A STUDY OF TYPENUMBER IN BOOK-EMBEDDING

Yen-Chi Chen, Hung-Lin Fu,and I-Fan Sun

March 30, 2000

Department of Applied Mathematics
National Chiao Tung University
Hsin Chu, Taiwan, R.O.C.

Abstract

The type of a vertex v in a p-page book-embedding is the $p \times 2$ matrix of nonnegative integers

$$\tau(v) = \begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix},$$

where $l_{v,i}$(respectively, $r_{v,i}$) is the number of edges incident to v that connect on page i to vertices lying to the left (respectively, to the right) of v. The typenumber of a graph G, $T(G)$, is the minimum number of different types among all the book-embeddings of G. In this paper, we disprove the conjecture by J. Buss et. al. which says for $n \geq 4$, $T(L_n)$ is not less than 5 and prove that $T(L_n) = 4$ for $n \geq 3$.

1 Introduction and Basic Properties

A book is a set of half-planes (the pages of the book) that share a common boundary line(the spine of the book). An embedding of a simple undirected graph of G(a pair of vertices are connected by at most one edge) in a *book* consists of an ordering of the vertices of G along the spine(horizontal line) of the book, together with an assignment of each edge of G to a page of the *book*, in which edges assigned to the same page do not cross.

There are three germane measure of the quality of a book-embedding: the thickness(number of pages) of the book, the individual and cumulative

ARS COMBINATORIA 62(2002), pp. 97-103

widths of the pages, and the number of distinct vertex types. Throughout of this paper we shall focus on the study of the third measure.

Given a p-page book-embedding of a graph G, each vertex v of G has an associated $p \times 2$ matrix of nonnegative integers, called its *type*,

$$\tau(v) = \begin{pmatrix} l_{v,1} & r_{v,1} \\ \vdots & \vdots \\ l_{v,p} & r_{v,p} \end{pmatrix},$$

where $l_{v,i}$(respectively, $r_{v,i}$) is the number of edges incident to v that connect on page i to vertices lying to the left (respectively, to the right) of v. Thus for each graph of order n and each p-page book-embedding, there are n types, one for each vertex and two types are different provided that the two matrices are not equal. It's interesting to know : among all the book-embeddings of G what is the minimum number of different *vertex types*, $T(G)$. For its application, the number of types in a book-embedding relates to the amount of logic necessary to realize fault-tolerant arrays of processors using one specific design methodology. The methodology views the desired array as an undirected graph, with vertices representing processing elements and edges representing communication links; the design process operates in two stages: first, the graph representing the desired array is embedded in a book; then, the book-embedding is converted to an efficient fault-tolerant layout of the array. The significance of the notation of vertex type is that the type of a vertex "tell" it what role to play in the fault-free processor array. Thus, the base-two logarithm of the number of vertex types is the number of control bits per processing element needed to configure the array to its fault-free format. [2] [3]

In any book-embedding of G, there are two specific vertex types in the embedding : *source* and *sink*. A nonzero vertex type for a vertex v is a *source* if all $l_{v,i} = 0$ and is a *sink* if all $r_{v,i} = 0$. Clearly, every book-embedding of a graph has at least one source and one sink.

Let $N_i = \{v \in V(G) \mid deg_G(v) = i\}$ and T_i be the different types counting all the vertices in N_i, i.e., $T_i = |\{\tau(v) | v \in N_i\}|$. By considering the relation between the degree of v and the $p \times 2$ matrix, $\tau(v)$, we have the followings.

Lemma 1.1. [1] *For each $v \in V(G)$, $deg_G(v) = \sum_{i=1}^{p}(l_{v,i} + r_{v,i})$, hence $\tau(u) \neq \tau(v)$ provided that $deg_G(u) \neq deg_G(v)$. Thus $T(G) = \sum_{i \in D_G} T_i$.*

Lemma 1.2. $\sum_{v \in V(G)} \sum_{i=1}^{p} l_{v,i} = \sum_{v \in V(G)} \sum_{i=1}^{p} r_{v,i}$

Lemma 1.3. [1] *A connected graph G with at least two vertices has typenumber two if and only if G is a star.*

A lattice graph $L_{m,n}$ is a graph with vertex set $\{(a,b) | a \in Z_m$ and $b \in Z_n\}$ and edge set $\{\{(a,b),(c,d)\} \mid a = c$ and $|b-d| = 1$ or $b = d$ and $|a-c| = 1\}$. While $m = 2$ or $n = 2$, it's called a ladder, denoted by $L_n(L_m)$.

