On Groups with Redundancy in Multiplication[1]

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Following the terminology in [2], we define a B_k-group to be a group G which satisfies the following condition:

If $\{a_1, \ldots, a_k\}$ is a k-subset of G, then $|\{a_i a_j | 1 \leq i, j \leq k\}| \leq \frac{k(k+1)}{2}$.

As in [2] and [3], we will use the notation $\{a_1, a_2, \ldots, a_k\}^2$ to denote $\{a_i a_j | 1 \leq i, j \leq k\}$.

Clearly all abelian groups are B_k-groups, as are all nonabelian groups of order $\leq \frac{k(k+1)}{2}$. The interesting problem is to determine which other nonabelian groups are B_k-groups. When $k = 2$, Freiman [4] showed that a nonabelian group is a B_2-group if and only if it is a Hamiltonian 2-group. It appears that this is the only value of k for which a complete characterization has been given, but Brailovsky [3] proved that when $k > 2$ a nonabelian B_k-group must be finite of order $\leq 2(k^3 - k)$. The corresponding notion of B_k-rings has been investigated by Bell and Klein in [2], and the same authors studied a related redundancy condition on rings in [1]. We would like to thank Howard Bell for several helpful conversations on this topic, and for providing us with a copy of [2].

In this note, we give a complete characterization of B_k-groups in the cases $k = 3, k = 4$. Specifically, we show that the only nonabelian B_k-groups in these cases are those of order $\leq \frac{k(k+1)}{2}$. We then give an example showing that this behaviour does not extend to $k = 5$.

The first half of the proof of the $k = 3$ case is essentially the same as the proof of Lemma 4.3 in [2].

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Theorem 1 *A nonabelian group G is a B_3-group if and only if G is isomorphic to S_3.*

Proof.
Assume that G is a nonabelian B_3-group. We will show first that if x, y are two noncommuting elements of G, then $< x, y > \cong S_3$.

To see this suppose first that $x^2 = 1$ and $y^2 \neq 1$. Note that $\{x, y, xy\}^2$ contains the 6 distinct elements $1, xy, y, yx, y^2, yxy$, so any other element in $\{x, y, xy\}^2$ must equal one of these. The only possibility for xy^2 is $xy^2 = yx$, while the possibilities for $xyxy$ are $xyxy = 1, yx$, or y^2. The latter two cases are incompatible with $xy^2 = yx$, so we are left with $xyxy = 1$ and $xy^2 = yx$. But this means that $y^3 = x^2y^3 = xyxy = 1$, and so $< x, y > \cong S_3$ as desired.

Next note that if $x^2 = 1$ and $y^2 = 1$, then $(xy)^2 \neq 1$ (since $xy \neq yx$). Since $< x, y > = < x, xy >$, we are in the case covered by the previous paragraph.

Finally assume $x^2 \neq 1$ and $y^2 \neq 1$. Since $\{1, x, y\}^2$ contains the 6 distinct elements $1, x, y, x^2, xy, yx$, we conclude that $y^2 = x^2$ in this case. But then consider $\{x, y, xy\}^2$. It contains the 6 distinct elements $x^2, xy, x^2y, yx, yxy, xy^2$. Hence xyx equals one of these elements, and the only possibility is $xyx = yxy$. Similarily we must have $xyxy = x^2$ or yx, but these are both incompatible with $xyx = yxy$. We conclude that this case is impossible.

Now let x, y be any two noncommuting elements of G. We have shown that $< x, y > \cong S_3$, and may assume that $x^2 = 1, y^3 = 1$ and $yx = xy^2$.

We wish to prove that $< x, y > = G$. Assume to the contrary that $z \notin < x, y >$. We may assume that x and z don't commute (otherwise replace z by yz). It follows from our earlier argument that $< x, z > \cong S_3$. Since $x^2 = 1$, either $z^3 = 1$ or $(xz)^3 = 1$. Replacing z by xz if necessary, we may assume that $z^3 = 1$. But then $\{x, y, z\}^2$ contains the 7 distinct elements $1, xy, xz, yx, y^2, yz, z^2$, and we have a contradiction. □

The proof for the $k = 4$ case follows along similar lines but is somewhat longer, primarily because of the proof of the following lemma (let D_n denote the dihedral group of order n and K_8 the quaternion group).

