to a block of size 3 containing 1_G and x_2. The irreducibility of X yields once again that the only block of size 3 that contains both 1_G and x_2 is the block $\{1_G, x_1, x_2\}$ that maps onto itself. We conclude that $\psi(x_1) = x_1$, and hence $\psi(x_i) = x_i$, for all $1 \leq i \leq k$. Applying Lemma 5, we see that $\psi = id_G$. □

The above lemma allows us to extend our results to generalized dihedral groups.

Lemma 9 *Let G be a generalized dihedral group. Then G can be represented as a regular full automorphism group of some combinatorial structure if and only if $G \not\cong \mathcal{Z}_2^2$.*

Proof. The abelian generalized dihedral groups are the elementary abelian 2-groups $\mathcal{Z}_2^n$. We have already seen in Lemma 7 that $\mathcal{Z}_2$ has a regular representation via an incidence structure. Further, $|\mathcal{Z}_2^2| = 4$, and thus, the only relevant blocks are those of size 2. It follows that the existence of an incidence structure $(\mathcal{Z}_2^2, \mathcal{B})$, such that $Aut(\mathcal{Z}_2^2, \mathcal{B}) = (\mathcal{Z}_2^2)_L$, would force the existence of a GRR for $\mathcal{Z}_2^2$ which is known not to exist (see Lemma 3). All the remaining elementary abelian 2-groups $\mathcal{Z}_2^n$, $n \geq 3$, allow the existence of an irreducible generating set X of size at least 3 and as such can be regularly represented as the full automorphism group of an incidence structure due to Lemma 8.

Either the kernel K of a non-abelian generalized dihedral group G is generated by an irreducible generating set $X = \{x_1, x_2, \ldots, x_k\}$, $k \geq 2$, or $K = \mathcal{Z}_p$, p a prime, in which case G is the dihedral group $\mathcal{D}_p$.

Consider first the case when $K = \langle x_1, x_2, \ldots, x_k \rangle$, $k \geq 2$, where $X = \{x_1, x_2, \ldots, x_k\}$ is irreducible. Let b be an element of order 2 from $G - K$ that normalizes K. Then $X' = X \cup \{b\}$ is an *irreducible* generating set for G of size at least 3 and, by Lemma 8, G admits a regular representation on an incidence structure.

The last case to consider is $G = \mathcal{D}_p$, $p \geq 3$, i.e., $G = \langle r, s \rangle$, $r^p = s^2 = 1$, $srs = r^{-1}$. Let $X = \{r, r^{-1}, s\}$ and $\mathcal{B}$ be the set of blocks

$$\{\, \{a, ax\} \mid a \in G, x \in X\} \cup \{\, \{a, ar, as\} \mid a \in G\}.$$

The proof of the identity $Aut(\mathcal{D}_p, \mathcal{B}) = (\mathcal{D}_p)_L$ follows essentially along the same lines as the proof for groups with at least 3 independent generators. Indeed, $\mathcal{D}_p \leq Aut(\mathcal{D}_p, \mathcal{B})$. Any $\psi \in Aut(\mathcal{D}_p, \mathcal{B})$ that maps $1_{\mathcal{D}_p}$ to $1_{\mathcal{D}_p}$ is a graph automorphism of $C(\mathcal{D}_p, \{r, r^{-1}, s\})$ and as such preserves distances and maps $\{r, r^{-1}, s\}$ to $\{r, r^{-1}, s\}$. The three blocks of size 3 that contain $1_{\mathcal{D}_p}$ are the blocks $\{1_{\mathcal{D}_p}, r, s\}$, $\{r^{-1}, 1_{\mathcal{D}_p}, r^{-1}s\}$, and $\{s, sr = r^{-1}s, 1_{\mathcal{D}_p}\}$. Since $sr = r^{-1}s$ is of distance 2 from $1_{\mathcal{D}_p}$, $\psi(\{r, s\}) = \{r, s\}$, which also implies $\psi(r^{-1}) = r^{-1}$. By [4] Proposition 2.3, $\psi(r) = r$, and the proof is complete. □

There are two more groups to consider to conclude our investigation. Neither of them possesses an irreducible generating set of size > 2.