Proposition 1.4. [1] *Restricting to 1-page book-embedding of L_n we have : $T(L_1) = 2$, $T(L_2) = 3$, $T(L_3) = 4$, and $T(L_n) = 5$, for $n \geq 4$.*

In [1], the authors conjectured that additional pages for L_n, $n \geq 4$, will not lower its typenumber below 5. Unfortunately, this conjecture is not true as we shall see in section 2.

2 Main Result

A *book − embedding graph* G_K corresponding to a book-embedding K of G is a colored (not necessarily proper) digraph such that (i) an arc $\overrightarrow{uv} \in A(G_K)$ if and only if $uv \in E(G)$ and u is to the left of v along the spine in the embedding where the two sides are fixed, and (ii) two arcs have the same color if and only if the corresponding edges are embedded by K on the same page.

For convenience, we use $G_K(\Pi)$ to denote the subgraph of G_K which is induced (edge) by the set of edges whose colors are in Π. If $\Pi = \{c_1, c_2, \cdots, c_k\}$, we also use $G_K(c_1, c_2, \cdots, c_k)$ to denote the subgraph. Now, we have the following results.

Lemma 2.1. *Let G_K be a book-embedding graph of G corresponding to a book-embedding K. Then, for each color c, each cycle in $G_K(c)$ is almost a directed cycle except exactly one arc.*

Proof. First, we observe that a directed path $(a_0, a_1, \cdots, a_t)$ in $G_K(c)$ gives the relation that a_i is to the left of a_j if and only if $0 \leq i < j \leq t$. Furthermore, any vertex v lies between a_i and a_{i+1} can be incident to neither a_0 nor a_t for any $i \geq 0$ and $i < j$ because they are at the same page in the book-embedding. Now, given a cycle C in $G_K(c)$ and choose $P = (a_0, a_1, \cdots, a_t)$ to be the longest directed path in C. If

(1). $t = 1$.
So $P = \overleftarrow{a_1a_2}\overrightarrow{a_2a_3} \cdots \overleftarrow{a_ta_1}$. Hence, the type number of this cycle C is 2, a contradiction to Lemma 1.3.

(2). $t \geq 2$ and the length of C is $t+1$.
Obviously, C is almost a directed cycle except exactly one arc.

(3). $t \geq 2$ and the length of C is large than $t+1$.
Let $v(\neq a_1)$ be incident to a_0 in C and v must be to the right of a_0 because P is the longest directed path. By the observation above, v must be to the right of a_t too. Similarly, let $v'(\neq a_{t-1})$ be incident to a_t in the cycle, we can obtain the relation that v' is to the left of a_1. So $v \neq v'$ and a cross happens between a_0v and $v'a_t$, a contradiction.

Then the only possibility is that the cycle C is almost a directed cycle except exactly one arc.

■

Now, we consider the graph induced by two colors.

Lemma 2.2. *Let G_K be a book-embedding graph of G corresponding to a book-embedding K. Then there doesn't exist a cycle C in $G_K(c_1, c_2)$ such that the arcs of the same color have the same orientation and the number of vertices which are incident with 2 colors edges are at least 4.*

Proof. Suppose not. Let C be such a cycle in $G_K(c_1, c_2)$ and C can be partitioned into

$$P_1(v_1, v_2)P_2(v_2, v_3)P_3(v_3, v_4) \cdots P_{k-1}(v_{k-1}, v_k)P_k(v_k, v_1)$$

such that P_i and P_{i+1} are two directed paths with the opposite orientations and colors. And obviously k is an even number and $k \geq 4$. By contracting each P_i to an arc $\overrightarrow{v_i, v_{i+1}}$(or $\overleftarrow{v_i, v_{i+1}}$), the cycle C becomes to a new cycle C'

$$\overrightarrow{v_1v_2}\ \overleftarrow{v_2v_3}\ \overrightarrow{v_3v_4} \cdots \overrightarrow{v_{k-1}v_k}\ \overleftarrow{v_kv_1}.$$