Lemma 2 *If x, y are noncommuting elements of a B_4-group, then $< x, y >$ is isomorphic to one of S_3, D_8, K_8, D_{10}.*

The main result follows reasonably directly from Lemma 2. Because of this, we will give the proof of Theorem 3 first and later outline the argument for Lemma 2.

Theorem 3 *A nonabelian group G is a B_4-group if and only if G is isomorphic to one of S_3, D_8, K_8, D_{10}.*

Proof of Theorem 3.
Let x, y be noncommuting elements from a nonabelian B_4-group G. Then $< x, y >$ is isomorphic to one of S_3, D_8, K_8, D_{10}. We will prove that $G =< x, y >$ in all cases.

First assume $< x, y > \cong D_8$ or D_{10}. We may also assume $x^2 = 1, y^4 = 1$ or $y^5 = 1$, and $yx = xy^{-1}$. Note that $\{x, y, xy\}^2$ contains the 8 distinct elements $1, xy, y, yx, y^2, yxy, xyx, xy^2$. Hence, if $z \notin < x, y >$ we would have 11 distinct elements in $\{x, y, xy, z\}$, giving a contradiction. So $< x, y >= G$ in this case.

Next assume $< x, y > \cong K_8$ and $x^4 = 1, y^2 = x^2, yx = xy^3$. Say $z \notin < x, y >$. We can assume that z does not commute with x (otherwise use yz). By Lemma 2, since $< x, z >$ contains an element of order 4 we know that $< x, z > \cong D_8$ or $< x, z > \cong K_8$. But if $< x, z > \cong D_8$, then $< x, z >= G$ by the previous paragraph. So we may assume $< x, z > \cong K_8$. It follows that $z^2 = x^2 (= y^2)$ and $zx = xz^3$. Now $\{x, y, xy\}^2$ contains the 7 distinct elements $x^2, xy, x^2y, yx, yxy, xyx, xy^2$. Also the elements xz, yz, xyz, zx are distinct, so $\{x, y, xy, z\}^2$ has 11 distinct elements and we have a contradiction. Again $G =< x, y >$.

Finally assume that $< x, y > \cong S_3$ and $x^2 = 1, y^3 = 1, yx = xy^2$. Say $z \notin < x, y >$. We may assume that z does not commute with x, and the only case not settled is where $< x, z > \cong S_3$. We may assume z is of order 3 (using xz if necessary), and so $zx = xz^2$. But now $\{x, y, xy, z\}^2$ contains all 6 elements of $< x, y >$ plus the 5 additional distinct elements xz, yz, xyz, z^2, zx. We again have a contradiction and conclude that $G = < x, y >$.
□

Now we return to the lemma.

Proof of Lemma 2.
This argument is divided into a number of cases, depending on the orders of x and y. Initially we will consider the situation where one of the generators (say x) is of order 2.

First assume that $x^2 = 1$ and that y is of order 8. Then $\{x, y, y^2, y^3\}^2$ contains the 9 distinct elements $1, xy, xy^2, xy^3, y^2$, y^3, y^4, y^5, y^6. Hence either yx or y^3x must be equal to one of the 9 elements listed. The only possibility for yx is $yx = xy^3$ (note $yx = xy^2$ implies $y^4x = xy^8 = x$), while the only possibility for y^3x is $y^3x = xy$ (note $y^3x = xy^3$ implies $yx = xy$) and this would then give $yx = xy^3$. So yx must equal xy^3 in either case. But then $\{x, y, xy, y^2\}^2$ contains the 11 distinct elements $1, xy, y, xy^2, yx, y^2, yxy, y^3, xyxy, y^2x, y^2xy$. We have a contradiction, so this case doesn't occur.