Lemma 10 *The groups $\mathcal{Z}_3^2$ and $\mathcal{Q}$ both admit a regular representation as a regular (full) automorphism group of an incidence structure.*

Proof. Let $X = \{e_1, e_2\}$ be the set of unit vectors of $\mathcal{Z}_3^2$. The blocks of the desired incidence structure on $\mathcal{Z}_3^2$ can be defined as follows : $\mathcal{B} =$

$$\{\, \{a, ax\} \mid a \in \mathcal{Z}_3^2, x \in X\} \cup \{\, \{a, ae_1, ae_2\} \mid a \in G\}$$
$$\cup \{\, \{a, ae_1, ae_1^{-1}, ae_1e_2\} \mid a \in G\}.$$

Any automorphism of the structure $(\mathcal{Z}_3^2, \mathcal{B})$ is a graph automorphism of $C(\mathcal{Z}_3^2, X \cup X^{-1})$, and $(\mathcal{Z}_3^2)_L \leq Aut(\mathcal{Z}_3^2, \mathcal{B})$. Let $\psi \in Aut(\mathcal{Z}_3^2, \mathcal{B})$ stabilize the identity. The only 3-blocks containing the identity are $\{1, e_1, e_2\}$, $\{e_1^{-1}, 1, e_1^{-1}e_2\}$ and $\{e_2^{-1}, e_2^{-1}e_1, 1\}$. Since both $e_1^{-1}e_2$ and $e_2^{-1}e_1$ are of distance 2 from the identity, the block $\{1, e_1, e_2\}$ must be fixed by ψ and $\{e_1, e_2\}$ maps to $\{e_1, e_2\}$. The only 4-blocks that contain the identity are $\{1, e_1, e_1^{-1}, e_1e_2\}$, $\{e_1^{-1}, 1, e_1, e_2\}$, $\{e_1, e_1^{-1}, 1, e_1^{-1}e_2\}$ and $\{e_2^{-1}e_1^{-1}, e_2^{-1}, e_2^{-1}e_1, 1\}$. Since the first of these four blocks can only be mapped to itself or to the third block due to distance considerations, we see that $\{e_1, e_1^{-1}\}$ maps to $\{e_1, e_1^{-1}\}$. Combining the two conditions for the image of e_1, we obtain $\psi(e_1) = e_1$, and consequently, $\psi(e_1^{-1}) = e_1^{-1}$, $\psi(e_2) = e_2$, and $\psi(e_2^{-1}) = e_2^{-1}$. It follows that $\psi = id_{\mathcal{Z}_3^2}$.

For the group $\mathcal{Q} = \{1, i, j, k, -1, -i, -j, -k\}$ take $X = \{i, j\}$ and the blocks $\mathcal{B} =$

$$\{\{a, ax\} | a \in \mathcal{Q}, x \in X\} \cup \{\{a, ai, aj\} | a \in G\} \cup \{\{a, -a, ai, -ai\} | a \in G\}.$$

The rest of the proof follows along the same lines as the above. □

The results we have obtained are summarized in the following theorem.

Theorem 1 *A finite group G can be represented as a regular full automorphism group of some incidence structure if and only if G is not one of the groups* $\mathcal{Z}_3$, $\mathcal{Z}_4$, $\mathcal{Z}_5$ *or* $\mathcal{Z}_2^2$.

References

[1] L. Babai, Finite digraphs with given regular automorphism groups, Period. Math. Hungar. Vol 11 (4) 1980, pp.257-271.

[2] C.D.Godsil, GRR's for non-solvable groups, Algebraic methods in graph theory, Vol. I., II. (Szeged, 1978), 221-239.

[3] W. Imrich, On graphical regular representations of groups, Colloq. Math. Soc. Janos Bolyai, Vol. 10, 1975, 905-925.

[4] L.A. Nowitz and M.E. Watkins, Graphical regular representations of non-abelian groups, I, Canadian J. Math., 24 (1972), 993-1008.

[5] L.A. Nowitz and M.E. Watkins, Graphical regular representations of non-abelian groups, II, Canadian J. Math., 24 (1972), 1009-1018.

[6] G.Sabidussi, On a class of fixed-point-free graphs, Proc. Amer. Math. Soc. 9 (1958), 800-804.

[7] M.E. Watkins, The state of the GRR problem, Proc. Second Czechoslovak Sympos., Prague, 1974, 517-522.