Since C is a subgraph of the book-embedding graph G_K, we can obtain a 2 page book-embedding of a cycle which is the underlying graph of C. And it's easy to see that the underlying graph of the contracted cycle C' can be embedded in this 2 page book and each vertex is either a source $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ or a sink $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. So the typenumber of this cycle is two. But it's not a star, a contradiction.

■

Theorem 2.3. $T(L_3) = 4$

Proof. By Figure 2, $T(L_3) \leq 4$. We suppose $T(L_3) < 4$. Since L_3 isn't a star, by lemma 1.3, $T(L_3) \geq 3$. Assume that $T(L_3) = 3$ and we have two cases to consider now.

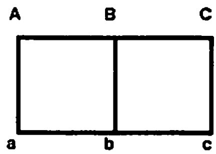

Figure 1: L_3

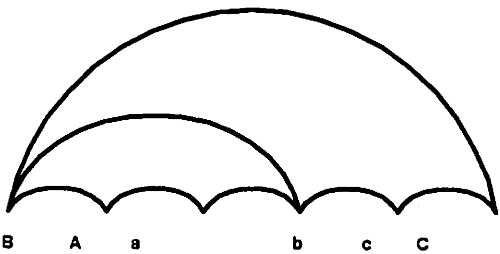

Figure 2: An book-embedding of $T(L_3)$

(1). $T_2 = 2, T_3 = 1$:

Since Aa, $Cc \in E(L_3)$, one of {A, a} is a source and the other is a sink. So is C, c. Since $Bb \in E(L_3)$, W.L.O.G., we may let $\sum l_{B,i} = \sum l_{b,i} = 2$. Hence $\sum\sum l_{v,i} \neq \sum\sum r_{v,i}$ and we have a contradiction.

(2). $T_2 = 1, T_3 = 2$:

The only vertex type of vertices of N_2 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then W.L.O.G. Figure 3 show the possible book-embedding graph only can be the following two types:

(i) (ii)

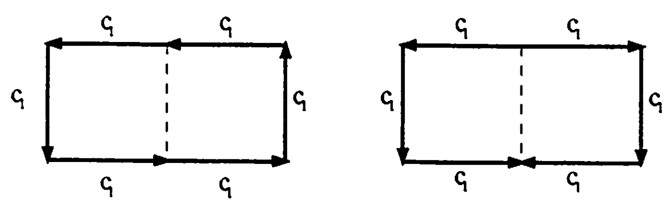

Figure 3: The six arcs must have the same color.

For each type, there exists a one-color cycle which is forbidden in Lemma 2.1, a contradiction. So we conclude that $T(L_3) = 4$.

■

Theorem 2.4. $T(L_n) = 4$, $n \geq 4$.

Proof. By Figure 4, $T(L_n) \leq 4$, since the vertex types are $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