Next assume that $x^2 = 1$ and that the order of y is greater than 6 but not equal to 8. In this case $\{x, y, y^2, y^3\}^2$ contains the 10 distinct elements $1, xy, xy^2, xy^3, yx, y^2, y^3, y^4, y^5, y^6$ (note $yx = xy^2$ implies $y = yx^2 = y^4$ while $yx = xy^3$ implies $y = y^9$). It follows that y^2x must equal one of these ten elements and the only possibilities are $y^2x = xy^2$ or $y^2x = xy^3$ (note $y^2x = xy$ implies $y = x^2y = y^4$). Similarly we must have $y^3x = xy^2$ or $y^3x = xy^3$. But $y^2x = xy^2$ and $y^3x = xy^3$ together imply $yx = xy$, while $y^2x = xy^3$ and $y^3x = xy^2$ give $y^6x = xy^9$ and $y^6x = xy^4$. Hence we obtain a contradiction and this case also cannot occur.

Now assume that $x^2 = 1$ and that y is of order 6. Hence $\{x, y, y^2, y^3\}^2$ contains the 9 distinct elements $1, xy, xy^2, xy^3, yx, y^2, y^3, y^4, y^5$. So either y^2x or y^3x must equal one of these nine elements. The only possibilities are $y^2x = xy^2$ or $y^3x = xy^3$ (for example $y^2x = xy^3$ would imply $y^4x = xy^6 = x$). But if $y^2x = xy^2$, then $\{x, y, xy, y^3\}^2$ contains the 11 distinct elements $1, xy, y, xy^3, yx, y^2, yxy, y^4, xy^2, y^3x, y^3xy$. Also, if $y^3x = xy^3$ then $\{x, y, xy, y^2\}^2$ contains the 11 distinct elements $1, xy, y, xy^2, yx, y^2, yxy, y^3, xy^3, y^2x, y^4$. Again we have a contradiction.

The next case is where $x^2 = 1$ and $y^5 = 1$. Observe that $\{x, y, xy, y^2\}^2$ contains the 10 distinct elements $1, xy, y, xy^2, yx, y^2, yxy, y^3, xy^3, y^4$ (note $yx = xy^2$ implies $y = yx^2 = y^4, xy^3 = yx$ implies $y = y^9$). Now y^2x must equal one of these elements, and the only possibility is $y^2x = xy^3$ (note $y^2x = xy$ implies $y = y^4, y^2x = xy^2$ implies $yx = xy$). But then $yx = y^6x = xy^4$, and so in this case we have $< x, y > \cong D_{10}$ which was one of the possibilities.

Next assume $x^2 = 1$ and y is of order 4. If $yx = xy^3$, then we have $< x, y > \cong D_8$, so assume this is not the case. But then similar reasoning to that seen before tells us that $\{x, y, xy, y^3\}^2$ contains the 11 distinct elements $1, xy, y, xy^3, yx, y^2, yxy, xy^2, x, y^3x, y^3xy$, and we have a contradiction.

We now assume $x^2 = 1$ and $y^3 = 1$. If $yx = xy^2$ then $< x, y > \cong S_3$, so assume that this is not the case. Then $\{x, y, xy, y^2\}^2$ contains the 10 distinct elements $1, xy, y, xy^2, yx, y^2, yxy, x, y^2x, y^2xy$. In this case, xyx and $xyxy$ must both equal elements which are already listed. But the only possibilities for xyx are $xyx = yxy$ or $xyx = y^2xy$, while the possibilities for $xyxy$ are $xyxy = yx$ or $xyxy = y^2x$. Checking case by case, we see that each combination of these possibilities leads to a contradiction.

If $x^2 = 1$ and $y^2 = 1$ then $(xy)^2 \neq 1$ (since $xy \neq yx$), so we can assume we are in one of the cases already considered.

To finish the argument, we need to handle cases where neither x nor y is of order 2.

First assume that $x^2 \neq 1, y^2 \neq 1$ and $x^2 = y^2$. If $yx = xy^3$, it will then follow that $y^3 = yx^2 = xy^3x = y^7$, so $y^4 = 1$ and $G \cong K_8$. Hence we may assume $yx \neq xy^3$. Then $\{x, x^3, y, xy\}^2$ contains the 10 distinct elements $x^2, x^4, xy, x^2y, x^3y, x^4y, yx, yx^3, yxy, xy^2$ (note $x^3 \neq 1$ since $x^2 = y^2$, also $yx^3 = xy$ implies $xyx = yx^4 = x^4y$). Hence xyx must be equal to some element in this list, and the only possiblity is $xyx = yxy$. But then xyx^3 is distinct from all elements in the list, and we have a contradiction.

Next assume that $x^2 \neq 1, y^2 \neq 1, x^2 \neq y^2$, and also that $xyx \neq y$ and $yxy \neq x$. We may also assume $(xy)^2 \neq 1$ or we would be in an earlier case. Consider the 12 elements $1, x, y, xy, x^2, x^2y, yx, y^2, yxy, xyx, xy^2, xyxy$ in $\{1, x, y, xy\}^2$. At least <u>two</u> of these must be equal to other elements in the list. However, given the conditions, the only possibilities are $yx = x^2y, yxy = x^2, xyx = yxy, xyx = y^2, xy^2 = yx, xyxy = yx$. The condition $yx = x^2y$ contradicts each of the other 5, and the same remark holds for $yx = xy^2$ and $xyxy = yx$. So we assume these do not hold. Next observe that $yxy = x^2$ and $xyx = yxy$ cannot be true at the same time, nor can $xyx = yxy$ and $xyx = y^2$. We are left with the possibility that $yxy = x^2$ and $xyx = y^2$. But in this case $\{x, y, xy, x^2\}^2$ contains the 11 distinct elements $x^2, xy, x^2y, x^3, yx, y^2, yx^2, xy^2, xyx^2, x^3y, x^4$, and so this case cannot occur.

The last set of cases all assume $x^2 \neq 1, y^2 \neq 1, x^2 \neq y^2$ and $xyx = y$. Once these possibilities have been settled, we will be finished because similar situations with $yxy = x$ are symmetrical. Note that $yx = x^{-1}y$ means that $y^n x = x^{-1}y^n$ whenever n is odd, and so forces the order of y to be even.

To begin, assume $x^2 \neq 1, y^2 \neq 1, x^2 \neq y^2, xyx = y$ and the order of y is greater than 4 but not equal to 8. Then $\{1, x, y, y^3\}^2$ contains the 11 distinct elements $1, x, y, y^3, x^2, xy, xy^3, yx, y^2, y^4, y^3x$ (note $x^2 = y^3$ implies $xy^3 = y^3x = x^{-1}y^3$, also either of $yx = xy^3, xy = y^3x$ implies that $x^2 = y^2$, also $y^4 = x^2$ implies $y^5 = yx^2 = x^{-2}y = y^{-3}$), and so this case can't occur.

Next consider the case where $x^2 \neq 1, y^2 \neq 1, x^2 \neq y^2, xyx = y$ and the order of y is 8. Now $\{1, x, y, y^3\}^2$ contains the 10 distinct elements $1, x, y, y^3, x^2, xy, xy^3, yx, y^2, y^3x$. So y^4 must be in this list, and the only possibility is $y^4 = x^2$. But then y^6 is distinct from all elements in the list, and we have a contradiction.

Finally, we assume $x^2 \neq 1, y^2 \neq 1, x^2 \neq y^2, xyx = y$ and the order of y equals 4. Then $\{x, y, xy, y^2\}^2$ contains the 9 distinct elements $x^2, xy, x^2y, xy^2, y^2, y^3, xyx, xy^3, 1$. It follows that either yx or yxy must be equal to one of the elements listed, and the only possibilities are $yx = x^2y$ or $yxy = x^2$. If $yx = x^2y$, then $x^3 = 1$ and $\{xy^2, y, x^2, x^2y\}^2$ contains the 11 distinct elements $xy^2xy^2, xy^3, xy^2x^2, xy^2x^2y, yxy^2, yx^2, yx^2y, x^2y, x, x^2y^2, x^2yx^2$ $(= x^2, xy^3, y^2, y^3, x^2y^3, xy, xy^2, x^2y, x, x^2y^2, y)$. On the other hand, if $yxy = x^2$, then $y^2 = x^3$ and $\{x, y, x^2, x^2y\}^2$ contains the 11 distinct elements $x^2, xy, x^3, x^3y, yx, yx^2, yx^2y, x^2y, x^4, x^2y^2$, x^2yx^2. So this case can't occur either.