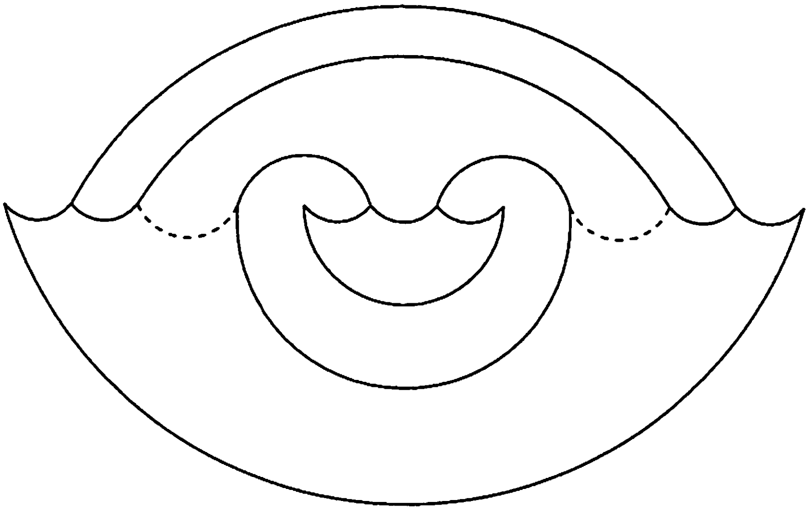

Figure 4: A book-embedding of $T(L_n)$

Assume that $T(L_n) < 4$. There are Three cases to consider:

(1). $T_2 = 1$, $T_3 = 1$ (i.e. $T(L_n) = 2$)
It's trivially impossible because L_n isn't a star.

(2). $T_2 = 2$, $T_3 = 1$ (i.e. $T(L_n) = 3$)
By the same reason as (1) in Theorem 2.3. $\sum\sum l_{v,i} \neq \sum\sum r_{v,i}$, a contradiction.

(3). $T_2 = 1$, $T_3 = 2$ (i.e. $T(L_n) = 3$)
Obviously, the only vertextype of vertices of N_2 is $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, and the only two vertex types of vertices of N_3 are a source and a sink.

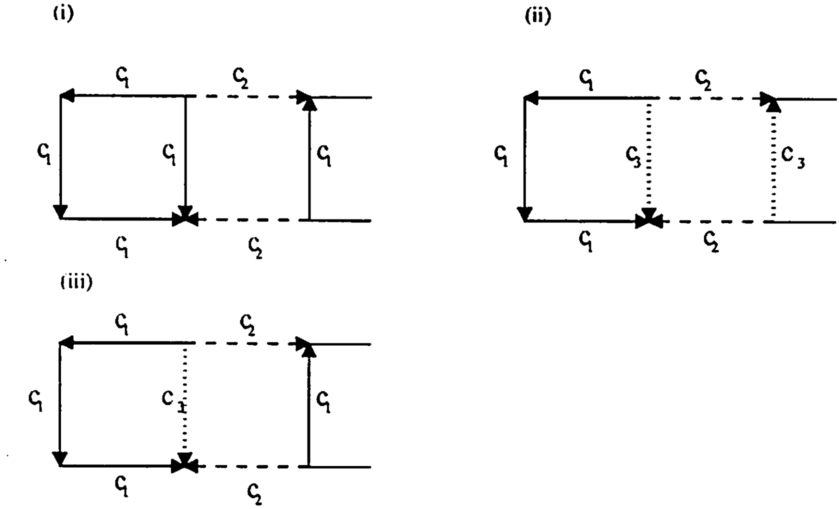

Figure 5: Each subgraph contains a two-color cycle forbidden in Lemma 2.2.

W.L.O.G., the possible book-embedding graph must contain one of the following three induced subgraphs :

Since, there exists a two-color cycle for each subgraph which is forbidden in Lemma 2.2, a contradiction. So $T(L_n) = 4$.

■

References

[1] J. F. Buss, A. L. Rosenberg, and J. D. Knott, *Vertex types in book-embedding*, SIAM J. Disc. Math. Vol.2 No 2, May 1989, pp. 156-175.

[2] F. R. K. Chung, F. T. Leighton, A. L. Rosenberg, *DIOGENES- A methodology for designing fault-tolerant propessor arrays*, in Proc. 13th Internat. conference on Faulty-Tolerent Computing. 1983, pp. 26-32.

[3] A. L. Rosenberg, *The DIOGENES approach to testable fault-tolerant arrays of processors*, IEEE Trans. Comput., 32(1983), pp.902-910.

Corresponding e-mail address : hlfu@math.nctu.edu.tw

Correction to "All c-Bhaskar Rao designs with block size 3 and $c \geq -1$ exist"

Spencer P. Hurd
The Citadel, Department of Mathematics and CS
Charleston, SC, 29409, hurds@citadel.edu

and

Dinesh G. Sarvate
University of Charleston, Department of Mathematics
Charleston, SC, 29424, sarvated@cofc.edu

Abstract: We correct an earlier theorem and reprove its consequences regarding c-BRD's with $v \equiv 5, 8 \pmod{12}$. The original conclusions remain valid.

Keywords: Bhaskar Rao Designs, c-BRD, BRD, BIBD.

It is necessary to correct the statements of Theorem 3 and its Corollary in the authors' paper "All c-Bhaskar Rao designs with block size 3 and $c \geq -1$ exist," Ars Combinatoria 59(2001), pp. 21-32 [3]. The statements of all the other theorems in the paper are correct, but two theorems, Theorems 10 and 13, require further argument which is based on a new result which we provide here in the form of a Lemma.

In the statement of Theorem 3, we hypothesized that v(v-1) is not 0 mod 12. What we proved, and mostly applied, required that v(v-1) is not 0 mod 4. With this correction, the proofs of Theorem 3 and the Corollary immediately following are correct as given. All the other theorems, Theorems 4 to 13 are correctly stated, and we indicate here where "mod 4" must replace "mod 12" in discussions or proofs. Theorem 10 requires one new easy case. The significant consequence is that Theorem 13, while correct, requires consideration of $v \equiv 5, 8$ mod 12.

The reader is referred to [3] for an introduction, notation, terminology, and Tables, all of which need no change. A c-BRD(v, k, λ) is a {0, 1, -1}-matrix whose rows have inner product c and such that the matrix of absolute values is the incidence matrix of a BIBD(v, k, λ). The full statements of the revised theorems are repeated here, and we take the opportunity to improve (smooth) the proof of Theorem 8 and to state Theorem 11 completely (one line of which was missing). We re-organize the proof of Theorem 10 to consider one new case,

Case 3 of the proof of Theorem 13 is new (and necessary), and the Lemma A below is a new result.

Theorem 3. *Suppose v(v-1) is not congruent to 0 mod 4. Then, for any c-BRD(v, 3, λ):*

(A) If c = 2s and λ = 2t, then s $\equiv$ t (mod 2).

(B) Suppose x = 1 or 5. If c = 2s+1 > 0 and λ = 6t+x, then s + t is even. If c = -2s - 1 < 0, then s + t is odd.

(C) Suppose λ = 6t + 3. If c = 2s + 1 > 0, then s + t is odd. If c = -2s - 1 < 0, then s + t is even. In particular, if c = -1, i.e., s = 0, then t must be even.

Corollary. *Suppose v(v-1) is not congruent to 0 mod 4. Then for any c-BRD(v, 3, λ), c $\equiv$ λ (mod 4).*

The proof of Theorem 3 is now correct as given in [3]. Lemma 4, Theorem 5, and Theorem 6 require no discussion here. The statements of Theorems 7-13 are correct, but in the proofs or discussions of Theorems 7 to 12, replace "mod 12" with "mod 4" in applying Theorem 3 or its Corollary.

Before re-proving Theorem 10 or Theorem 13, we establish the following Lemma which contains new results.

Lemma A *Whenever v $\equiv$ 0, 1 (mod 4), a 4-BRD(v, 3, 6) exists.*

Proof: First, for v = 4, 9, or 12, the incidence matrix for a 4-BRD(v, 3, 6) may be formed by juxtaposition (placing side-by-side) of the incidence matrices of a 0-BRD(v, 3, 2) and a 4-BRD(v, 3, 4), i.e., a BIBD(v, 3, 4). A 4-BRD(5, 3, 6) is obtained from a 1-BRD(5, 3, 3) and a 3-BRD(5, 3, 3). A 4-BRD(8, 3, 6) is obtained from the following signed difference family:

(∞, -0, 1), (∞, 0, 2), (∞, 0, 3), (1, 2, -4),
and 4 copies of (1, 2, 4).