The proof is complete. □

We will close with an example showing that Theorems 1 and 3 do not extend to the case $k = 5$. Specifically, we present a nonabelian B_5-group of order $16 > \frac{5(6)}{2}$.

Example 4.
Let $G = K_8 \times C_2$. We will show that G is a B_5-group. To do this, it will be useful to note that the center $Z(G)$ is an elementary abelian 2-group of order 4, and that one particular element of $Z(G)$, which we will denote h, has the property that $x^2 = h$ for all noncentral elements x of G. In addition, if a product xy of noncentral elements x, y is noncentral in G then $xy = yxh$.

Assume to the contrary that G is not a B_5-group. This means that we can choose distinct elements a, b, c, d, e in G such that $\{a, b, c, d, e\}^2 = G$.

Observe that each of a^2, b^2, c^2, d^2, e^2 must equal 1 or h. In particular, this means that there are at least 3 repeated products among these squares. If $a^2 = 1$ and $b^2 = 1$, then a and b would have to be central, and this would lead to 7 more repeated products in $\{a,b,c,d,e\}^2$. We now have a contradiction to $\{a,b,c,d,e\}^2 = G$ (since there are 25 products), so we can assume from now on that at most one of a,b,c,d,e is central in G.

First assume that one of these elements is central, i.e. $a^2 = 1$ and $b^2 = c^2 = d^2 = e^2 = h$. We now have 7 repeated products in $\{a,b,c,d,e\}^2$ (namely $c^2, d^2, e^2, ba, ca, da, ea$). Since $\{a,b,c,d,e\}^2 = G$ and $|Z(G)| = 4$, some product of different noncentral elements must be in $Z(G)$ and not equal to 1 or h - by relabelling if necessary we can assume bc is this product. But then $bc = cb$, so we have an eighth repeated product. In addition, some other such product must equal the fourth element of $Z(G)$, and this gives a ninth repeated product. If this product involves b or c, we would be able to construct yet another central product and would have a tenth repeated product and a contradiction (e.g.if bd is central, then so is $cd = (cb)b^2(bd)$). So the only possibility is that $de \in Z(G)$. But now $(ce)(db) = c(ed)b = (cb)(ed) = h$, since it is the product of the two elements of $Z(G)$ which are different from 1 and h. But for this to happen in G, it must be the case that either ce and db are central or $ce = db$. In either case, we have a tenth repeated product, and hence a contradiction.

We are left with the case where $a^2 = b^2 = c^2 = d^2 = e^2 = h$. So now we have 4 repeated products in $\{a,b,c,d,e\}^2$. In this situation, the three central elements other than h must all be obtainable in $\{a,b,c,d,e\}^2$. Some element in $\{a,b,c,d,e\}$ must be used twice in these products - by relabelling, we can assume ab and ac are central. But then, as seen earlier, $bc(= ba(a^2)ac)$ is also central. So now we have 7 repeated products. Since ab, ac and bc are all different, we may assume that $ab = 1$ and it follows that $b = ah$. Next observe that if ad or ae were central, then we would be able to find additional central elements as above, getting more repeated products and a contradiction. So we can assume that none of these products is central. But then $ad = dah = db, da = adh = bd, ae = eah = eb$, and again we have 10 repeated products.

This completes the proof. □

The argument in Example 4 shows that $|\{a, b, c, d, e\}^2| \leq 14$ when $G = K_8 \times C_2$. It is easy to see that the bound of 14 is best possible.

References

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