We note that, if v $\equiv$ 0, 1 (mod 4), a PB(v, {4, 5, 8, 9, 12}, 1) exists [1]. The Lemma now follows from Theorem 1 of [2] which states: : *Let PB(v; K; λ) be any pairwise balanced design. If there exists a c-BRD(k', k, μ) for each k' in K, then there exists a cλ-BRD(v, k, $\lambda\mu$).*

Theorem 8. *If λ $\equiv$ 1, 5 (mod 6), then the necessary conditions are sufficient for (-1)-BRD(v, 3, λ) to exist.*

Proof: By Table 1, we note v $\equiv$ 1, 3, 7, 9 (mod 12). First suppose v $\equiv$ 1, 9 (mod 12). We construct the incidence matrix of a (-1)-BRD(v, 3, 6t+1) by juxtaposing, or placing side-by-side, the incidence matrices for 3t-1 copies of a 0-BRD(v, 3, 2) and one copy of a (-1)-BRD(v, 3, 3). If λ = 6t+5, use two more 0-BRD(v, 3, 2). On the other hand, suppose v $\equiv$ 3, 7 (mod 12), i.e., v(v-1) $\neq$ 4m for any m. For this case, the proof is given in [3].

Theorem 10 The necessary conditions are sufficient for the existence of 5-BRD(v, 3, 6t+3) and 3-BRD(v, 3, 6t+3)

Proof: For 3-BRD(v, 3, 6t+3), use the first paragraph of Section 4 replacing mod 12 with mod 4. For 5-BRD(v, 3, 6t+3), there are 3 cases. Case 1: $v \equiv 1, 9 \pmod{12}$. Use the argument given for $v(v-1) \equiv 0 \pmod{12}$. Case 2: $v \equiv 3, 7, 11 \pmod{12}$. Use the remaining argument given, noting $v(v-1) \neq 4t$. Case 3 (*new*): $v \equiv 5 \pmod{12}$. Here create a 5-BRD(v, 3, 6t+3) by juxtaposition of a 4-BRD(v, 3, 6) from Lemma A, (t-1)-copies of a 0-BRD(v, 3, 6), and one 1-BRD(v, 3, 3).

We re-state Theorem 11 to correct a misprint.

Theorem 11. *The necessary conditions are sufficient for the existence of a 3-BRD(v, 3, λ).*

Theorem 13. *Suppose $c \geq -1$. Then, the necessary conditions are sufficient for the existence of all c-BRD(v, 3, λ).*

Proof: The cases $c = -1$ (Theorem 9) and $c = 0$ (i.e., for BRD's) are done. We need only consider positive c. We divide the argument into 3 cases.

Case 1: Assume $v(v-1) \equiv 0 \pmod{12}$. Use the proof given.

Case 2: Assume v(v-1) is not congruent to 0 mod 4. Use the proof given.

Case 3: Assume $v \equiv 5, 8 \pmod{12}$. (This is the new case.) It is only necessary here to show how to construct the incidence matrix for the needed c-BRD for those examples not given in Tables 5 and 6 in [3]. Consequently, here $\lambda - c$ is even but not a multiple of 4 as those cases are in the referenced tables.

First suppose $v \equiv 5 \pmod{12}$. It follows that $\lambda \equiv 0, 3, 6, 9 \pmod{12}$. Constructions for all these are in Table 1. In the table, *m means use m copies. We note that, in general, existence of the c-BRD's (and the underlying BIBD's) in Table 1 follow from Theorems 5, 6 in [3], Table 1 in [3], and the new Lemma A. A 1-BRD(v, 3, 3) exists whenever $v \equiv 1 \pmod{4}$ [2]. Now suppose $v \equiv 8 \pmod{12}$. In this case, $\lambda \equiv 0, 6 \pmod{12}$. The ingredients for the c-BRD are in the first two sections of Table 1.

Acknowledgments: The authors wish to thank Amanda Roberts for bringing the difficulty with Theorem 3 to our attention, and to Malcolm Greig for a helpful reading of the papers and for supplying the difference family construction in the Lemma A.

References:

1. H. Hanani, The existence and construction of balanced incomplete block designs, *Ann. Math. Statis.* **32**(1961), p.361-386.